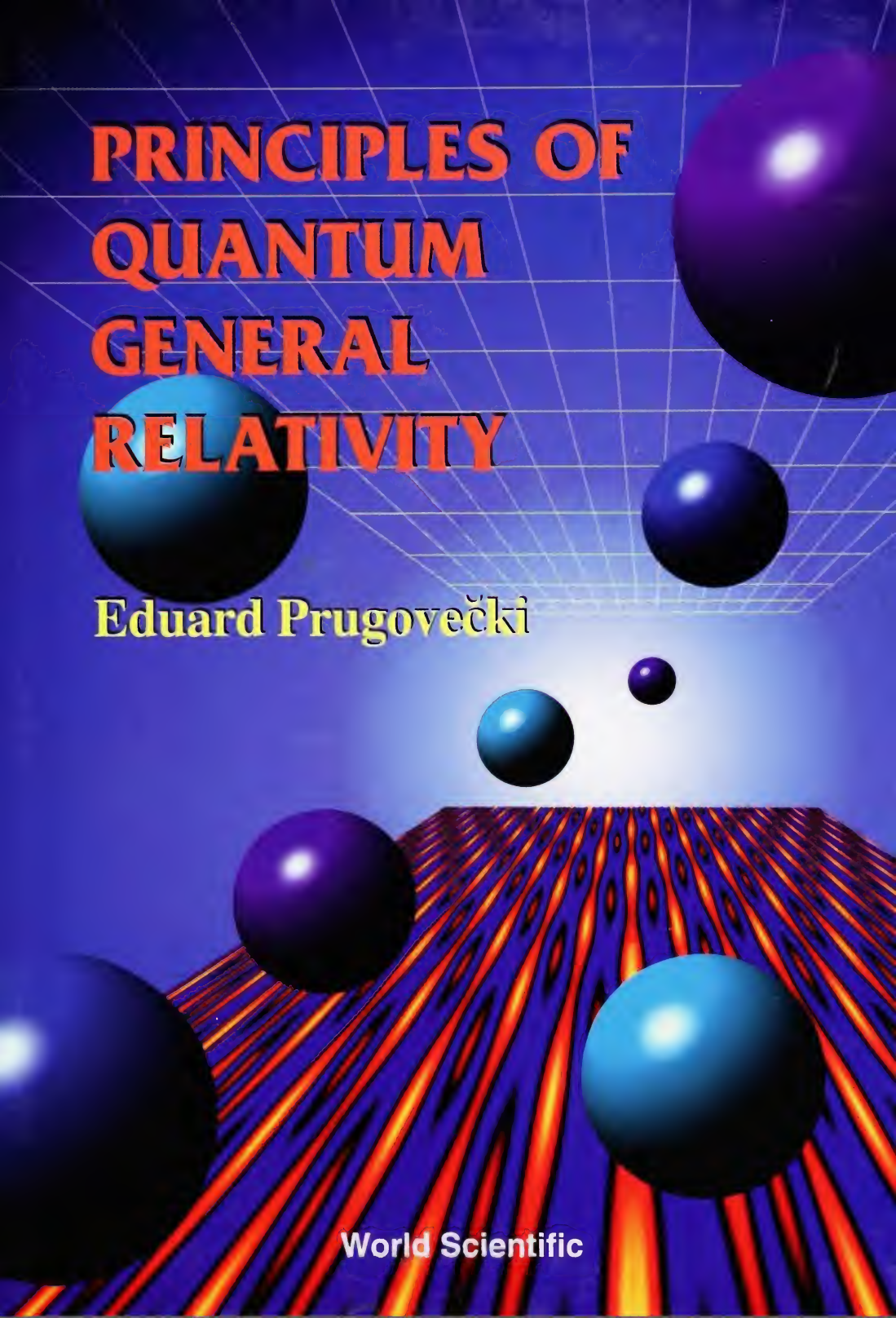
PRINCIPLES OF QUANTUM GENERAL RELATIVITY

Eduard Prugovečki



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Eduard Prugovečki is a mathematical physicist, who acquired his graduate training at Princeton University, finishing his Ph.D. there in 1964. After spending an additional year in Princeton as research associate, he became Postdoctoral Fellow at the Theoretical Physics Institute in Edmonton, Canada. In 1967 he joined the University of Toronto, where he became Professor in 1975. He has written numerous articles on the mathematical foundations and methods in quantum mechanics, quantum field theory in flat and curved spacetime, and quantum gravity. He has published three monographs dealing with various aspects of this research: *Quantum Mechanics in Hilbert Space* (Academic Press, 1971; 2nd ed., 1981), *Stochastic Quantum Mechanics and Quantum Spacetime* (Reidel, 1984), and *Quantum Geometry* (Kluwer, 1992).

Errata for
Principles of Quantum General Relativity

Location	*Original expression*	*Corrected expression*
p. 34, Eq. (3.5b):	$\boldsymbol{u},\boldsymbol{u}^*$	$\boldsymbol{u}',\boldsymbol{u}^{*\prime}$
p. 43, 1st line from the top:	Footnotes 17 and 18	Footnotes 18 and 19
p. 81, 1st line above Eq. (3.18):	confidence of	confidence functions of
p. 115, Eq. (1.11):	$\boldsymbol{u}' = \boldsymbol{u}\cdot(b,\Lambda)$	$\boldsymbol{u}' = \boldsymbol{u}\cdot(b,\Lambda)^{-1}$
p. 117, Eqs. (2.1) & (2.7):	iv^k	$i\ell v^k$
p. 124, Eq. (3.5):	$\varphi(q-x,v)$	$\varphi(q+x,v)$
p. 124, Eq. (3.5):	iv^k	$i\ell v^k$
p. 125, Eq. (3.7):	$\partial\varphi_{x;\mathcal{L}}(\zeta)/\partial x^\mu$	$-\partial\varphi_{x;\mathcal{L}}(\zeta)/\partial x^\mu$
p. 125, Eq. (3.9b):	$\boldsymbol{s}(\mathcal{L};x)\cdot(b(x),\Lambda(x))$	$\boldsymbol{s}(\mathcal{L};x)\cdot(b(x),\Lambda(x))^{-1}$
p. 127, Eq. (4.14):	$M^{21}_{\boldsymbol{s}(x)} = -M^{12}_{\boldsymbol{s}(x)}$	$M^{03}_{\boldsymbol{s}(x)} = -M^{30}_{\boldsymbol{s}(x)}$

p. 128, Eq. (4.16):	$-i\tilde{\boldsymbol{\theta}}^i_s(\boldsymbol{X})$	$+i\tilde{\boldsymbol{\theta}}^i_s(\boldsymbol{X})$
p. 128, Eq. (4.17):	$+i\tilde{\boldsymbol{\theta}}^i_s(\boldsymbol{X})$	$-i\tilde{\boldsymbol{\theta}}^i_s(\boldsymbol{X})$
p. 129, Eqs. (4.18)-(4.19):	Ψ^{ζ}_x	Ψ^{ζ}_x
p. 129, Eq. (4.20):	$-i\tilde{\boldsymbol{\theta}}^i$	$+i\tilde{\boldsymbol{\theta}}^i$
p. 134, Eq. (6.3):	$K^*_\gamma(x'',\zeta'';x',\zeta')$	$K^*_\gamma(x',\zeta';x'',\zeta'')$
p. 135, Eq. (6.5):	$K^{(\ell)}(x''+q'',v'';x'+q',v')$	$K^{(\ell)}(x'+q'',v'';x'+q',v')$
p. 135, Eq. (6.6):	$K^{(\ell)}(x_N,\zeta_N;x_{N-1},\zeta_{N-1})$	$K^{(\ell)}(x_N,v_N;x_{N-1},v_{N-1})$
p. 148, 2nd line below Eq. (8.5):	in (2.18)	in (8.5)
p. 148, 2nd line from the bottom:	encountered all by	encountered by
p. 149, 2nd line below Eq. 8.8):	by path integral	by the path integral
p. 164, Eq. (3.2):	$-i\tilde{\boldsymbol{\theta}}^i$	$+i\tilde{\boldsymbol{\theta}}^i$
p. 164, Eq. (3.3):	$-i\tilde{\boldsymbol{\theta}}^i_s(\boldsymbol{X})$	$+i\tilde{\boldsymbol{\theta}}^i_s(\boldsymbol{X})$
p. 172, Eq. (4.17):	$-i\delta x^j_n$	$+i\delta x^j_n$
p. 173, Eq. (4.20):	$-i\delta x^j_n$	$+i\delta x^j_n$

p. 174, Eq. (4.24a):	$-X_n^{\ i}$	$+X_n^{\ i}$
p. 174, Eq. (4.28):	$-X^i(\tau)$	$+X^i(\tau)$
p. 179, 1st line above Eq. (5.15):	and (5.10)	and (5.11)
p. 179, Eq. (5.15):	$dx/\sqrt{x-2}$	$\sqrt{1-2/x}\,dx$
p. 180, 2nd line below Eq. (5.22):	with the	with a subbundle of the
p. 201, 2nd line above Eq. (8.2):	to following	to the following
p. 225, Eq. (4.5):	$-i\tilde{\boldsymbol{\theta}}_s^i(\boldsymbol{X})$	$+i\tilde{\boldsymbol{\theta}}_s^i(\boldsymbol{X})$
p. 251, Eq. (5.2):	$-i\tilde{\boldsymbol{\theta}}^i$	$+i\tilde{\boldsymbol{\theta}}^i$
p. 277, 4th line from the top:	of all matter	of matter
p. 277, 3rd line in Footnote 4:	It not made	It is not made
p. 280, 11th line from the top:	Ashtekar and Stachel, 1961	Ashtekar and Stachel, 1991
p. 282, 2nd line above Eq. (3.12a):	any generalized Lorenz	the proper Lorenz
p. 283, 4th line from the top:	generalized Lorenz gauge conditions	proper Lorenz gauge condition
p. 304, Eq. (7.7):	$-i\tilde{\boldsymbol{\theta}}_s^j$	$i\tilde{\boldsymbol{\theta}}_s^j$
p. 310, last line:	in typical fibre	in the typical fibre

p. 312, Eq. (8.27a):	$C_\alpha(X(t))$	$\bar{\partial}_\alpha + C_\alpha(X(t))$
p. 312, Eq. (8.27b):	$\overline{C}_\alpha(X(t))$	$\partial_\alpha + \overline{C}_\alpha(X(t))$
p. 314, Eq. (9.5):	$-i\tilde{\boldsymbol{\theta}}_s^j$	$+i\tilde{\boldsymbol{\theta}}_s^j$
p. 318, Eq. (9.29):	$-i\left(\delta x_n^j P_j - \tilde{\boldsymbol{\omega}}_{ab}(\delta x_n)M^{ab}\right)$	$+i\left(\delta x_n^j P_j + \frac{1}{2}\tilde{\boldsymbol{\omega}}_{ab}(\delta x_n)M^{ab}\right)$
p. 323, 3rd line from the bottom:	Bibliopolis, Naples	World Scientific, Singapore
p. 324, 4th line from the top:	*SpacedTime*	*Space-Time*

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To those who share Dirac's vision

"The research worker,
in his efforts to express the fundamental laws of Nature in mathematical form,
should strive mainly for mathematical beauty."

P. A. M. Dirac (1939)

"One must seek a new relativistic quantum mechanics and
one's prime concern must be to base it on
sound mathematics."

P. A. M. Dirac (1978)

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Preface

The primary goal of the present monograph is to explain and analyze, in a manner accessible to readers with a graduate-student level of knowledge and understanding of general relativity and quantum theory, the principles of a quantum-geometric unification of these two disciplines, formulated in a manner that can be justifiably said to supply the foundations for a quantum counterpart of classical general relativity.

In view of the many attempts, carried out in the course of the past several decades, at a consistent unification of general relativity with quantum theory, no contemporary theoretical framework can lay the claim, without qualifications, to the title "quantum general relativity." On the other hand, what the main *principles* underlying such a framework should be emerges very clearly from the work of the founders of general relativity and quantum theory. Indeed, the principles underlying the classical theory of general relativity were enunciated by Einstein from the outset, and were further clarified in his later writings; whereas those of quantum theory were formulated, often explicitly, but sometimes only implicitly, by Bohr, Born, Dirac, Heisenberg, and other well-known founders of quantum mechanics in many of their early writings. Therefore, what the basic principles of *quantum* general relativity should be can be discerned by examining and analyzing those writings.

It is the intent of the various chapters in the present monograph to study all the main physical and mathematical implications of the principles that emerge from those writings, examine their mutual compatibility at the conceptual level, and describe a geometric framework capable of incorporating them in a consistent manner, by taking advantage of recent advances in certain areas of mathematics and theoretical physics. Clearly, such a framework should also be capable of incorporating theoretical models whose numerical predictions would be in agreement with those of experimentally well-supported conventional models in relativistic quantum theory. It will be shown that such a task can be achieved by taking advantage of key geometric aspects and results in the theory of gauge fields and of coherent states developed during the last two decades, by incorporating into quantum mechanics and quantum field theory the idea of fundamental length along the epistemic lines first suggested by Born and Heisenberg, and by extrapolating the classical concept of local frame of reference into a concept of *quantum* local frame of reference, into which such a fundamental length is embedded from the outset.

The introduction of a fundamental length in quantum physics was advocated already in the 1930s by several of the best known founders of quantum mechanics. However, for various historical reasons (which will be discussed in the first chapter of this monograph), it was paid scant attention by the subsequent generation of theoretical physicists. It was only during the last decade that string theory revived the interest in this idea by proposing the Planck length as a basic parameter related to the extension of superstrings.

On the other hand, as opposed to the historical development of general relativity as well as of quantum theory, string theory itself emerged from phenomenological rather than

from foundational considerations. As a consequence, its underlying physical principles were not clearly enunciated from the outset, and some are still obscure. Hence, the present monograph will concentrate on a purely geometric formulation of quantum general relativity into which only well-established gauge groups are incorporated. In other words, the considered quantum geometries are based only on those principles of general relativity and quantum theory that have survived the test of time. On the other hand, these quantum geometries incorporate into their very structure a fundamental length in such a manner that, in the nonrelativistic and the special relativistic regime, conventional results can be recovered in the sharp-point limit that is obtained as that length is allowed to tend to zero.

A very general quantum-geometric framework for general relativity was expounded in a recent monograph of the present author. In contradistinction, the present volume sets emphasis on the implementation of fundamental physical principles rather than on mathematical generality, as it strives to bring out to the fore the conceptual simplicity of the ideas underlying those principles. These fundamental physical principles, and their method of implementation, will be explained at a level accessible to students who have already taken introductory courses in quantum mechanics and general relativity, but have not yet necessarily acquired the background knowledge in fibre and superfibre bundle theory, Krein spaces, gauge groups and gauge fields, coherent states, and other physically or mathematically advanced topics required in the formulation of quantum geometries. This means that the basic concepts of these mathematical disciplines will be explained in this monograph in a *physics* context, by first describing the physical motivation for their introduction, and then elaborating only on those mathematical aspects that are of an immediate physical relevance. Those readers interested in their more advanced mathematical aspects will be directed, in the main text or in footnotes, to suitable publications.

This does not mean, however, that mathematical rigor has been in any way compromised. Rather, the mathematical material that is actually presented in this monograph is formulated in the mathematically sound manner advocated by Dirac as a methodological guide to the formulation of a "new relativistic quantum mechanics," and with the use of precise notation meant to bring out in strong relief the underlying physical and mathematical nuances. Although this kind of notation is standard in advanced mathematics textbooks, at first glance some of it might be unfamiliar to those readers primarily used to the typical textbooks in relativistic quantum physics. To assist that kind of reader, an extensive list of symbols has been provided. With its help, a more careful inspection should reveal that the additional information contained in the mathematically precise notation used in those formulae, as compared with their often sketchier formulation that is customary in typical textbooks on relativistic quantum mechanics and quantum field theory, clarifies not only their mathematical but also their physical meaning. In general, considerable emphasis is set on the kind of precise notation and rigorous argument that is long overdue in some of the fields with which this monograph is dealing.

Great emphasis is set also on the principle of mathematical beauty and simplicity. This principle was championed by Einstein throughout his life. As a methodological guide, it had its advocates also in Poincaré and Weyl, but its strongest champion was again Dirac. Indeed, as stated in a recent scientific biography of Dirac: "For Dirac this principle of mathematical beauty was partly a methodological moral and partly a postulate about

nature's qualities. It was clearly inspired by the theory of relativity, the general theory in particular, and also by the development of quantum mechanics." (Kragh, 1990, p. 277.)

These twin methodological principles of mathematical soundness and beauty, advocated by Dirac, have extensively guided the approach to quantum general relativity presented in this monograph. Of course, the application of the second of these principles obviously required a discriminating attitude, geared to taking into account the predictions of those conventional quantum field theoretical models that have received indubitable experimental support. With that stipulation in mind, it became possible to apply Dirac's principle of mathematical beauty and simplicity to narrow down to unique choices the wide range of alternatives to the various key features in the formulation of quantum general relativity, which in previous publications of this author were presented from a perspective that emphasized mathematical generality. In particular, the adoption of this methodological principle influenced the choice of quantum-geometric propagators, and the formulation of quantum-geometric causality. This has a profound effect on the mathematical form of the implementation of the concept of quantum-geometric propagation of fields, which is central to the present formulation of quantum general relativity. As a consequence, in the special relativistic context the formal agreement with conventional quantum field theory becomes very straightforward at the perturbative level, since it involves primarily the removal of finite normalization factors that become infinite in the sharp-point limit in which the adopted fundamental length ℓ is allowed to tend to zero.

For the benefit of the reader who desires a concise overview of the final outcome of the present formulation of quantum general relativity, the principal ideas of the framework for geometric quantum gravity that emerges by the end of the last chapter of the present monograph can be summarized as follows:

1. The central concept of this framework is that of quantum frame and superframe bundle. The frames and superframes in such bundles take over the role played by the local Lorentz frames of classical general relativity. Due to their informational completeness, these local quantum frames and superlocal quantum superframes are capable of taking over the role played by complete sets of (compatible) observables in orthodox quantum mechanics.

2. In the quantum geometric regime all the counterparts of the constraints that emerge from classical Yang-Mills field theories and from classical general relativity are geometrized by means of gauge groups, which inject them into Maurer-Cartan structural equations that govern the construction of connection forms in principal quantum frame and superframe bundles.

3. The elements of all massive quantum frame bundles possess an operational interpretation at the microlevel, obtained by replacing the test particles of classical relativity with quantum test bodies (i.e., geometro-stochastic excitons).

4. The quantum-geometric evolution of fields in mutual local interactions within a quantum spacetime supermanifold is described by a perpetually ongoing process of creation and annihilation of the geometro-stochastic excitons associated with them.

5. In the resulting framework for geometric quantum gravity, matter and radiation quantum fields in free fall propagate by parallel transport along stochastic paths; those paths are the limits of broken paths corresponding to time-ordered segmentations of a quantum gravitational spacetime supermanifold.

6. The causal time-ordering in a quantum gravitational supermanifold is intrinsic, since it is implicit in the (local) proper time marked by the massive constituents of the quantum frames. This *proper* time emerges from an adaptation to geometro-stochastic excitons of de Broglie's (1923, 1924) original idea that a *natural* time is inscribed in all matter in existence, since each elementary quantum object of rest-mass m can be viewed as a *natural* clock with mean period $T = 2\pi/m$ in Planck *natural* units.

7. The quantum general relativistic covariance principle is embedded in a quantum gravitational supergroup, which incorporates the semidirect product of two types of subgroups of gauge transformations: one type pertains to the metric equivalence classes of quantum superframes that are interrelated by supergauge transformations originating from superoperator representations of the diffeomorphism group, and gives rise to equivalence classes of *mean* Lorentzian metrics; the other type describes changes of quantum superframes within the equivalence class for each of these mean metrics, and is provided by superunitary representations of the Poincaré group.

8. The strong equivalence principle is embedded in the above type of Poincaré gauge invariance, as well as in the mode of the quantum-geometric propagation of quantum fields, which takes place by parallel transport along the arcs of the broken paths that are the horizontal lifts of geodesics of the Levi-Civita connection in each of the metric equivalence classes. These geodesics lie in the base manifold resulting from the natural fibration of the quantum spacetime supermanifold into superfibres of quantum superframes lying above the various points in that base manifold – which can be viewed as a "classical" spacetime manifold that labels the mean stochastic locations of geometro-stochastic excitons and of the fields producing or annihilating them.

9. The quantum superposition principle is embedded in the path integrals that describe the outcome of the quantum-geometric propagation of all quantum superfields along all possible stochastic causal paths. The causality of these paths is embodied in the above-described features of quantum propagation, which reflect a microcausal time-ordering and an adaptation of the geodesic postulate to quantum general relativistic propagation. Their stochasticity is *not* due to the presence of probability measures over paths, as in classical stochastic processes; rather, it is due to the fact that, in constructing the limits leading to the quantum-geometric propagators of quantum fields and superfields, the superpositions of propagators for parallel transport are taken, with purely geometric weighting factors, over *all* possible broken causal paths consisting of geodesic arcs.

The author would like to thank the organizers of the *XIIIth Workshop on Geometric Methods in Physics*, in Bialowieza, Poland, for the opportunity to present the main ideas in this monograph for professional consideration and discussion prior to its publication. He also thanks Professors J. A. Brooke and W. Drechsler for discussions and correspondence on some of the topics treated in it. Furthermore, he would like to express his appreciation to M. Clayton, E. Reyes and M. Sangani-Monfared for comments and valuable assistance with the proof-reading of parts of the manuscript. The comprehensive help with its over-all preparation which he received from his wife, Margaret R. Prugovečki, and the moral support she extended to him throughout the many years of research leading to the results presented in this monograph, were truly invaluable.

List of Symbols

The following list contains primarily symbols whose physical and mathematical meaning remains unchanged throughout this monograph. If page numbers are cited, they refer to the page where the symbol in question is defined, or it first appears.

$\langle \mid \rangle$	inner product in a Hilbert or pseudo-Hilbert space
$(\mid)_J$	J-inner product in a pseudo-Hilbert space (i.e., Krein space)
$[\ ,\ ]$	commutator bracket
$\{\ ,\ \}$	anticommutator bracket *or* Poisson bracket
$\{\quad\}$	set of elements represented by the symbols appearing between such brackets
$\{\ \mid\ \}$	set of elements satisfying all the relations appearing to the right of the bar
$:\quad:$	indicates the normal ordering of the quantum fields between the colons
$:=$	equality, where the left-hand side is *by definition* equal to the right-hand side
$\approx$	approximate equality
$\cong$	equivalence relation
$\equiv$	identity relation
$\varnothing$	the empty set
$\oplus$	direct sum of Hilbert or pseudo-Hilbert spaces (or their closed subspaces)
$\otimes$	tensor product of vector spaces or of (pseudo-)Hilbert spaces
$\times$	group direct product, *or* Cartesian product of sets, *or* ordinary product
$\times_{\mathbf{G}}$	$\mathbf{G}$-product between principal bundle and a typical fibre
	denotes Euclidean, *or* in a Minkowski inner product, *or* right-action
$\bullet$	denotes inner product in a Lie algebra $L(\mathbf{G}_0)$
$\wedge$	denotes a group semidirect product, *or* a wedge product of n-forms
$\mapsto$	map of an element of some set into another element of some set
$\leftrightarrow$	bijective (i.e., one-to-one *and* onto) map between two sets
$\rightarrow$	map of a set into another *or* notation for convergence
$\in$	"belongs to" symbol for elements of a given set
$\notin$	"does not belong to" symbol for elements outside a given set
*	denotes complex conjugation, *or* adjoints of operators, *or* duals
$\bar{}$	denotes complex conjugation, *or* adjoints of bispinors, *or* of supernumbers
†	denotes duals of some topological vector space
$\circ$	denotes the composition of mappings, so that, e.g., $(\phi \circ \psi)(x) := \phi(\psi(x))$
$\mathbf{A}$	a complete set $\{A_1,...,A_n\}$ of operators or observables, p. 71
$\mathbf{A}_{\alpha\,\mu}$	infinitesimal generators of a representation U_x of $\mathbf{G}$ within a fibre, p. 45
$\mathbf{AutP}$	family of automorphisms of a principal bundle $\mathbf{P}$, p. 258
$\mathbf{Aut_M P}$	family of vertical automorphisms of a principal bundle $\mathbf{P}$, p. 258

$\boldsymbol{A}^{\boldsymbol{s}}$	gauge potential for a choice $\boldsymbol{s}$ of section of $(\mathbf{P}, \boldsymbol{\Pi}, \mathbf{M}, \mathbf{G})$, pp. 46, 255
$\mathcal{A}$	Berezin-Dirac superfibre bundle, p. 222
$\hat{\mathcal{A}}$	fermion field-mode superfibre bundle, p. 229
ad_h	adjoint action of $\mathbf{G}$ onto itself determined by $h \in \mathbf{G}$, p. 259
$\boldsymbol{ad}\,\mathbf{P}$	bundle associated to a principal bundle $\mathbf{P}$ by the adjoint action of $\mathbf{G}$, p. 259
$a_\alpha(x)$	element of a set of canonical annihilation operators above $x \in \mathbf{M}$, p. 167
$(\boldsymbol{a}(x), \boldsymbol{e}_i(x))$	element of the Poincaré frame bundle $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, p. 113
(a, Λ)	Poincaré transformation, p. 102
B	Borel set in $\mathbf{R}^n$, p. 70
$\mathbf{B}$, $\overline{\mathbf{B}}$	standard Dirac superfibre and its conjugate superfibre, p. 219
$\mathcal{B} \otimes \overline{\mathcal{B}}$	standard Berezin-Dirac superfibre, p. 220
$\mathcal{B}_{\boldsymbol{u}} \otimes \overline{\mathcal{B}}_{\boldsymbol{u}}$	BFP (Berezin-Faddeev-Popov) ghost-antighost superfibre, p. 294
$b_j(\zeta)$	gauge-fixing functions of generalized Lorenz gauges for gravitons, p. 282
$(b, \mathbf{a}, \mathbf{v}, R)$	element of the Galilei group, pp. 64, 78
$C^{(\pm)}(\boldsymbol{u}; \zeta)$	ghost annihilation and creation operators, p. 294
$C(\mathbf{P})$	family of all connection forms on a given principal bundle $\mathbf{P}(\mathbf{M}, \mathbf{G})$, p. 261
c	ghost field in classical Yang-Mills theory, p. 262
$\boldsymbol{d}$	exterior differentiation form, pp. 46, 129
$\boldsymbol{df}$	covector (1-form) associated with the smooth function f on $\mathbf{M}$, p. 24
$\mathbf{df}\,\mathbf{df}^*$	measure for functional integration in Hilbert fibres, p. 166
$\boldsymbol{df}\bar{\boldsymbol{f}}\boldsymbol{df}$	measure for functional integration in Berezin fibres, p. 223
$\boldsymbol{dx}^\mu$	element of the holonomic linear coframe dual to the linear frame $\{\boldsymbol{\partial}_\mu\}$, p. 24
$d\Omega(u)$	invariant measure element on the 4-velocity hyperboloid $\boldsymbol{V}$, p. 115
$d\Omega_m(k)$	invariant measure element on the mass hyperboloid $\boldsymbol{V}_m$, p. 96
$d\Omega_0(k)$	invariant measure element on the light cone $\boldsymbol{V}_0$, p. 234
$d\Sigma(q,p)$	measure element on reference hypersurface in phase space, p. 103
$d\Sigma(\zeta)$	measure element on reference hypersurface with $\zeta = (q, v)$, p. 117
$d\tilde{\Sigma}(\zeta)$	formally renormalized measure element for the mass-0 case, p. 245
$d\sigma(x)$	Riemannian measure element on spacetime reference hypersurface, p. 185
$d\sigma^\mu(x)$	component $n^\mu d\sigma(x)$ of $d\sigma(x)$ in the direction of the normal $\boldsymbol{n}$, p. 97
$d\hat{\sigma}^\mu$	formally renormalized component of measure element, p. 245
$\mathrm{D}\varphi_n$	functional "measure" in action-based path integrals, p. 187
$D\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$	Dirac frame bundle over a Lorentzian manifold $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$, p. 214
$\mathcal{D}\varphi_n$	functional "measure" in path integration for boson fields, p. 184
$\mathcal{D}\psi_n$	functional "measure" in path integration for fermion fields, p. 231
Δ	the Laplacian operator in configuration space variables, p. 66
$\Delta_{\mathbf{q}}$	the Laplacian operator in stochastic space $\mathbf{q}$-variables, p. 66
$\Delta_F(x)$	2-point Green function for spin-0 particles, p. 161
$\Delta_F(x)$	Feynman 2-point function for spin-0 particles, p. 161
$\Delta_x^{(+)}$	local quantum fluctuation amplitude for zero spin, p. 118
∇	covariant differentiation operator-form, pp. 44, 46, *or*...
∇	quantum connection, p. 129

$\nabla_{\boldsymbol{X}}$ covariant differentiation operator, pp. 42, 44, 126
$\boldsymbol{\nabla}$ QFT connection, p. 164
$\boldsymbol{\nabla}_{\boldsymbol{X}}$ covariant differentiation operator for a QFT connection, p. 164
$\partial_\alpha = \partial/\partial x^\alpha$ spacelike tangent to a coordinate line , p. 58
$\partial_\mu = \partial/\partial x^\mu$ tangent to the coordinate line of some chart (M_α, ϕ_α) around $x \in \mathbf{M}$, p. 24
$\boldsymbol{\partial}_{\boldsymbol{X}}$ exterior differentiation operator, pp. 45, 123, 127
$\boldsymbol{E}$ generic POV measure, p. 74
$\boldsymbol{E}^{\mathbf{A}}$ spectral measure for a complete set $\mathbf{A}$ of operators (observables), p. 70
$\boldsymbol{E}^{\mathbf{P}}$ spectral measure for momentum operators (observables), p. 70
$\boldsymbol{E}^{\mathbf{Q}}$ spectral measure for position operators (observables), p. 70
$\boldsymbol{E}_\xi$ POV measure associated with the resolution generator ξ, p. 80
$\mathcal{E}$ Fock bundle, p. 159
$\hat{\mathcal{E}}$ field-mode bundle for a quantum-geometric boson field $\boldsymbol{\varphi}$, p. 178
$\mathcal{E}^\uparrow$ Gupta-Bleuler multi-photon bundle, p. 248
$(\mathbf{E}, \boldsymbol{\pi}, \mathbf{M}, \mathbf{F})$ C^∞-fibre bundle over $\mathbf{M}$, p. 35
$(\mathbf{E}, \boldsymbol{\pi}, \mathbf{M}, \mathbf{F}, \mathbf{G})$ fibre bundle associated with a principal bundle $(\mathbf{P}, \boldsymbol{\Pi}, \mathbf{M}, \mathbf{G})$, p. 37
$\{\boldsymbol{e}_i(x)\}$ element of the general linear frame bundle $GL\mathbf{M}$, p. 28, *or* ...
$\{\boldsymbol{e}_i(x)\}$ element of the Lorentzian frame bundle $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, p. 35
$\boldsymbol{\varepsilon}_{(\alpha)}(k)$ linear polarization tetrad element in the momentum representation, p. 239
$\boldsymbol{\varepsilon}^{(\mu)}(\boldsymbol{u};k)$ linear polarization cotetrad element for gravitons, p. 285
$\boldsymbol{\varepsilon}^{\mu\nu\zeta}$ linear polarization frame element for gravitons, p. 286
$\boldsymbol{\eta}$ the Minkowski metric of special relativity, p. 34
$\eta^{(\ell)}$ generic special relativistic resolution generator, p. 103
η_{ij} components of the Minkowski metric of special relativity, p. 29
$\eta^{(\ell)}$ fundamental special relativistic resolution generator, p. 108
$\mathbf{F}$ typical fibre of an associated (quantum) bundle, pp. 36, 115
$\mathbf{F}^\dagger$ dual of a typical fibre of a quantum bundle, p. 130
$\overline{\mathbf{F}}$ complex conjugate of a typical fibre of a quantum bundle, p. 130
$\hat{\mathbf{F}}$ standard exciton-mode fibre, p. 178
$\mathbf{F}_{\boldsymbol{u}}$ graviton ghost fibre, p. 293
$\mathbf{F}_x$ fibre above $x \in \mathbf{M}$ of a quantum bundle, p. 116
$\mathcal{F}$ typical Fock fibre, pp. 158, 212
$\hat{\mathcal{F}}$ standard field-mode fibre, p. 178
$\mathcal{F}_n$ n-boson typical fibre, p. 158
$\hat{\mathcal{F}}_n$ standard n-boson field-mode fibre, p. 178
$\mathcal{F}_x$ Fock fibre above $x \in \mathbf{M}$, p. 159
$\hat{\mathcal{F}}_x$ field-mode fibre above $x \in \mathbf{M}$, p. 181
$\mathbf{f}$ element of a boson Fock fibre above $x \in \mathbf{M}$, p. 165
$\boldsymbol{f}$ element of a single-fermion superfibre above $x \in \mathbf{M}$, p. 222, *or* ...
$\boldsymbol{f}$ element of a photon fibre above $x \in \mathbf{M}$, p. 249, *or* ...
$\boldsymbol{f}$ element of a graviton fibre above $x \in \mathbf{M}$, p. 281
$\boldsymbol{f}^{\mathrm{T}}$ transversal component of $\boldsymbol{f}$ in a single-photon fibre, p. 249

$\boldsymbol{f}^{\mathrm{TT}}$ TT-component of $\boldsymbol{f}$ in a single-graviton fibre, p. 287
$\hat{\mathbf{f}}$ element of an exciton-mode fibre above $x \in \mathbf{M}$, p. 183
f element of a typical Fock fibre, pp. 158, 212
$\hat{f}$ element of a standard exciton-mode fibre, p. 178
f Schwartz test function, p. 155
$f_{ij}(\zeta)$ components of graviton wave functions, p. 282
$\boldsymbol{\Phi}_\alpha$ local trivialization map of $(\mathbf{P}, \boldsymbol{\Pi}, \mathbf{M}, \mathbf{G})$, p. 36
$\boldsymbol{\Phi}_{\mathbf{f}}$ second-quantized boson frame in $\mathcal{F}_x$, p. 165
$\boldsymbol{\Phi}_f$ indefinite-metric coherent state in a fibre $\mathcal{K}_x$ or $Z_{\boldsymbol{u}}$, pp. 249, 292
$\boldsymbol{\Phi}_c$ BFP ghost superframe element, p. 294
$\boldsymbol{\Phi}_f$ standard Berezin-Dirac superframe element, p. 230
$\boldsymbol{\Phi}^{\boldsymbol{u}}$ proper state vector at the origin of a quantum frame $Q^{\boldsymbol{u}}$, pp. 81, 104
$\boldsymbol{\Phi}_x^{\ \boldsymbol{u}}$ local quantum frame element, p. 118
ϕ collective notation for all "matter" fields in the ADM method, p. 59
ϕ_0 initial data for all the nongravitational fields in the ADM method, p. 59
φ special-relativistic wave function on stochastic phase space, p. 102
ϕ vertical automorphism in a principal bundle $\mathbf{P}$, p. 258
$\varphi(x;\zeta)$ Klein-Gordon quantum frame field, p. 159
$\varphi^{(-)}(x;\zeta)$ boson annihilation operator in $\mathcal{F}_x$, p. 159
$\varphi^{(+)}(x;\zeta)$ boson creation operator in $\mathcal{F}_x$, p. 159
$\varphi^{(-)}(\mathbf{f})$ smeared boson annihilation operator in $\mathcal{F}_x$, p. 165
$\varphi^{(+)}(\mathbf{f})$ smeared boson creation operator in $\mathcal{F}_x$, p. 165
$\phi(x)$ conventional scalar quantum field, p. 155
$\phi(f)$ smeared (Wightman) scalar quantum field, p. 155
ϕ_α coordinate map between a 4-dimensional manifold $\mathbf{M}$ and $\mathbf{R}^4$, p. 22
$\boldsymbol{\varphi}(x,\boldsymbol{v})$ quantum-geometric boson field, p. 177
$\boldsymbol{\phi}_\alpha$ local trivialization map of $(\mathbf{E},\boldsymbol{\pi}, \mathbf{M},\mathbf{F},\mathbf{G})$, p. 38
$\phi^*\boldsymbol{\omega}$ pull-back of a covector $\boldsymbol{\omega}$, p. 26
$\phi_*\boldsymbol{X}$ push-forward of a vector $\boldsymbol{X}$, p. 26
Ψ element of some typical quantum fibre, pp. 114, 164, 214
$\boldsymbol{\Psi}$ element of a fibre in some quantum bundle, pp. 114, 159, 214, 223
Ψ_0 vacuum state vector in a typical fibre, p. 214
$\boldsymbol{\Psi}_{0;x}$ local vacuum state vector, pp. 158, 214
$\boldsymbol{\Psi}_{n;x}$ n-boson local state vector, p. 159
$\boldsymbol{\Psi}_x^{\ \boldsymbol{u}}$ coordinate wave function, p. 115
ψ diffeomorphism, generically between two manifolds, pp. 26, 258
ψ nonrelativistic wave function on stochastic phase space, p. 78
$\psi^{(-)}(\zeta,s)$ fermion annihilation operator in a Fock-Dirac typical fibre $\mathcal{F}$, p. 159
$\psi^{(+)}(\zeta,s)$ fermion creation operator in a Fock-Dirac typical fibre $\mathcal{F}$, p. 159
$\mathbf{G}$ structure group of a principal bundle $(\mathbf{P}, \boldsymbol{\Pi}, \mathbf{M}, \mathbf{G})$, p. 32
$\boldsymbol{G}$ quantum metric over a Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$, p. 132
$GA\mathbf{M}$ the general affine frame bundle over $\mathbf{M}$, p. 122

$GA^*\mathbf{M}$	the general affine coframe bundle over $\mathbf{M}$, p. 275
$GL\mathbf{M}$	the general linear frame bundle over $\mathbf{M}$, p. 28
$GL^*\mathbf{M}$	the general linear coframe bundle over $\mathbf{M}$, p. 29
$\mathrm{GL}(4,\mathbf{R})$	the general linear group of 4×4 real and invertible matrices, p. 32
$\mathcal{G}(\mathbf{P})$	generic gauge group of the second kind, p. 257
$\mathcal{G}(\mathbf{M},g^{\mathrm{L}})$	CGR gauge group of the second kind, p. 276
$\mathbf{g}^{\mathrm{M}}$	mean quantum gravitational field, pp. 278, 282
$\mathbf{g}^{\mathrm{Q}}$	quantum gravitational superfield, p. 309
$\boldsymbol{g}$	generic metric on $\mathbf{M}$, p. 49
$\boldsymbol{g}^{\mathrm{L}}$	Lorentzian metric on $\mathbf{M}$, p. 29
$\boldsymbol{g}^{\mathrm{R}}$	Riemannian (i.e., "Euclidean") metric, p. 137
g	element of the structure (Lie) group $\mathbf{G}$ of a principal bundle, p. 32
g^{L}	equivalence class of Lorentzian metrics, p. 272
$\mathbf{g}_{\alpha\beta}(x)$	transition function in a fibre bundle $(\mathbf{E}, \boldsymbol{\pi}, \mathbf{M}, \mathbf{F})$, p. 37
Γ	nonrelativistic classical phase space, p. 62
Γ^i_{jk}	connection coefficients for the Levi-Civita connection in $GL\mathbf{M}$, pp. 49, 54
$\Gamma^\lambda_{\mu\nu}$	Christoffel symbols for the Levi-Civita connection in $GL\mathbf{M}$, p. 49, 54
$\boldsymbol{\gamma}$	Riemannian 3-metric on in the ADM method, p. 56
$\boldsymbol{\gamma}_0$	initial-data 3-metric on $\boldsymbol{\Sigma}_0$, p. 59
γ	smooth curve $\{x(t) \mid a \le t \le b\}$ in $\mathbf{M}$, p. 23
γ^*	horizontal lift to the total space $\mathbf{P}$ of a curve γ in $\mathbf{M}$, p. 43
$\gamma^\uparrow$	horizontal lift from a base-segment $\mathbf{S}$ to a base-supersegment $\boldsymbol{S}$, p. 311
γ^k	Dirac matrix, p. 213
$H(t)$	total Hamiltonian, p. 93
$H_I(t)$	interaction Hamiltonian, p. 93
H_0	free Hamiltonian, p. 89
$H_{\boldsymbol{u}}\mathbf{P}$	the horizontal space at $\boldsymbol{u} \in \mathbf{P}$ of a connection on $(\mathbf{P}, \boldsymbol{\Pi}, \mathbf{M}, \mathbf{G})$, p. 41
$\mathcal{H}$	Hilbert space, p. 72
χ_B	characteristic function of a subset B of some given set, p. 70
χ_x	confidence function of a stochastic value (x,χ_x), pp. 76, 81
$\mathrm{ISO}(3,1)$	the full Poincaré group, p. 95
$\mathrm{ISO}_0(3,1)$	the restricted Poincaré group, often called just the Poincaré group, p. 95
$\mathrm{ISO}^\uparrow(3,1)$	the orthochronous Poincaré group, p. 233
J	fundamental symmetry of a J-inner product in a Krein space, p. 237
$\boldsymbol{J}_{\boldsymbol{u}}$	fundamental symmetry of a J-inner product in a graviton fibre $\mathbf{Z}_{\boldsymbol{u}}$, p. 282
$J^\mu_\eta(q)$	charge current associated with a resolution generator η, p. 105
$\mathbf{j}_{\boldsymbol{u}}$	quantum probability 3-current with respect to a Galilei frame $\boldsymbol{u}$, p. 86
$j^\mu(x)$	component of the Klein-Gordon current, p. 98
$j_{\boldsymbol{s}}{}^k(x)$	quantum field-mode probability current, p. 190
$\boldsymbol{j}_a(x)$	3-momentum density operators of a quantum-geometric field $\boldsymbol{\varphi}$, p. 177
j_a	proper 3-momentum components of nongravitational sources, p. 59
$j^\mu_\eta(q)$	probability current associated with a resolution generator η, p. 104
$\mathbf{K}$	Krein space in general, *or* as a Krein typical fibre, pp. 236, 245

$\tilde{\mathbf{K}}$ Krein space of photon wave functions in momentum representation, p. 237
$\tilde{\mathbf{K}}^L$ Lorenz subspace of the Krein space $\tilde{\mathbf{K}}$, p. 239
$\tilde{\mathbf{K}}^0$ null subspace of the Krein space of photon $\tilde{\mathbf{K}}$, p. 239
$\boldsymbol{K}$ extrinsic curvature of a reference hypersurface $\boldsymbol{\Sigma}_t$, p. 58
$\boldsymbol{K}_0$ initial-data extrinsic curvature for $\boldsymbol{\Sigma}_0$, p. 59
K geometro-stochastic propagator for single bosons, p. 138, *or* ...
K quantum-geometric propagator for free-fall fields, pp. 184, 232
K_F Feynman special-relativistic propagator for spin-0 particles, pp. 109, 161
K_γ propagator for parallel transport in a quantum or QFT bundle, pp. 134, 168
$K^{(\ell)}$ phase space special-relativistic propagator for spin-0 particles, p. 109
$K(\mathbf{q}'',\mathbf{p}'',t'';...)$ nonrelativistic stochastic phase space propagator, p. 91
$K(\mathbf{x}'',t'';\mathbf{x}',t')$ nonrelativistic Feynman propagator, p. 89
$\mathcal{K}$ quantum-geometric propagator for interacting boson fields, p. 207, *or* ...
$\mathcal{K}$ Krein-Gupta-Bleuler typical multi-photon fibre, p. 248
$\mathcal{K}_x$ Krein-Gupta-Bleuler fibre above $x \in \mathbf{M}$, p. 249
$\mathbf{k}$ 3-momentum variable, p. 65,
$\boldsymbol{k}$ 4-momentum variable, pp. 96, 160
k relativistic momentum coordinate 4-tuple, p. 234
$L(\mathbf{G})$ the Lie algebra of a structure group $\mathbf{G}$, p. 42
$L^2(\mathbf{R}^3)$ Hilbert space of Lebesgue square-integrable functions on $\mathbf{R}^3$, p. 65
$L^2(\mathbf{R}^n)$ Hilbert space of Lebesgue square-integrable functions on $\mathbf{R}^n$, p. 78
$L^2(V_m^+,d\Omega_m)$ the Hilbert space of the special-relativistic momentum representation, p. 96
$\boldsymbol{LM}(\boldsymbol{g})$ principal bundle of frames orthonormal in a generic metric $\boldsymbol{g}$, p. 47
$\boldsymbol{LM}(\boldsymbol{g}^L)$ Lorentz frame bundle for the Lorentzian metric $\boldsymbol{g}^L$ on $\mathbf{M}$, p. 29
$\boldsymbol{LM}^\uparrow(\boldsymbol{g}^L)$ subbundle of $\boldsymbol{LM}(\boldsymbol{g}^L)$ with frames having future oriented $\boldsymbol{e}_0$-vectors, p. 273
$\boldsymbol{LM}_0(\boldsymbol{g}^L)$ subbundle of $\boldsymbol{LM}(\boldsymbol{g}^L)$ having space and time oriented frames, p. 48
$\mathcal{L}$ global Lorentz frame (i.e., Minkowski frame) $\{\boldsymbol{e}_i(O)|i = 0,1,2,3\}$, p. 35
$\mathcal{L}_0$ Lagrangian density of free-fall quantum-geometric fields, pp. 187, 232
$\mathcal{L}_I$ interaction Lagrangian density of quantum-geometric fields, p. 203
ℓ fundamental length, pp. 78, 85
Λ Grassmann algebra, p. 217
Λ^0 even-grade Banach subalgebra of Λ, p. 218
Λ^1 odd-grade Banach subspace of Λ, p. 218
$\Lambda^{(m,n)}$ (m,n)-dimensional superspace, p. 298
$\mathbf{M}$ 4-dimensional differential manifold, p. 22
$\mathbf{M}^s$ domain of definition of a section $\boldsymbol{s}$ of some principal bundle over $\mathbf{M}$, p. 34
$\hat{\mathbf{M}}^{jk}_s$ infinitesimal Lorentz transformations in a local Lorentz frame $\boldsymbol{s}(x)$, p. 48
$\tilde{\mathbf{M}}^{jk}_s$ infinitesimal Lorentz transformations in a local Poincaré frame $\boldsymbol{s}(x)$, p. 12
$(\mathbf{M},\boldsymbol{g})$ Riemannian of pseudo-Riemannian manifold with metric $\boldsymbol{g}$, p. 49
$(\mathbf{M},\boldsymbol{g}^L)$ Lorentzian manifold, i.e., a manifold $\mathbf{M}$ with a Lorentzian metric $\boldsymbol{g}^L$, p. 47
$(\mathbf{M},g^L)$ CGR gauge orbit, p. 272
$\boldsymbol{M}$ supermanifold, p. 298

$\boldsymbol{M}^{jk}$ angular momentum operators of quantum fields, p. 172
M_α the domain of a chart (M_α,ϕ_α) in a 4-dimensional manifold $\mathbf{M}$, p. 22
M^{jk} infinitesimal generators of local Lorentz transformations, p. 129
M^4 four-dimensional Minkowski space, also denoted by $(\mathbf{R}^4,\eta)$, p. 101.
(M_α,ϕ_α) a chart in a 4-dimensional manifold $\mathbf{M}$, p. 22
N_a ADM shift function, p. 57
$\mathcal{N}_x$ neighborhood of some point $x \in \mathbf{M}$, p. 23
$\boldsymbol{n}(x)$ normal to a reference hypersurface, p. 177
$n^\mu(x)$ component of a $\boldsymbol{n}(x)$, p. 177
O(3,1) the group of 4×4 real matrices representing Lorentz transformations, p. 34
O_α open set in the Euclidean space $\mathbf{R}^4$, p. 22
$\boldsymbol{\Omega}^{\boldsymbol{\omega}}$ curvature 2-form associated with the connection form $\boldsymbol{\omega}$, p. 255
$\boldsymbol{\omega}$ connection form on a principal bundle $(\mathbf{P}, \boldsymbol{\Pi}, \mathbf{M}, \mathbf{G})$, pp. 42, 48
$\boldsymbol{\omega}$ covector (1-form) at some point $x \in \mathbf{M}$, p. 24
$\boldsymbol{\omega}^a$ Cartan connection forms associated with a basis in $\boldsymbol{L}(\mathbf{G})$, p. 42
$\boldsymbol{\omega}_{ij}$ connection coefficients (1-forms) associated with $\mathcal{Y}_{ij}$, p. 48
$\tilde{\boldsymbol{\omega}}_s^{jk}$ connection 1-forms in bundles associated with $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, p. 123
$\boldsymbol{\omega}^{(\varphi)}$ generalized affine connection form, p. 122
$\boldsymbol{\omega}^{(\theta)}$ affine connection form, p. 122
$\mathbf{P}$ total space of principal bundle, p. 32, *or* ...
$\mathbf{P}$ complete set $\{P^1, P^2, P^3\}$ of momentum operators, p. 71
$\mathbf{P}_\eta$ projector onto the relativistic subspace with resolution generator η, p. 103
$\mathbf{P}_\xi$ projector onto the nonrelativistic subspace with resolution generator ξ, p. 79
P_j canonical momentum operators, pp. 62, 67, 93, *or*...
P_j infinitesimal generators of local spacetime translations, p. 129
$\boldsymbol{P}_{j;s}$ infinitesimal generators of spacetime translations for quantum fields, p. 170
$\hat{\boldsymbol{P}}_{j;s}$ energy-momentum density operators for quantum fields, p. 177
$P\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ Poincaré frame bundle over a Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$, p. 273
$P\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$ subbundle of $P\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ with frames having future oriented $\boldsymbol{e}_0$-vectors, p. 273
$P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ subbundle of $P\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ with space and time oriented frames, p. 112
$P^*\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$ metrized coframe bundle, p. 275
$(\mathbf{P},\boldsymbol{\Pi}, \mathbf{M},\mathbf{G})$ generic principal bundle, p. 32
$\mathbf{p}$ 3-momentum component in a phase-space variable $\boldsymbol{\zeta} = (\mathbf{q},\mathbf{p})$, p. 78
p 4-momentum component in a phase-space variable $\zeta = (q,p)$, p. 102
$\boldsymbol{\Pi}$ projection map in a principal bundle $(\mathbf{P}, \boldsymbol{\Pi}, \mathbf{M}, \mathbf{G})$, p. 32
$\boldsymbol{\Pi}^{-1}(x)$ the fibre above $x \in \mathbf{M}$ in a principal bundle $(\mathbf{P}, \boldsymbol{\Pi}, \mathbf{M}, \mathbf{G})$, p. 32
$\boldsymbol{\pi}$ projection map in a bundle $(\mathbf{E},\boldsymbol{\pi}, \mathbf{M},\mathbf{F})$ or $(\mathbf{E},\boldsymbol{\pi}, \mathbf{M},\mathbf{F},\mathbf{G})$, pp. 35, 37
$\boldsymbol{\pi}^{-1}(x)$ fibre in $(\mathbf{E},\boldsymbol{\pi}, \mathbf{M},\mathbf{F})$ or $(\mathbf{E},\boldsymbol{\pi}, \mathbf{M},\mathbf{F},\mathbf{G})$ over some $x \in \mathbf{M}$, pp. 35, 37
$\mathbf{Q}$ the complete set $\{Q^1,Q^2,Q^3\}$ of position operators, p. 71
Q^j canonical position operators, pp. 62, 67. 93
$Q\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ quantum frame bundle over a Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$, p. 113
$\mathcal{Q}^{\boldsymbol{u}}$ global quantum frame associated with a classical frame $\boldsymbol{u}$, pp. 81, 104
$\mathcal{Q}^{\boldsymbol{u}}_\ell$ local quantum frame associated with a local classical frame $\boldsymbol{u}$, pp. 113, 117

$\boldsymbol{q}$	component of a Poincaré gauge variable $\boldsymbol{\zeta} = \boldsymbol{q} - i\ell\boldsymbol{v}$, p. 170
q	component of a (stochastic) phase-space variable $\zeta = (q,p)$, p. 102, *or*...
q	component in a Poincaré gauge 4-tuple $\zeta = q - i\ell v$, p. 115
R	Riemann curvature scalar field, p. 53, *or*...
R	a rotation of a spatial 3-frame, represented by an element of SO(3), p. 64
$\boldsymbol{R}$	Ricci tensor field, p. 53
$\mathbf{R}^4$	four-dimensional Euclidean space, p. 22
$\mathbf{R}^n$	n–dimensional Euclidean space, p. 62
$\mathbf{R}(\boldsymbol{X},\boldsymbol{Y})$	curvature operator form, p. 53
$(\mathbf{R}^4,\boldsymbol{\eta})$	four-dimensional Minkowski space, also denoted as $\boldsymbol{M}^4$, pp. 35, 101
$R^i{}_{jkl}$	Riemann curvature tensor field components, p. 53
R_g	the right action map determined by $g \in \mathbf{G}$ in $\mathbf{P}$, p. 41
$R_{k\,\Lambda}$	Wigner rotation, p. 23
$\rho(x)$	proper energy density operator, pp. 177, 231, 253
$\rho^{\text{tot}}(X)$	superlocal total proper energy density operator, p. 317
ρ	proper energy density of nongravitational sources, p. 59
$\rho_{\boldsymbol{u}}$	quantum probability density with respect to a Galilei frame $\boldsymbol{u}$, p. 84
S	scattering operator, p. 209
$\mathbf{S}$	base-segment, p. 277
$\boldsymbol{S}$	base-supersegment, p. 303
$S(\varphi)$	action for quantum-geometric boson fields, p. 187
$S(\psi)$	action for quantum-geometric fermion fields, p. 232
$S_x^{(+)}$	spinorial quantum fluctuation amplitude, p. 120
S^{jk}	spin operators in a local Lorentz transformation, p. 129
$S\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$	affine spin-frame bundle over a Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$, p. 214
SO(3,1)	special orthogonal group of O(3,1)-matrices of determinant one, p. 48
$\mathrm{SO}_0(3,1)$	subgroup of SO(3,1) not incorporating spacetime reflections, p. 48
$\mathbf{S}^{(+)}_{U(x)}$	local quantum fluctuation amplitudes for fermions and antifermions, p. 214
$(\mathbf{S},g^{\mathbf{M}})$	QGR gauge orbit, p. 297
$\mathcal{S}(\mathbf{R}^4)$	Schwartz space of test functions, p. 155
$so(3,1)$	Lie algebra of $\mathrm{SO}_0(3,1)$, p. 48
$\mathbf{s}$	cross-section of a Dirac quantum frame bundle $D\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, p. 222
$\boldsymbol{s}$	moving frame or, more generally, section of some principal bundle, p. 34
$\boldsymbol{s}(x)$	element above $x \in \mathbf{M}$ in a section $\boldsymbol{s}$ of some principal bundle over $\mathbf{M}$, p. 34
$\boldsymbol{s}_\gamma$	inertial (Lorentz) moving frame along a timelike geodesic γ, p. 51
$\boldsymbol{s}(\mathcal{L})$	cross-section of $L\mathbf{M}(\boldsymbol{\eta})$ identifiable with the global Lorentz frame $\mathcal{L}$, p. 35
$\mathbf{s}_0(\mathcal{L})$	cross-section of $P\mathbf{M}_0(\boldsymbol{\eta})$ identifiable with a global Lorentz frame $\mathcal{L}$, p. 116
s	spin index in a Dirac bispinor, p. 215
Σ_m	reference hypersurfaces in stochastic phase space, p. 101
Σ_0	initial-data Cauchy surface in $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$, p. 56
Σ_t	reference hypersurfaces in a foliation of $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$, p. 56
$\Sigma(P^*\mathbf{S}^{\uparrow}(g^{\mathbf{M}}))$	family of all cross-sections of $P^*\mathbf{S}^{\uparrow}(g^{\mathbf{M}})$, p. 303
σ	maximal spacelike hypersurface within the Minkowski space $\boldsymbol{M}^4$, p. 101

$\boldsymbol{\sigma}_x^u$ soldering map for a quantum bundle or a QFT bundle, pp. 115, 164
$\boldsymbol{\sigma}_x^U$ soldering map for a Dirac quantum frame bundle, p. 214
$\mathbf{T}(X,Y)$ torsion operator, p. 49
$\boldsymbol{T}$ either a generic tensor in $T^{r,s}\mathbf{M}$, or a stress-energy tensor, pp. 25, 53
T_{jk} stress-energy tensor components, p. 171
$T\mathbf{M}$ the tangent bundle over a 4-dimensional differential manifold $\mathbf{M}$, p. 28
$T^*\mathbf{M}$ the cotangent bundle over a 4-dimensional differential manifold $\mathbf{M}$, p. 28
$T^{r,s}\mathbf{M}$ the (r,s)-tensor bundle over a 4-dimensional differential manifold $\mathbf{M}$, p. 28
$T_e\mathbf{G}$ tangent space of $\mathbf{G}$ above the unit element $e \in \mathbf{G}$, p. 42
$T_x\mathbf{M}$ tangent space to a manifold above the point $x \in \mathbf{M}$, p. 24
$T_x{}^*\mathbf{M}$ cotangent space to $\mathbf{M}$ above the point $x \in \mathbf{M}$, p. 24
$T_u\mathbf{P}$ the space tangent at $\boldsymbol{u} \in \mathbf{P}$ to the total space of $(\mathbf{P}, \Pi, \mathbf{M}, \mathbf{G})$, p. 41
$T_u\Pi^{-1}(x)$ space tangent to $\Pi^{-1}(x)$, equal to the vertical space $V_u\mathbf{P}$ at $\boldsymbol{u} \in \mathbf{P}$, p. 41
$\tau_\gamma(x'',x')$ operator for parallel transport within a principal bundle, p. 43
$\tau_\gamma(x'',x')$ operator for parallel transport within an associated bundle, pp. 43, 164
$\mathcal{T}$ time-translation group, p. 103
$\boldsymbol{\Theta}$ Maurer-Cartan 2-form of a Lie group $\mathbf{G}$, p. 256
$\boldsymbol{\Theta}_g$ the value assumed by the Maurer-Cartan 2-form $\boldsymbol{\Theta}$ at $g \in \mathbf{G}$, p. 256
ϑ_α, $\bar{\vartheta}_\alpha$ generators of a Grassmann algebra, p. 217
θ^α, $\bar{\theta}^\alpha$ odd-valued variables in a Grassmann algebra, pp. 219, 299
$\tilde{\boldsymbol{\theta}}_s^i$ connection 1-forms in bundles associated with $P\mathbf{M}_0(\boldsymbol{g}^L)$, p. 123
$\{\boldsymbol{\theta}^i(x)\}$ element of a moving coframe, or a vierbein (i.e., cotetrad field), p. 29
$(\theta,b,\mathbf{a},\mathbf{v},R)$ element of central extension of the Galilei group, p. 65
$U(g)$ representation of a structure group $\mathbf{G}$, p. 37
$U(b,\mathbf{a},\mathbf{v},R)$ representation of the Galilei group, pp. 65, 78
$U(a,\Lambda)$ representation of the Poincaré group for single bosons, pp. 102, 115
$\boldsymbol{U}(a,\Lambda)$ Fock or Krein space representation of the Poincaré group, pp. 158, 248
$\mathrm{U}(a,A)$ representation of $\mathrm{ISL}(2,\mathbf{C})$ in a standard Dirac superfibre, p. 221
$\mathbf{U}(a,A)$ representation of $\mathrm{ISL}(2,\mathbf{C})$ in a standard Berezin-Dirac superfibre, p. 221
$\boldsymbol{u}(x)$ a linear frame, or a vector tetrad, above $x \in \mathbf{M}$, p. 28
$\boldsymbol{u}^*(x)$ a linear coframe, or a covector tetrad, dual to $\boldsymbol{u}(x)$, p. 29
$\{\mathbf{V}_\rho(\boldsymbol{u})\}$ frames within the fibres of $\mathbf{E}$ associated to an element $\boldsymbol{u} \in \mathbf{P}$, p. 45
$\boldsymbol{V}$ 4-velocity hyperboloid, p. 213
$\boldsymbol{V}^+$ forward 4-velocity hyperboloid, pp. 113, 115
$\boldsymbol{V}_m^+, \boldsymbol{V}_m^-$ forward and backward mass hyperboloid for rest mass m, p. 96
$\boldsymbol{V}_0$ light cone, p. 234
$\boldsymbol{V}_0^+, \boldsymbol{V}_0^-$ forward and backward light cone, respectively, p. 234
$V_u\mathbf{P}$ the vertical space at $\boldsymbol{u} \in \mathbf{P}$, equal to $T_u\Pi^{-1}(x)$, p. 41
$\mathbf{v}$ 3-velocity in Galilean transformations, p. 64
$\boldsymbol{v}$ 4-velocity vector, 4-velocity vector field, p. 50, *or*...
$\boldsymbol{v}$ component of a Poincaré gauge variable $\boldsymbol{\zeta} = \boldsymbol{q} - i\ell\boldsymbol{v}$, p. 170
v component of a coordinate 4-tuple $\zeta = q - i\ell v$, p. 115
$\mathbf{W}$ Weyl-Klein typical fibre, p. 267
$\mathbf{W}(0)$ Weyl-Lorenz typical subfibre, p. 267

W_ξ unitary transform from momentum representation space to $\mathbf{P}_\xi L^2(\mathbf{R}^6)$, p. 80
$\mathcal{W}$ Yang-Mills typical fibre, p. 268
$\mathbf{w}_\alpha$ element of an orthonormal basis in a typical fermion fibre, p. 166
w_α element of an orthonormal basis in a typical boson fibre, p. 166
$\mathbf{w}_\alpha$ element of an orthonormal basis in a fibre above $x \in \mathbf{M}$, p. 166
$\mathbf{X}$ vector tangent to the total space $\mathbf{P}$ of a principal bundle $(\mathbf{P},\Pi, \mathbf{M},\mathbf{G})$, p. 41
$\boldsymbol{X}$ vector tangent to a generic manifold or $\mathbf{M}$, p. 24
$\mathbf{x}$ point in a 3-dimensional Euclidean space, p. 64, *or*...
$\mathbf{x}$ point in a 3-dimensional spacelike (reference) hypersurface, p. 56
x point in a 4-dimensional manifold a Lorentzian manifold $\mathbf{M}$, pp. 22, 29
x^μ coordinate of x in some chart in $\mathbf{M}$, p. 22
ξ nonrelativistic resolution generator for $\mathbf{P}_\xi L^2(\mathbf{R}^6)$, p. 79
$\mathcal{Y}_a$ basis elements in $T_e\mathbf{G}$, pp. 42, 254
$\mathcal{Y}_{ij}$ basis elements in $T_e\mathrm{SO}_0(3,1)$, p. 48
$\mathbf{Z}_u$ single-graviton fibre above $\boldsymbol{u} \in P^*\mathbf{S}^\uparrow(g^\mathrm{M})$, p. 281
$\mathcal{Z}_u$ quantum gravitational fibre above $\boldsymbol{u} \in P^*\mathbf{S}^\uparrow(g^\mathrm{M})$, p. 292
$Z_{\ell,m}$ exciton-mode localization renormalization constant, p. 179
$\tilde{Z}_\ell$, $\tilde{Z}_{\ell,m}$ local quantum state normalization constants, pp. 108, 115
$\hat{Z}_\ell$, $\hat{Z}_{\ell,m}$ charge current renormalization constants, pp. 105, 117, 170
$\boldsymbol{\zeta}$ Poincaré gauge variable $\boldsymbol{\zeta} = \boldsymbol{q} - i\ell\boldsymbol{v}$, p. 118
ζ stochastic phase-space variable $\zeta = (q,p)$, p. 105, *or*...
ζ 4-tuple $\zeta = q - i\ell v$ of Poincaré-gauge coordinate variables, p. 115

Chapter 1

Survey of Principal Historical Developments

For a good understanding of general relativity and of quantum theory at the *foundational* level it is not sufficient to study only contemporary textbooks on these subjects. Rather, it is also necessary to study original sources, and to try to understand them in their historical context. This chapter is meant to provide the reader with a guide to those sources, and to show how such studies can reveal the presence of still unsolved or only partially solved foundational problems in key areas within these two most fundamental disciplines of twentieth-century physics. These foundational difficulties can set impediments to a significant progress in the understanding of these two disciplines, and, therefore, to their consistent unification. The gradual removal of these difficulties, resulting from the systematic analyses and careful extrapolations carried out in the next four chapters, will supply the launching pad for the formulation of a purely geometric formulation of quantum general relativity.

1.1. From Special to General Relativity

The concept of inertial frame is essential to Newtonian classical mechanics, since Newton's axioms are not valid even in the classical nonrelativistic regime in arbitrary frames of reference, but only in inertial ones. The group of coordinate transformations resulting in Newtonian mechanics from changes of inertial frames and of initial values for the time parameter was eventually named the Galilei group. By its very construction, this fundamental group leaves invariant all of Newton's laws, but it does not leave invariant the Maxwell equations for classical electromagnetic fields.

The work of H.A. Lorentz and H. Poincaré on the dynamics of classical models for the electron, carried out towards the end of the last century and during the first few years of this century, had pinpointed the group of coordinate transformations that leaves the Maxwell equations invariant. Eventually that group became known as the inhomogeneous Lorentz group, and was renamed the Poincaré group in more recent years. However, it was the seminal 1905 paper "On the Electrodynamics of Moving Bodies" by A. Einstein that provided the physical interpretation of this group which was destined to become universally accepted – namely that of a group of *spacetime* coordinate transformations, resulting from changes of inertial frames of reference and of the corresponding families of "standard" clocks. Each such family of clocks was synchronized by means of light signals, and could be used for the measurement of time *relative* to the frame in relation to which it was at rest.

As is well-known, Einstein did not achieve this revolutionary conceptualization of spacetime by developing new mathematical concepts, but by adopting an epistemologically new point of view towards the ancient concepts of space and time, which he founded on purely operational considerations. The foundational cornerstone of these considerations

was provided by the (later named) concept of *Lorentz frame*, defined by him as a physical system "of three rigid material lines, perpendicular to one another, and issuing from a point . . . [taken in conjunction] with a rigid measuring-rod and a number of clocks" (Einstein, 1905, p. 43) stationary with respect to the three axis of the adopted spatial frame. Einstein then *postulated* that these clocks were to be synchronized by means of light signals, and that the speed of light was independent of the choice of inertial frame. From these special-relativistic postulates he then rederived the Lorentz group as a group resulting from changes of Lorentz frames. Hence, whereas Lorentz, Poincaré and other leading researchers into the electrodynamics of the electron accepted the Newtonian conceptual framework, and merely tried to develop new *models* for the electron within the conventional Euclidean geometry for space and time, Einstein adopted an epistemological approach which was later shown by Minkowski to lead to a new kind of geometry, which was based on an indefinite metric, and eventually became universally known under the name of Minkowski space.

Within the resulting *physical* geometry of Minkowski space, the key idea of *space-time* was conceptualized as providing an insuperable intertwining of the operationally defined concepts of space and time. This fact was boldly emphasized by Minkowski in the very opening of his historically famous 1908 talk on the subject: "Henceforth space by itself, and time by itself, are doomed to fade away into the mere shadows, and only a kind of union of the two will preserve an independent reality." (Minkowski, 1908, p. 75.)

On the other hand, Einstein's deep faith in the power of sound epistemology was to be soon tested, as he was confronted with an apparently decisive negative verdict on his theory from a leading experimentalist of that time. Indeed, the experimental results[1] of Kaufmann (1905, 1906) seemed to provide indubitable support for Abraham's (1902, 1903) theory of the electron, and to contradict the theories of both Lorentz and Einstein. The ensuing situation is recounted as follows in a historical study of this subject: "We know what Einstein did when he heard about Kaufmann's results – one of the foremost experimentalists in Europe disproving this unknown person's work. Einstein did not respond for nearly two years. Finally, ... [in 1907] Einstein wrote that he had not found any obvious errors in Kaufmann's article, but that the theory that was being proved by Kaufmann's data was a theory of so much smaller generality than his own, and therefore so much less probable, that he would prefer for the time being to stay with it. Actually, it took until 1916 for a fault in Kaufmann's experimental equipment to be discovered." (Holton, 1980, p. 92.)

Of course, by 1916 Einstein had just successfully completed his protracted search after a general theory of relativity – to which we shall refer as *classical general relativity*, or *CGR*. In developing CGR he adopted Riemannian geometry as the mathematical vehicle best suited for embedding his relativistic principles into a theory of gravity, but retained his operational outlook towards the concept of spacetime, as seen from the following quotation: "All our space-time verifications invariably amount to a determination of space-time coincidences. If, for example, events consisted merely in the motion of material points, then ultimately nothing would be observable but the meetings of two or more of

[1] A detailed contemporary recounting of Kaufmann's experiment can be found in Chapter 6 of (Zahar, 1989), together with Planck's logical analysis, which reinterpreted it as a test between competing methodologies, rather than a decisive test between Abraham's (1902) theory and Einstein's (1905) relativity theory *per se*.

these points. Moreover, the results of our measurings are nothing but verifications of such meetings of the material points of our instruments with other material points, coincidences between the hands of a clock and points on the clock dial, and observed point-events happening at the same place at the same time." (Einstein, 1916, p. 117.)

It was this operationally-founded epistemology, in combination with his logico-analytic methodology, that prompted Einstein to adopt as fundamental to CGR the principles of general covariance and of strong equivalence.

The principle of general covariance eventually turned out to be the more controversial of the two, and even to this day it sometimes gives rise to physical or mathematical dead ends. In fact, the protracted intellectual struggle in which Einstein was engaged while trying to arrive at a satisfactory formulation of CGR caused him to temporarily abandon the principle of general covariance. This delayed the completion of his general relativistic programme by several years even after, as the result (Pais, 1982) of his many conversations about Riemannian geometry with his friend and eventual co-author Marcel Grossmann, he had already embraced that mathematical framework as best suited for that programme.

Indeed, in their joint 1913 article, which eventually became known as their *Entwurf* paper, Einstein and Grossmann argued as follows: assume that a nongravitational stress-energy tensor vanishes identically within some region[2] L of a pseudo-Riemannian manifold $\mathbf{M}$ describing spacetime, but it was fixed and nonzero outside L; then a coordinate transformation *inside* L could change the components of the metric tensor inside L despite the fact that the components of the stress-energy tensor remain unchanged under such a transformation within the entire spacetime manifold $\mathbf{M}$.

Einstein and Grossmann took this to imply that "if therefore... one maintains that the [metric tensor] should be completely determined by the [nongravitational stress-energy tensor], then one is forced to restrict the choice of the coordinate system" (Einstein and Grossmann, 1913, p. 260) – namely to give up the principle of general covariance. This "hole" argument later led some general relativists to the impression that there was a lack of understanding on the part of Einstein as to what "covariance" under coordinate transformation actually meant. However, as we shall see in Sec. 2.1, and as the more recent historical studies by Norton (1987, 1989) and Stachel (1989) demonstrate, the "hole" argument actually established a most fundamental conclusion about the structural nature of general relativistic theories. That feature can be expressed in contemporary mathematical language by saying that all CGR theories have to be invariant under global diffeomorphisms of $\mathbf{M}$. This invariance is nowadays taken by some general relativists to provide the most adequate formulation of the principle of general covariance.

On the other hand, the formulation of this principle originally adopted by Einstein, and even emphasized by him by means of italics, ran as follows:

"*The general laws of nature are to be expressed by equations which hold good for all systems of co-ordinates, that is, are co-variant with respect to any substitutions whatever (generally co-variant).*" (Einstein, 1916, p. 117.)

This formulation, however, exposed Einstein to the later criticism of Kretschmann

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[2] The letter L stands for "Loch" in the German original, which has the meaning of "hole" in English. Hence this entire argument has become known under the name of the "hole" argument. Its historical origins and significance for general relativity have been extensively discussed by Norton (1987, 1989) and by Stachel (1989).

(1917), who pointed out that one can take *any* system of differential equations expressed in some fixed coordinate chart, and then transform those equations to other coordinate charts by using the algebra of tensor transformations (cf. Chapter 2), and thus satisfy the above formulation of the principle of general covariance. Thus, in the words of a contemporary philosopher of science, "the principle of general covariance has no physical content whatever: it specifies no particular physical theory; rather, it merely expresses our commitment to a certain style of formulating physical theories. For the same reason, general covariance has nothing at all to do with the relativity of motion." (Friedman, 1983, p. 55.)

In other words, from Einstein's original formulation of the principle of general covariance, this principle emerges as a *mathematically* necessary condition due to the "style" of the CGR framework, but not as a *physically* meaningful condition of the kind envisaged by Einstein – namely one that would affect the *physical* content of general relativistic theories in a manner analogous to the way in which Lorentz covariance affected his formulation of special relativistic theory in the course of determining its operational interpretation. As a consequence, this key principle is only very casually mentioned in some of the best known and most influential textbooks on CGR, with the sometimes cited reason being that it "fathered half a century of confusion." (Misner *et al*, 1973, p. 431.)

As will be shown in the next chapter, this "confusion" could have been avoided if a consistently fibre-theoretical point of view had been adopted after H. Cartan, largely inspired and motivated by classical general relativity, set down in the 1920s and 1930s the foundations of the fibre-theoretical framework to differential geometry that was destined to provide, at least amongst mathematicians, *the* modern approach to this subject. Indeed, this approach brings to the fore the concept of *principal frame bundle* over a differential manifold; moreover, in the presence of Lorentzian metrics, such as the ones used in CGR, it singles out orthonormal frames that play the physical role of local Lorentzian frames. In turn, such frames provide the means of a natural realization of the following dictum, which to a large extent motivated Einstein's original formulation of the principle of general covariance: "*The laws of physics must be of such a nature that they apply to systems of reference in any kind of motion.*" (Einstein, 1916, p. 113.)

On the other hand, the principle of strong equivalence singles out *inertial* moving Lorentz frames, so that "absolutely accelerating and rotating frames are just as distinguishable from inertially moving frames in general relativity as in all previous theories." (Friedman, 1983, p. 208.) Indeed, this principle was the one that Einstein liked to illustrate by his well-known *gedanken* experiments carried out by an "observer" imagined to be totally isolated within an elevator in free fall, within whose confines the laws of special relativity would appear to him to hold true. Consequently, the strong equivalence principle was originally formulated by Einstein as follows:

"For infinitely small four-dimensional regions the theory of relativity in the restricted sense is appropriate, if coordinates are suitably chosen." (Einstein, 1916, p. 118.)

Nowadays this is taken to assert that "*in any and every local Lorentz frame [in free fall] all the (nongravitational) laws of physics must take on their familiar special relativistic forms*" (Misner *et al.*, 1973, p. 386). However, when it comes to its precise formulation, Friedman's foundational analysis of the various shades of mathematical meaning of this basic general relativistic principle concludes with the following observations: "Standard

formulations of the [strong] principle of equivalence characteristically obscure [the] crucial distinction between first-order laws and second-order laws by blurring the distinction between 'infinitesimal' laws, holding at a single point, and local laws, holding on a neighborhood of a point. ... What the principle of equivalence says, then, is that special relativity and general relativity have the same 'infinitesimal' structure, not that they have the same local structure." (Friedman, 1983, p. 202.) At the mathematical level, this is a most fundamental distinction since, as we shall see in the next chapter, the "infinitesimal structure" of a tangent bundle is related to internal features of its fibres, but not to the topology of its base manifold, which is reflected only in its "local structure."

Thus, in the ultimate analysis, although Einstein's intentions in adopting the principles of general covariance and of strong equivalence as fundamental to CGR were clearly stated, the subsequent mathematical embodiment in standard literature of both these principles has given rise to a considerable amount of confusion, and sometimes even misrepresentation. This confusion can be, however, eliminated by understanding the basic tenets of Einstein's epistemology, and combining that epistemology with those modern mathematical tools that are best suited for its precise and faithful realization. This will be the task carried out in Chapter 2, as a preamble to the application of these methods to the formulation of a *quantum general relativity* (*QGR* for short) that represents an extrapolation of CGR, and is based on adaptations of these basic principles to the quantum regime.

1.2. Geometry as Part of Physical Theory

While working at the beginning of this century on the dynamics of the electron, H. Poincaré not only discovered many of the properties of the inhomogeneous Lorentz group (later named after him), but also discovered many important results in special relativity. Actually, in a recent historical study it is argued "that, in 1905, Poincaré had gone far beyond the results obtained by Einstein; that, in 1900, he had already given the operational definition of clock synchronization which is usually, but incorrectly, attributed to Einstein; that he had enunciated the Lorentz covariance principle and founded the relativity programme of the Lorentz group; that he had corrected the transformation for electric density proposed by Lorentz in 1904, then used this new rule to transform the whole electromagnetic field." (Zahar, 1989, p. 150.) Moreover, several years before Einstein published his 1905 paper on special relativity, Poincaré examined at great length in his monograph *Science and Hypothesis* the possibility of adopting non-Euclidean geometries in the description of physical space. However, his ultimate thesis was that "experience ... tells us not which is the truest geometry, but which is the most *convenient*" (cf. Feigle and Brodbeck, 1953, p. 180), and Poincaré eventually concluded that Euclidean geometry fulfilled such a "convenience criterion." (cf. Jammer, 1969, pp. 165 and 209.)

Consequently, although some historians of science later ascribed the discovery of relativity theory to Poincaré (Whittaker, 1951; Zahar, 1989), the almost universal consensus nowadays is that the credit belongs to Einstein. And indeed, whereas Einstein developed the special theory of relativity from operational considerations based on light signals, which provided an interpretation of Lorentz transformations as part of a theory of the (later to be named) Minkowski space, Poincaré's treatment primarily centered on the electromag-

netic phenomena themselves. Furthermore, although with a different outlook he might have made the intellectual leap that eventually led Einstein, via the equivalence principle, to the general theory of relativity, his conventionalism blinded him to the possibilities inherent in Riemannian geometry viewed as framework for physical theories, despite his outstanding expertise in this and other areas of mathematics. On the other hand, Einstein totally rejected from the outset Poincaré's conventionalistic point of view, and eventually stated very decisively that "the question whether the structure of [the spacetime] continuum is Euclidean, or in accordance with Riemann's general scheme, or otherwise, is . . . a physical question which must be answered by experience, and not a question of mere convention to be selected on practical grounds." (cf. Feigle and Brodbeck, 1953, p. 193.)

To uncover the reasons behind this essential difference of opinion, which has given rise in the history of science to two diametrically opposed epistemologies, one has to trace the historical development of the concept of geometry to its very roots in the culture of ancient Babylonia, Egypt and Greece. In Babylonia and Egypt geometric ideas arose in response to practical problems encountered in land surveying (Lorenzen, 1987), so that when those ideas became adopted in ancient Greece, they became collectively described by the term "geo-metry" – whose original Greek meaning is "earth-measurement." In keeping with this fact, Aristotle and some of the ancient Greek philosophers regarded the concepts of geometry as rooted in the physical world. On the other hand, Plato and other very influential philosophers in ancient Greece, viewed them as a reflection of a world of pure ideas (Maziarz and Greenwood, 1968). Consequently, after Euclid compiled and systematized in his *Elements* the rich body of interrelated definitions, axioms and theorems eventually named after him, it became commonplace to regard that body of knowledge as an abstract discipline whose value and validity transcended the realm of experience.

This attitude culminated in the *Critique of Pure Reason*, first published by I. Kant in 1781, where it was claimed that Euclidean geometry enjoyed an exclusively *a priori* status. The philosophical arguments underlying this claim were not only uncritically accepted by the most outstanding of Kant's contemporaries, but when, towards the end of the eighteenth century, Gauss discovered non-Euclidean geometries, he did not dare publish his results since he feared the ridicule and condemnation of his peers (Kline, 1980).

On the other hand, Gauss proved to be at the epistemological level a true predecessor of Einstein, as he became involved in geodetic goniometry and related professional activities that clearly reflected his empirical approach to geometry (Jammer, 1969; Reich, 1977). In fact, he even attempted to establish observationally the deviations from 180° in the sum of the angles of a large triangle formed by light rays travelling between observational towers on three mountain peaks. Nowadays we know that such a deviation exists, but the 15" discrepancy measured by Gauss was within his observational error bounds. Nevertheless, by 1830 he overcame his qualms and publicly announced his conviction that geometry is *not* an *a priori* science. Thus, he implicitly confirmed that, as a contemporary historian of mathematics points out, it was only Kant's "inability to conceive of another geometry [that] convinced him that there could be no other." (Kline, 1980, p. 76.)

On the other hand, despite the independent discoveries of non-Euclidean geometries by Gauss, Bolyai and Lobachevski, most of the mathematicians and physicists of the last century did not display a great interest in this subject matter, or in the related question as to

whether Euclidean geometry is indeed the physical geometry of space. Around the middle of the last century Riemann provided the only notable exception to this professional disinterest. Thus, in his famous 1854 inaugural lecture (attended by Gauss) he publicly expressed his belief in treating geometry as an empirical science, since "the theorems of geometry cannot be deduced from general notions of quantity, but that those properties which distinguish Space from other conceivably triply extended quantities can only be deduced from experience." (Cf. Spivak, 1979, p. 135.)

Despite the very positive attitudes of such outstanding mathematicians as Gauss and Riemann on the issue of geometry viewed as part of physical science, the change in the intellectual climate was very slow: "Non-Euclidean geometry and its implications about the truth of geometry were accepted gradually by mathematicians [in the course of the last century], but not because the arguments for its applicability were strengthened in any way. Rather the reason was given in the early 1900s by Max Planck, the founder of quantum mechanics: 'A new scientific truth does not triumph by convincing its opponents and making them see light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it.' " (Kline, 1980, p. 88.)

On the other hand, Riemann prepared the ground for the advent of general relativity when he pointed out that all data supporting a choice of physical geometry "are – like all data – not logically necessary, but only of empirical certainty, [i.e.] they are hypotheses; one can therefore investigate their likelihood, which is [for Euclidean geometry] certainly very great within the bounds of observation, and afterwards decide upon the legitimacy of extending them beyond the bounds of observation, both in the direction of the immeasurably large, and in the direction of the immeasurably small." (Cf. Spivak, 1979, p. 136.)

Viewed in this light, Poincaré's conventionalism represented historically the ultimate effort from an outstanding thinker to retain what was mathematically most familiar, and therefore intellectually most comfortable for a conservative predisposition, by forcing *any* observation pertaining to geometric relationships to conform to a *preconceived* set of ideas (Wilder, 1967). As pointed out by philosophers of science, the premises of such a conventionalistic outlook cannot be rejected on either logical or empirical grounds alone, since the "propositions of a physical geometry cannot be tested until a method of measurement has been chosen." (Pap, 1962, p. 112.) In other words, the tests of the validity of a given choice of *physical* geometry, i.e., of a geometry viewed as part of a physical theory, depend crucially on the underlying theory of measurement – in particular on the *choice* of test bodies, reference frames, measuring devices, signals, etc., and on the *assumed* properties and behavior of all these physical entities.

This fact emerges very clearly from Einstein's 1905 and 1916 key papers on relativity theory. Although he did not spell out all the details, it is clear from those writings that he considered his test particles to be pointlike massive objects, that he viewed his reference frames as consisting of macroscopic "rigid rods" and "standard clocks," that he considered that light signals provided the most fundamental means of communication within his measurement-theoretical scheme, and that he envisaged that all these objects behaved in a strictly deterministic manner.

On the other hand, the *actual* existence of "rigid rods" would contradict the relativity principle, since it would allow for the instantaneous transmission of signals from one end

of such a rod to the other. However, Marzke and Wheeler (1964) proved that the postulate of their existence was *not* required by the operational interpretation of general relativity. Later studies by Kronheimer and Penrose (1967), Ehlers *et al.* (1972-73), Woodhouse (1973), and others, confirmed the feasibility of deriving the geometric structure of general relativity from a consistent set of axioms about the behavior of these physical objects viewed as *classical* objects.

That still left open, however, the question of whether there indeed exist in nature objects which display, either in a strict sense, or at least in an arbitrarily close approximation, the classical behavior postulated by Einstein. For example, in the well-known collection of essays honoring Einstein's seventieth birthday, H. Margenau voiced the criticism that "particles may not be regarded as points but as structures of finite size ..., which threatens the validity of [relativistic] causal description" (Schilpp, 1949, p. 259); whereas K. Menger pointed out that all our empirical knowledge is statistical in nature, so that a thorough geometrization of physics might require "the introduction of probability into the foundations of geometry" (Schilpp, 1949, p. 472). Moreover, even after many years of extensive correspondence with Einstein, Max Born remained convinced that "the mathematical concept of a point in a continuum has no direct physical significance." (Born, 1955, p. 3). Eventually, Born arrived at the following epistemic conclusion, which has an immediate bearing on this issue: "Statements like 'A quantity x has a completely definite value' (expressed by a real number and represented by a point in the physical continuum) seem to me to have no physical meaning. Modern physics has achieved its greatest successes by applying a principle of methodology that concepts whose application requires distinctions that cannot in principle be observed are meaningless and must be eliminated." (Born, 1956, p.167.)

These epistemic observations actually do not contradict the views of Einstein, since in the same essay in which he implicitly criticized Poincaré's conventionalism, Einstein also clearly stated "that the proposed [by himself] physical interpretation [of Lorentzian geometry] breaks down when applied immediately to spaces of sub-molecular order of magnitude." (Cf. Feigl and Brodbeck, 1953, p. 193.) In view of this fact, when one tries to extend the domain of validity of relativity theory to the microscopic domain, one is faced with the following two clear-cut alternatives:

1) To view the choice of geometry at the subatomic level conventionalistically, as Poincaré did with Euclidean geometry at the macroscopic level, and to therefore consider the adoption of Minkowski geometry, or, more generally, of Lorentzian geometries, to be at the microscopic level a matter of pure convention dictated by convenience, which, as such, transcends empirical verification.

2) To face the fact that, according to generally accepted quantum principles, the behavior of matter at the subatomic level is *not* in accordance with the deterministic behavior postulated by the operational interpretations of these classically-founded geometries, and that therefore new types of geometric structures, which incorporate quantum principles, have to be introduced as one extrapolates relativity theory to the micro-domain.

The developments in high-energy physics during the second half of this century have tended to favour the first alternative. Clearly, however, if the same fundamental epistemic

principles hold true in all of physics, then it is only the second of these two alternatives that is epistemologically sound.

1.3. Quantum Theory and the Idea of Fundamental Length

After Bohr's old quantum mechanics transcended in 1925 into the modern quantum mechanics of Heisenberg and Schrödinger, and after Born proposed his fundamentally statistical interpretation of this new theoretical framework, it was soon realized by some physicists that radical modifications of the classical concept of spacetime were required if a reconciliation of relativity theory and the just emerged new quantum theory were to be achieved. Hence, after the introduction by Heisenberg (1927) of the uncertainty principle in nonrelativistic quantum mechanics, papers dealing with schemes for a "quantization of spacetime" related to elementary space and time uncertainties began appearing (Ruark, 1928; Flint and Richardson, 1928; Fürth, 1929; Landau and Peierls, 1931; Glaser and Sitte, 1934; Flint, 1937).

On the other hand, as far as nonrelativistic quantum theory was concerned, there was no foundational need for any new geometries, since by that time it was already universally agreed that nonrelativistic physics provided approximations that were satisfactory under certain physical conditions, wherein Euclidean geometry was acceptable not only as the most convenient computational tool, but also as the most adequate geometric framework. Hence, it was in the realm of relativistic quantum field theory that the need for new ideas soon became evident not only on epistemic grounds, but even on mathematical grounds.

Remarkably, the quantization of the electromagnetic field was undertaken by Born and Jordan (1925) already in the very same year when matrix mechanics was first formulated by Heisenberg. However, the generally acknowledged founder of quantum electrodynamics (QED) is deemed to be Dirac, since his was the first attempt at developing a method of quantization of the electromagnetic field which strived to be consistent with the basic tenets of special relativity. In the introduction to his pioneering 1927 paper on this subject, Dirac summarized the problems he encountered in this task, and his degree of success at coping with them, as follows:

"Hardly anything has been done up to the present on quantum electrodynamics. The question of the correct treatment of a system in which forces are propagated with the velocity of light instead of instantaneously, of the production of an electromagnetic field by a moving electron, and of the reaction of this field on the electron have not yet been touched. In addition, there is a serious difficulty in making the theory satisfy all the requirements of the restricted principle of relativity, since a Hamiltonian function can no longer be used. This relativity question is, of course, connected with the previous one, and it will be impossible to answer any one question completely without at the same time answering them all. However, it appears possible to build a fairly satisfactory theory of the emission of radiation and of the reaction of the radiation field on the emitting system on the basis of kinematics and dynamics which are not strictly relativistic. This is the main object of the present paper. The theory is non-relativistic on account of the time being counted throughout as a c-number, instead of being treated symmetrically with the space co-ordinates. . . . The mathematical development of the theory has been made possible by the author's

general transformation theory of the quantum matrices. Owing to the fact that we count time as a c-number, we are allowed to use the notion of the value of the dynamical variable at any instant of time. This value is a q-number, capable of being represented by a generalized 'matrix' according to many different matrix schemes, some of which may have continuous ranges of rows and columns, and may require the matrix elements to involve certain kinds of infinities (of the type given by the δ functions)." (Dirac, 1927, pp. 244-246.)

It was these "infinities" that were to plague quantum field theory from its beginning to the present day. They prompted Bohr to consider already in the early 1930s the possibility of existence of a fundamental length in nature (Bohr, 1980). In particular, in a letter to Dirac, he expressed the following opinion: "I . . . believe firmly the solution of the present troubles [with divergences] will not be reached without a revision of our general physical ideas still deeper than that contemplated in the present quantum mechanics." (Cf. Cassidy, 1992, p. 287.)

It was Heisenberg (1930a) who first attempted such a revision of QED by postulating a lattice structure for spacetime. Bohr, however, soon rejected that idea, since it led to a loss of charge conservation. Nevertheless, throughout the remainder of his professional life Heisenberg made various attempts at introducing a fundamental length in quantum theory. Amongst the best known are the papers that led Heisenberg (1938, 1943) to the concept of *S*-matrix. This work inspired other researchers (March, 1937; Markov, 1940) to propose a statistical concept of metric, and led Born (1938-1949) to propose and develop a theory of reciprocity between spacetime and 4-momemtum variables, based on the idea of a quantum metric operator that incorporated a fundamental length in its very definition. Interestingly, some aspects of this idea were later adapted by Yukawa (1950, 1953) to his formulation of non-local quantum field theory; this in turn led to the formulation of a relativistic concept of harmonic oscillator (Katayama and Yukawa, 1968; Takabayashi, 1967, 1979) that enabled Feynman *et al* (1971), as well as others, to formulate a relativistic quark model for mesons. On the other hand, as we shall see in Chapter 3, by imposing on Born's quantum metric operator a type of boundary condition that is different from that used after its identification with the Hamiltonian for a relativistic harmonic oscillator, its ground state assumes a basic significance for all the quantum geometries considered in this monograph.

In the 1940s the lattice-type approach to the introduction of a fundamental length in quantum physics was revived by Snyder (1947) in the form of "quantized spacetimes." For a while such discrete types of spacetimes were pursued by a host of researchers (Yang, 1947; Flint, 1948; Schild, 1948; Hill, 1950; Das, 1960; Gol'fand, 1963; Kadyshevskii, 1963). However, that idea eventually sank into oblivion. In fact, during the last few years several researchers (Bombelli *et al.*, 1987; Finkelstein, 1989; Gudder, 1988; Isham, 1990) independently proposed mathematical schemes for discrete models of spacetime without making any references to the aforementioned earlier work.

There is no doubt that mathematically closely related schemes for making numerical approximations based on the introduction of lattices in gauge field theories have provided powerful computational techniques. However, from a foundational physical perspective, the principal difficulty with introducing in relativistic quantum physics a fundamental length based on such discrete-spacetime formulations lies in the violations of special relativistic as well as general relativistic covariance principles: any particular choice of lattice in

Minkowski space is bound to lead to a breakdown of Lorentz covariance, since it favors the spatio-temporal directions dictated by the adopted choice of lattice points; when such a procedure is applied in the general relativistic regime, it violates the principle of general covariance[3], since it favors those coordinate systems that are used when a Lorentzian spacetime continuum is "quantized" by singling out in it a lattice of points.

The common weak point of all of the above cited proposals for quantum spacetimes resides in their lack of a clear-cut theory of measurement, that could point the way to empirically well founded geometric ideas. If they were present, such ideas would have either enabled the injection of the fundamental principles of relativistic invariance, causality and locality into these frameworks, or they would have produced compelling principles that would transcend into the aforementioned ones in the macroscopic regime, where classical special and general relativity have received most convincing experimental confirmation. But no such fundamentally new epistemic ideas emerged, since most of the above work on "quantized spacetimes" was of a formal mathematical nature. This contributed to the fact that, as discussed in the next section, after the achievement of the well-known numerical successes of renormalization theory in the late 1940s, the interest in the idea of fundamental length began to suffer a rapid decline.

On the other hand, at the foundational level, Riemann's thesis cited in the preceding section, which stipulates that it is necessary to modify conventional geometric concepts as one applies them "both in the direction of the immeasurably large, and in the direction of the immeasurably small," remains very pertinent to the developments in the present monograph. Indeed, whereas classical general relativity denies the legitimacy of merely extending the bounds of validity of Euclidean geometry "in the direction of the immeasurably large," and requires instead the adoption of Lorentzian geometries, the existence of intrinsic quantum fluctuations at the microlevel removes the bounds of validity not only of Euclidean but also of Lorentzian geometries as we move "in the direction of the immeasurably small" – with all contemporary evidence indicating that the role of "immeasurably small" is played by the Planck length and time, i.e., that it occurs at approximately 10^{-33} cm and 10^{-44} sec (DeWitt, 1962; Mead, 1964; Blokhintsev, 1973; Namsrai, 1986; Markov *et al.*, 1988).

The empirical interpretation of Lorentzian geometries provided by Einstein (1905, 1916) was purely classical in the sense that it was related to measurement-theoretical procedures with test bodies which were assumed to behave deterministically. Therefore, in later chapters we shall argue in favor of replacing, at least at the microlevel, such classical geometries with geometries which are quantum. Their "quantum" nature will not be manifested in some artificial "discretization" scheme, but rather by the fact that they allow a probabilistic interpretation which embodies, from the outset, Heisenberg's uncertainty principle for position-momentum and time-energy measurements, as well as the limitations imposed on spatio-temporal relationships when Planck's length is regarded as an *absolute* lower limit for the operational distinguishability of spacetime locations. Epistemologically, this type of approach to the idea of fundamental length is very much in the spirit advocated by Heisenberg (1976) in his last paper, where he pointed out that the conceptual procedure

[3] As will be explained and discussed in Chapter 2, there are additional facets to the principle of general covariance, related to diffemorphism invariance and to Lorentz gauge invariance. However, the introduction of a discrete spacetime structure renders the other facets meaningless, since differentiable structures are essential to them.

of "subdividing" spacetime into smaller and smaller regions might *gradually lose its very meaning* as we approach "immeasurably small" orders of magnitude.

1.4. Localizability and Renormalizability in Quantum Field Theory

As we saw in the preceding section, Dirac was aware of the problem of divergencies from the very time he founded QED. The persistence of these problems, despite many efforts to eliminate them, eventually led him to the following conclusion: "Present-day atomic theories involve the assumption of localizability, which is sufficient but it is very likely to be too stringent. The assumption requires that the theory shall be built up in terms of dynamical variables that are each localized at some point in space-time, two variables localized at two points lying outside each other's light-cones being assumed to have zero P.b. [i.e., Poisson bracket]. A less drastic assumption may be adequate, e.g., that there is a fundamental length λ such that the P.b. of two dynamical variables must vanish if they are localized at two points whose separation is space-like and greater than λ, but need not vanish if it is less than λ." (Dirac, 1949, p. 339.)

Subsequent developments in conventional quantum field theory (QFT) proved that the relaxation of the local (anti)-commutativity postulate in the form proposed above by Dirac was actually unfeasible. At the purely mathematical level, the reasons for this emerge very clearly from a recent monograph which deals systematically with all the general principles of conventional QFT: "The postulate of local commutativity is one of the most restrictive principles of quantum field theory. Misgivings could arise concerning the experimental justification of this postulate: we have no special reason for supposing that the measurement of a component of a Hermitian field at some point has no influence on the value of the components of this field at another point separated from the first point by an arbitrarily small spacelike interval. It turns out, however, that the property of local commutativity can be *proved* if we make the assumption, which at a first glance seems to be weaker, that the fields commute only at sufficiently large spacelike separations. It follows from this that if the remaining requirements of relativistic quantum theory hold in the non-local theory, then, roughly speaking, the commutator of the fields must be non-zero everywhere. It is therefore no surprise that the attempts to introduce 'non-locality in the small' at the same time require a rejection of some other requirements of the Wightman formalism, for example, 'renormalizability'." (Bogolubov *et al.*, 1990, p. 373.)

The above mentioned "Wightman formalism" merely embodies, in a mathematically rigorous form, principles which are taken for granted in conventional QFT. Hence, the arguments presented in the above quotation demonstrate that a physically and mathematically consistent introduction of fundamental length in relativistic quantum physics requires a more drastic revision of conventional ideas on QFT than that envisaged by Dirac in 1949.

However, these limitations did *not* represent the main reason why Dirac's suggestion was subsequently ignored. Rather, the main reasons why this and some other key epistemic ideas and guidelines of the founders of quantum mechanics were ignored in the post-World War II era are traceable to some of the facts described in a historical essay on developments in QFT by S. S. Schweber (1986a). In particular, it is pointed out there that "the workers of the 1930s, particularly Bohr and Dirac, had sought solutions to the

problems [of quantum field theory] in terms of revolutionary departures. ... The solution advanced by Feynman, Schwinger, and Dyson was at its core conservative: it asked to take seriously the received formulation of quantum mechanics and special relativity and to explore the content of [their] synthesis. A generational conflict manifested itself in the contrast between the revolutionary and conservative stances of the pre- and post-World War II theoreticians." (Schweber, 1986a, p. 299.)

This sharp demarcation line between pre- and post-World War II attitudes towards foundational problems in QFT can be traced to differences of opinion over the theoretical treatment of cosmic rays, for which Heisenberg advocated in the late 1930s a fundamental length of the order of 1 fermi – i.e., 10^{-13} cm. As pointed out in a recent biography by Cassidy (1992), depicting Heisenberg's life and science, some well-known physicists, who were to play very important roles during post-World War II developments in quantum physics, reacted very negatively to that proposal. The crux of the debated issues is depicted as follows: "While Heisenberg's [approach to cosmic ray showers and their field-theoretic implications] attracted seekers of a new and even more radical quantum revolution – mainly the leaders of the first revolution, Heisenberg, Pauli, and their remaining Central European collaborators – the Anderson-Neddermeyer paper confirmed the Bethe-Heitler theory of absorption for the soft component and catalyzed the calculation of 'cascade' showers using QED. For the supporters of this rival account of high-energy particle creation – mainly Oppenheimer, Heitler, and their collaborators in England and the United States – no revolution was necessary at all: QED would do just fine." (Cassidy, 1992, p. 361.)

An illustration of this generational conflict can be also found in a recent treatise on Pauli's philosophy: "Pauli and Heisenberg formulated the basis of modern quantum field theory in 1929. Pauli remarked in several connections, however, that quantum field theory is not satisfactory in its present form. He saw in it deep and fundamental problems which evidently specifically involve the concepts of space and time, but was unable to propose anything more suitable to replace the present theory. ... He was not satisfied with the instrumentalist attitude [of post-World War II theoreticians] either, but laid great stress on the search for a conception of reality in which a picture of the microworld as atomic theory gives us would also fit." (Laurikainen, 1988, p. 19.)

The crux of this fundamental difference of opinion between the pre- and post-World War II generations of theoretical physicists emerges most clearly from Dirac's plethora of critical statements on the subject. Thus, soon after the renormalization programme was launched, he declared: "Recent work by Lamb, Schwinger and Feynman and others has been very successful ... but the resulting theory is an ugly and incomplete one." (Dirac, 1951) And, as extensively documented in his recent biography (Kragh, 1990), throughout the remainder of his life he never wavered in the verdict that "these [renormalization] rules, even though they may lead to results in agreement with observations, are artificial rules, and I just cannot accept that the present foundations [of relativistic quantum field theory] are correct." (Dirac 1978a, p. 20.) In fact, in his very last paper, published posthumously under the title "The Inadequacies of Quantum Field Theory", Dirac reiterated the following: "Just because the results [of conventional renormalization theory] happen to be in agreement with observation does not prove that one's theory is correct. After all, the Bohr theory was correct in simple cases. It gave very good answers, but still the Bohr theory had

the wrong concepts. Correspondingly, the renormalized kind of quantum theory with which physicists are working nowadays is not justifiable by agreement with experiments." (Dirac, 1987, p. 196.)

It has to be emphasized, however, that the purely instrumentalist attitude of many leading post-World War II theoreticians towards the foundational problems of quantum theory, extensively discussed and criticized by Popper (1982a), and clearly reflected in the widespread acceptance of the conventional post-World War II renormalization programme as the panacea of all the ills encountered by QFT in the pre-World War II era, was not wholeheartedly shared by its two main progenitors, namely by J. Schwinger and R. P. Feynman.

In fact, Schwinger himself explained very clearly as follows why the renormalization programme, which he co-founded, was not, in the ultimate analysis, totally satisfactory: "The post-war developments of quantum electrodynamics have been largely dominated by questions of formalism and technique, and do not contain any fundamental improvement in the physical foundations of the theory. . . . The observational basis of quantum electrodynamics is self-contradictory. . . . The localization of charge with indefinite precision requires for its realization a coupling with the electromagnetic field that can attain arbitrarily large magnitudes. *The resulting appearance of divergences, and contradictions, serves to deny the basic measurement hypothesis.* ... To limit the magnitude of interactions while retaining the customary coördinate description is contradictory; since *no mechanism is provided for precisely localized measurements.* . . . But, we may ask, is there a fatal fault in the structure of field theory? Could it not be that the divergences – apparent symptoms of malignancy – are only spurious by-products of an invalid expansion in powers of the coupling constant and that renormalization, which can change no physical implication of the theory, simply rectifies this mathematical error? This hope disappears on recognizing that *the observational basis of quantum electrodynamics is self-contradictory. . . . A convergent theory cannot be formulated consistently within the framework of present space-time concepts.*" (Schwinger, 1958, pp. xv-xvi – italics added.)

Similarly, in a recent well-documented biography dealing with Feynman's life and science, it is pointed out that "Feynman himself remained nearly as uncomfortable [with conventional renormalization theory] as Dirac." (Gleick, 1992, p. 348.) Feynman had expressed this discomfort on various occasions, and most notably in his 1965 acceptance speech for the Nobel prize awarded to him for his contributions to the renormalization programme in quantum electrodynamics, where, with his characteristic directness, he stated that "I believe there is really no satisfactory quantum electrodynamics, but I'm not sure. . . . Therefore, I think that the renormalization theory is simply a way to sweep the difficulties of the divergences of electrodynamics under the rug." (Feynman, 1966, p. 707.) More than twenty years later, in an interview in which he discussed mainly the shortcomings of superstring theory as an effective tool for dealing with the same kind of infinities as in QED, he made some remarks which clarify why he was not totally certain of his earlier assessment of renormalization theory: "[I]t's also possible that electrodynamics is not a consistent theory . . . . If we didn't have a mathematically consistent theory, we would have to learn more about nature and find out what modifications to electrodynamics would be necessary. . . . I must say right away that I never thought trying to get rid of the

infinities would be a good way to discover correct physical laws, and I was wrong. I've often been wrong in guessing the best way to proceed." (Feynman, 1989, p. 199.)

1.5. Quantum Field Theory in Curved Spacetime

The first attempt at formulating quantum field theory in a curved spacetime can be traced to Schrödinger (1939), but a canonical scheme for the second-quantization of fields in curved spacetime was first systematically studied by L. E. Parker (1968, 1971). Since that time this subject grew to the extent that it became the topic of monographs (cf., e.g., Birrell and Davies, 1982; Fulling, 1989) and entire conferences (cf., e.g., Audretsch and de Sabbata, 1990), despite the fact that, as pointed out in one of these monographs: "*There does not exist a quantum field theory formalism in an arbitrary curved spacetime*. This problem is deep-rooted and arises from the fact that standard formalisms of field theory require a preferential slicing in spacetime." (Narlikar and Padmanabhan, 1986, p. 277.)

More specifically, the review article "Quantum Field Theory in Curved Spacetime" by B. S. DeWitt (1975) lists, in a section entitled "Failure of conventional procedures," the following difficulties:

"This [canonical quantization procedure of fields in curved spacetime] is just as in conventional particle physics. The trouble with it is: *it's wrong*. It is not wrong in a technical mathematical sense. It simply provides a grossly inadequate foundation for the theory. Here are just some of the situations where it fails:

1. There may be no Killing vector at all, timelike or spacelike. This is the generic situation. How to deal with it is unknown, except possibly when there is an approximate Killing vector that becomes exact asymptotically. ...

2. There may be a global Killing vector, but it may not be everywhere timelike. ...

3. Spacetime may be stationary only in limited regions. If each region possesses complete Cauchy hypersurfaces then a local timelike Killing vector field may be set up in each and vacuum defined for each [such region]. ..." (DeWitt, 1975, p. 302).

However, even in those cases where there is a global timelike Killing vector field, so that the Lorentzian spacetime manifold is stationary but not static, foundational difficulties remain, since the prediction of conventional QFT in such curved spacetimes is that *inertial* detectors will register spontaneous pair creation *ex nihilo* .

This phenomenon is usually compared (Birrell and Davies, 1982) with some of the results obtained when the canonical quantization of fields in Minkowski space is *formally* carried out in Rindler coordinates, rather than in the Minkowski coordinates used by Dirac (1927) in founding QFT. Indeed, such a purely formal adaptation of the canonical second-quantization scheme then leads to spontaneous pair creation *ex nihilo* that is not part of the energy-conserving and well established phenomenon of pair creation for quantum fields in Minkowski space, but rather "suggests a rather surprising viewpoint of this [spontaneous Rindler] radiation: it seems as though the detector is excited by swallowing part of the vacuum fluctuation of the field in the region of spacetime containing the detector. This liberates the correlated fluctuations in a noncausally related region of [Minkowski] spacetime to become a real particle." (Unruh and Wald, 1984, p. 1055)

Thus, according to this type of analysis, in Minkowski spacetime "a uniformly accel-

erated observer will "see" thermal radiation (Davies, 1975; Unruh, 1976) even though the field is in a vacuum state and, as far as inertial observers are concerned, no particles are detected whatever." (Birrell and Davies, 1982, p. 54.) On the other hand, it is conceded that, possibly, "basing one's treatment of these [pair creation *ex nihilo*] concepts on the considerations of accelerated observers is a fraud, because inertial observers occupy a special status in most physical theory." (*ibid.*, p. 55.)

Indeed, ever since the inception of classical mechanics by Newton, inertial frames have enjoyed a special status in physics. In particular, the laws set by Newton were presumed to hold true *only* in inertial frames, with *fictitious* forces (such as Coriolis forces, centrifugal forces, etc.) making their appearance only if a transition to accelerated frames were performed. This special status of inertial frames and inertial observers has been retained in special relativity – and in fact it has been implicitly transferred by Einstein also to general relativity (cf. Friedmann, 1983, Chapter V). Thus, the physical and conceptual problems encountered by conventional QFT in curved spacetime have roots that run all the way into the foundations of general relativity. They, therefore, deserve special attention, and the remainder of this section will be devoted to them.

The contention that *ex nihilo* particle production is an observer dependent phenomenon raises the question whether "observers" can transcend into "creators." In particular, the point of view that "Rindler particles," as well as all the other "particles" allegedly observable *only* by select classes of noninertial observers in Minkowski spacetime, actually exist, has to contend also with the following fundamental epistemological difficulty: if *something* is presumed to be literally created out of what is *physically* nothing, what is there to prevent the creation in this manner of *any* kind of object? In other words, what is there to prevent the creation of Rindler particles of *any* mass, spin, charge, etc.?

Despite its obvious nature, this question does not appear to have been raised and discussed in the literature. This might be because the Fock vacuum of such models is mentally identified with the "dressed vacuum" of quantum field theories for interacting fields which, on account of the customary renormalization procedures, is inhabited by an infinity of "bare" particles. However, the above arguments concerning Rindler particles apply to *free* quantum fields in Minkowski space, so that the "dressed vacuum" of quantum field theories for interacting fields has no bearing on them. Consequently, even if some form of *apparently* spontaneous particle production were observed in high-energy particle accelerators, that still would not represent a true test for the existence of Rindler particles.

Indeed, the hallmark of such a "particle" is not only its violation of *local* energy conservation, reflected by the prediction that it "liberates the correlated fluctuations in a noncausally related region of [Minkowski] spacetime to become a real particle" (Unruh and Wald, 1984, p. 1055), but also the *indiscriminate nature of Rindler particle production*, whereby each species of particle has an equal chance of being produced literally out of a vacuum, rather than as a result of an energy-conserving collision process. As a corollary of these manifestations, it might even appear that a Rindler perpetuum mobile could be created, whereby unlimited amounts of energy could be produced by the simple expedient of accelerating any material object, of however small rest mass, to sufficiently high acceleration in relation to any inertial frame – such as a (terrestrial) laboratory frame.

The only way out of this ultimate paradox is to suggest that in a particle accelerator

the energy for producing the Rindler particles, which an accelerated micro-detector would register, might come from the accelerator itself, since as it accelerates the micro-detector, "the work done by [it] ... supplies the missing energy that feeds into the [quantum] field via the quanta emitted from the detector." (Birrell and Davies, 1982, p. 55.) But in that case the whole Rindler particle production phenomenon is due to the fact that an open system, capable of receiving energy from the outside (i.e., from the accelerator), was treated as a closed system – whereas the physically *correct* treatment should have incorporated from the outset all the quantum fields created by the accelerator during the acceleration process. In that case the same argument cannot be applied to a similar micro-detector in *free* fall[4], despite all the manifest formal analogies between Rindler particle production and the particle creation *ex nihilo*, which are very much stressed in some of the literature on the subject. Indeed, if the strong equivalence principle of general relativity is correct, then *"observers" in free fall are truly inertial* – and as such they are not to be viewed as being accelerated by an external gravitational field, as is the case in Newtonian mechanics.

On the other hand, if it is assumed that a freely falling observer in curved spacetime is, contrary to Einstein's point of view, an accelerated observer, the question arises: in relation to what is he accelerated? The only *mathematical* answer to that question that can be found in the literature is that, in the case of curved spacetimes that are *asymptotically* flat, an observer in free fall is accelerated in relation to an observer in the *fictional* flat spacetime that asymptotically merges with the considered curved spacetime. However, since no acceptable cosmological models of our universe as a whole are asymptotically flat, even such an *ad hoc* "solution" does not provide a truly satisfactory answer.

These fundamental difficulties of the conventional framework for quantum fields in curved spacetime extend also to its formulation of the vacuum state. Indeed, it is again acknowledged in standard literature on the subject that "as far as Minkowski space is concerned, the [Fock] vacuum is a strong candidate for the 'correct' or 'physical' vacuum – the experiences of the accelerated observers being 'distorted' by the effects of their non-uniform motion. The trouble is that when gravitational fields are present, inertial observers become free-falling observers, and in general no two free-falling detectors will agree on a choice of vacuum." (Birrell and Davies, 1982, p. 55.)

We can infer from all this that the above mentioned "lack of agreement" between various free-falling detectors is due to the fact that in CGR all inertial observers are *local* observers, whereas the only "vacuum" considered in conventional quantum fields in curved spacetime is a *global* vacuum. Hence, as part of the solution to all of the above fundamental inconsistencies, we shall offer from Chapter 5 onwards a quantum geometry framework for QFT based on fibre bundles whose fibres are Fock spaces, so that *all* Fock vacuum states will be local rather than global.

1.6. From Canonical Quantum Gravity to Superstrings

The opinion that Einstein's theory of gravity has to be quantized is very widely held, but there are exceptions: Møller (1952) and Rosenfeld (1957) have argued that the gravitational

[4] Cf. Sec. 2.6 for a discussion of the geodesic postulate and of the strong equivalence principle governing all free-fall conditions in CGR.

field should not be quantized at all, and more recently arguments to the same effect have been presented in a monograph which ends with the following conclusion:

"The limitations on measurement of gravitational fields are, due to the geometrical character of gravity, necessarily limitations on the size of spacetime regions over which the Bohr-Rosenfeld quantum measurement procedure is, in principle, feasible. One way of expressing this fact is to say that supplementary to the well-known nonlocalizability of classical, i.e., non-quantized gravitational fields (making the $L_0 \to 0$ limit for lengths L_0 [of test bodies] physically senseless), there exists a finite limit ℓ_P [i.e., the Planck length] on the localizability of quantized gravitational fields. This is owed to the fact that localizability is inconsistent with Heisenberg's uncertainty principle taken in conjunction with the basic tenets of general relativity." (von Borzeszkowski and Treder, 1988, p. 100.)

Whereas an inconsistency between Heisenberg's uncertainty principle and such basic tenets of general relativity as the general covariance and strong equivalence principles is indeed in evidence if classical Lorentzian geometries are adopted for the description of spacetime, we shall demonstrate in Chapter 8 that this is no longer the case if appropriate *quantum* geometries are adopted instead. Hence, the simple and direct argument in favor of quantizing gravity that was given by Dirac remains in effect: "There is no experimental evidence for the quantization of the gravitational field, but we believe quantization should apply to all fields of physics. They all interact with one another, and it is difficult to see how some could be quantized and others not." (Dirac, 1968, p. 539.)

Dirac's (1950, 1958, 1959) pioneering work on Hamiltonian dynamics with constraints had prepared the ground for the canonical quantization of gravity, but Dirac himself did not further pursue this quantization since he eventually reached the following conclusion: "There does not exist any general method for handling [in quantum gravity] quadratic quantities in the δ-function, free from inconsistencies. ... The problem of the quantization of the gravitational field is thus left in a rather uncertain state. If one accepts Schwinger's plausible [but nonrigorous] methods, the problem is solved. But one cannot be happy with such methods without having a reliable procedure for handling quadratic expressions in the δ-function." (Dirac, 1968, p. 543.)

The origin of some of these and other inconsistencies can be traced to the presence of constraints in the canonical formulation of gravity. Thus, in trying to adapt the *non*-relativistic canonical quantization scheme to the general relativistic regime, it has been postulated that "in the quantum version of a general-relativistic theory only observables should play a role." On the other hand, in CGR "only first-class variables are observable" (Bergmann and Komar, 1980, p. 243) – where, in accordance with Dirac's (1959, 1964) classification, a first-class canonical variable is defined as one whose Poisson brackets with the Hamiltonian constraints vanish identically. However, as pointed out by Bergmann and Komar (1980) in Sec. 5 of their review paper on the subject,[5] such a definition removes the status of "observable" from the most basic of Einstein's actually observable quantities in CGR, namely from the metric tensor. To rectify this glaring inconsistency with orthodox

[5] Cf. also Bergmann's articles in (Bergmann and de Sabbata, 1986) and (de Sabbata and Melnikov, 1988). In Sec. 2.2 we shall discuss a concept of "metrization" of bundles of consisting of frames of reference, which can be described in operational terms that are physically tantamount to measurements of classical metrics.

theory, the artificial concept of "quasiobservable" has to be introduced, so that the non-relativistic position operators can be then reinstated at least to the partial status of "quasiobservables," and so that the conjecture can be then advanced that in canonical gravity the twelve canonical variables are "quasiobservable."

In light of the renewed interest in canonical quantum gravity due to the discovery of new variables (Ashtekar, 1991), the question has to be therefore asked as to whether some new kind of "observables" in the sense of "smooth functions on the phase space of the theory [which, if physical] have vanishing Poisson brackets with the constraints" (Rovelli, 1991b, p. 301) might exist in CGR, despite the fact that in the case of "the vacuum Einstein equations in the spatially compact case, *not a single physical observable is known explicitly as a function of phase space variables.*" (Smolin, 1991, p. 447.) This question is especially pertinent in light of the fact that the italicized part of this statement represents a reiteration of the observation that "one obvious difficulty with this [kind of] approach is that thus far no one has been able to discover a single classical observable" (Bergmann and Komar, 1980, p. 246), which would be represented by a first class variable on the phase space of a CGR theory. Since the latter statement was actually made more than a decade prior to the previous one, this lack of explicit examples is obviously not due to the lack of serious attempts at trying to find such "physical observables." Rather, it reflects some deeper features of general relativistic theories, intimately related to their underlying physical nature.

Of course, the motivation behind the renewed search for some "physical observables" invariant under the diffeomorphism group *Diff***M** of a Lorentzian spacetime manifold **M** does not reside in the interpretation of CGR, but in their conceived usefulness to the new approach to the canonical quantization procedure based on "loop representations." These types of approaches are based on purported "quantum observables" represented by "diffeomorphism-invariant operators", for which, however, "the problem is that we do not know how to make a correspondence between any of them and the classical diffeomorphism-invariant observables." (Smolin, 1991, p. 468.) On the other hand, the *true* observables of CGR, which have served Einstein (1916) in the prediction of *all observable* effects in CGR, are *not* the elements of the Lorentzian spacetime manifold, *nor* its differential-manifold structure, and *generically not* even the holonomic metric components $g_{\mu\nu}$. As discussed by M. Friedman (1983), these are all indispensable *mathematical* objects, reflecting the "style" in which CGR is formulated, but none of them have any direct physical significance. Rather, the basic observables in CGR are the *non*holonomic metric components g_{ij} with respect to classical *local* frames, the components of various tensors, such as the stress-energy, angular momentum, matter and nongravitational radiation fields with respect to the same frames, and the *relative* positions of the test particles which are in the immediate neighborhoods of the locations of such frames – with those in free-fall playing an especially important role as providers of the best-known observable predictions in CGR that are given a prominent place in all textbooks on general relativity.

An entirely different type of approach to the quantization of gravity (DeWitt, 1967) has become known under the name of covariant quantum gravity. B. S. DeWitt, who has made fundamental contributions to both the canonical and the "covariant" approach, has remarked in one of his key 1967 papers on the subject that "... no rigorous mathematical link has thus far been established between the canonical and covariant theories. In the case

of infinite worlds it is believed that the two theories are merely two versions of the same theory, expressed in different languages, but no one knows for sure." In essence, DeWitt's 1967 assessment has remained valid to the present day.

The "covariant" approach to quantum gravity assumes an already given Lorentzian manifold, so that the components of the quantum metric field can be then viewed as equal to a sum whose first term is a "classical" Lorentzian metric on that manifold (called in that context the "background metric"); whereas the second term is deemed to represent a "quantum correction," and is formally viewed as operator-valued function on the given manifold **M**. Since such a decomposition is obviously not invariant under the diffeomorphism group ***Diff*M**, covariant quantum gravity is actually not general relativistically covariant in the modern sense (discussed in Sec. 2.2). Nevertheless, in the 1970s this approach, with a choice of background metric equal to the Minkowski metric, enjoyed greater popularity than the canonical approach, since general coordinate invariance could be then *formally* embedded into the framework via a non-Abelian gauge group, so that the *formal* perturbative techniques based on the Feynman rules developed for Yang-Mills theories could be applied to it (Duff, 1975, 1981; van Nieuwenhuizen, 1977).

Many questions could be raised regarding the physical justifiability as well as the mathematical validity of this approach. However, from the conventional point of view its fatal fault lies in the nonrenormalizability of the terms in its formal perturbation expansion, so that the following conclusion was ultimately reached: "The failure to combine the particle physics version of quantum theory and general relativity poses a fundamental problem, since gravitation undeniably exists as a force in nature. Either our quantum theory must be modified, or other gravitational models should be considered, or we must leave gravitation unquantized, which might be inconsistent according to a Bohr argument." (van Nieuwenhuizen, 1977, p. 24).

The later work of Goroff and Sagnotti (1986) on two-loop formal perturbative expansions in pure quantum gravity confirmed the impossibility of arriving at even formally renormalizable expressions. In the meantime, hopes were entertained that supergravity (Wess and Bagger, 1983; West, 1986) might be able to supply models constructed in the same vein, and which would be renormalizable in the conventional sense. However, those expectations were not fulfilled (cf. Ashtekar, 1991a, p. 7).

For these reasons, during the last decade most of the hopes of researchers in quantum gravity were pinned on superstring theory. This was due to the fact that, at the level of formal perturbation theory, the vertices of diagrams in string theory contain exponential nonlocal factors that cause loops to converge in the Euclidean regime. However, it turned out that superstring perturbation theory encounters a breakdown of (its originally presumed) uniqueness after the compactification to four spacetime dimensions is carried out (Narain, 1986). Moreover, it was ultimately proven that its perturbation series not only is *not* convergent, but it is not even Borel summable (Gross and Perival, 1988).

In view of these and other problems, by the end of the 1980s, the following essential points were made in a review of quantum gravity, written by an advocate of string theory: "It is true that we are not (yet) able to address the physically more interesting questions in quantum gravity. But this is mainly due to lack of technique (and probably also lack of some deep understanding of the fundamental physical principles)." (Alvarez, 1989, p. 602.)

On the other hand, the route for eventually arriving at such a "deep understanding of fundamental physical principles" cannot be supplied by string theory itself, since there is a huge methodological gap between the manner in which string theory originally emerged from the Veneziano model in the late 1960s, and was subsequently developed in the course of the 1970s and 1980s, and the manner in which classical general relativistic gravity theory was created and developed by Einstein. The existence of this gap is actually pointed out in one of the main textbooks on string theory:

"For a theory that makes the claim of providing a unifying framework for all physical laws, it is the supreme irony that [superstring] theory itself appears so disunited, ... [with] the fundamental physical and geometric principles that lie at [its] foundation... still unknown. ... By contrast, when Einstein first discovered general relativity, he started with physical principles, such as the equivalence principle, and formulated them in the language of general covariance." (Kaku, 1988, pp. viii, 5-6.)

This monograph is therefore devoted to arriving at an understanding, at a physical as well as mathematical level, of those fundamental principles in general relativity and quantum theory that can lead to a purely geometric framework for their unification. This framework will be based on a systematic but very careful extrapolation to the quantum regime of the principles enunciated by Einstein in developing classical general relativity, and of an extrapolation to the general relativistic regime, carried out in the same spirit, of the principles enunciated by Bohr, Born, Dirac, Heisenberg, and other well-known founders of quantum mechanics as they developed fundamental theoretical results that have withstood the test of time. For, as will be seen in the subsequent chapters of this monograph, the mathematical tools that were not available to them when they formulated those principles have been made available by developments in the theories of gauge fields, gauge groups, coherent states, fibre and superfibre bundles, supermanifolds, etc., carried out during the past couple of decades.

Chapter 2

Classical Frame Bundles in General Relativity

Fibre bundle theory has become a standard tool in contemporary mathematical formulations of differential geometry. It emerged from the work of E. Cartan (1923-35) on connections and "moving frames," which was originally developed by him with general relativity in mind (cf. Cartan, 1985). The fibre-theoretical formulation of the geometries based on the Lorentzian manifolds used in classical general relativity (CGR) provides not only mathematical advantages, but it also contributes to a better understanding of fundamental general relativistic principles, such as those of general covariance and of strong equivalence. Furthermore, it leads to the clarification of the role and nature of gauge invariance in CGR, which at the present time is still a matter of dispute between various schools of thought. The connection forms on principal bundles of classical local frames, required in the fibre-theoretical formulation of CGR, determines the parallel transport in associated bundles which, in conjunction with the geodesic postulate, is essential for arriving at all the main predictions of CGR. Hence, their introduction also provides a bridge between CGR and the theories of gauge fields developed during the last two decades, which is essential to the development of the geometric framework for quantum general relativity (QGR) presented in the subsequent six chapters.

2.1. General Covariance under Coordinate Transformations

In modern differential geometry a (*differentiable* or *differential*) *manifold* is defined as a topological space[1] which can be covered with a family of charts that constitute an atlas (cf. Spivak, 1979; Isham, 1989; Nakahara, 1990). Thus, if $\mathbf{M}$ is a 4-dimensional manifold used in CGR, then $\mathbf{M}$ can be covered with a family of open sets M_α, so that there is a homeomorphism[2] between each M_α and some open subset O_α in the Euclidean space $\mathbf{R}^4$,

$$\phi_\alpha : x \mapsto (x^0,x^1,x^2,x^3) \in O_\alpha \subset \mathbf{R}^4 , \qquad x \in M_\alpha \subset \mathbf{M} , \tag{1.1a}$$

which assigns coordinates x^μ, $\mu = 0,1,2,3$, to each $x \in M_\alpha$; moreover, the resulting family of charts (M_α,ϕ_α) constitutes an atlas in the sense that it has the following basic property: if the domains of definition M_α and M_β of any two charts intersect, so that to each point $x \in M_\alpha \cap M_\beta$ is assigned two sets of coordinates, namely those in (1.1a) as well as those assigned by some other chart (M_β,ϕ_β) by means of a homeomorphism

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[1] A *topological space* is a set X which contains a family of subsets, called *open* sets, to which X itself as well as the null set $\varnothing$ belong; moreover, that family has to be such that the union of a finite or infinite number of open sets, as well as the intersection of any *finite* number of open sets, is again an open set.

[2] A *homeomorphism* between two open sets O_1 and O_2, which might in general belong to different topological spaces, is a one-to-one continuous map of O_1 onto O_2 which takes open subsets of O_1 into opens subsets of O_2.

$$\phi_\beta : x \mapsto (x^{0'},x^{1'},x^{2'},x^{3'}) \in O_\beta \subset \mathbf{R}^4 , \qquad x \in M_\beta \subset \mathbf{M} , \tag{1.1b}$$

then the *coordinate-transformation maps* between those two charts,

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(M_\alpha \cap M_\beta) \to \phi_\beta(M_\alpha \cap M_\beta) \subset O_\beta , \tag{1.2a}$$

represent *smooth* mappings; in other words, the four functions which express the new coordinates $x^{\nu'}$, $\nu = 0,1,2,3$, in terms of the old coordinates x^μ, $\mu = 0,1,2,3$, and which give rise to the map in (1.2a), so that

$$\phi_\beta \circ \phi_\alpha^{-1} : (x^0,x^1,x^2,x^3) \mapsto (x^{0'},x^{1'},x^{2'},x^{3'}) , \tag{1.2b}$$

are infinitely differentiable, namely they possess partial derivatives of all orders in all their four variables[3].

For the understanding of general relativity as a *physical* framework, it is essential to realize that the role of coordinates in the manifolds used in CGR is purely mathematical, since they are essential to the definition of differential geometric concepts that involve "smoothness," but generally the coordinates in (1.1) do not play any physical role. The opposite is the case with all the Minkowski coordinates in special relativity, since a direct physical meaning is imparted to them by the operational procedures discussed in Sec. 1.1.

This distinction was not fully reflected by Einstein's (1916) original formulation of the principle of general covariance, cited in Sec. 1.1, so that it exposed it to the criticism of Kretschmann (1917) to the effect that a principle based on such a formulation was of no physical significance. Indeed, saying that "the general laws of nature are to be expressed by equations which hold good for all systems of co-ordinates" (Einstein, 1916, p. 117) is true in CGR by virtue of the mere fact that those laws are formulated within a differential geometric framework, and does not represent a *physical* postulate that could be experimentally tested, as is the case with special relativistic invariance under Lorentz transformations.

This fact becomes especially evident within modern formulations of differential geometry, since such formulations are basically coordinate-independent once the concept of "smoothness" has been introduced as follows: a real-valued function $f(y)$ defined at all the points y in a neighborhood $\mathcal{N}_x$ of some $x \in \mathbf{M}$ is said to be smooth if it is such that, when coordinates are assigned to all $y \in M_\alpha \cap \mathcal{N}_x$ by some chart (M_α,ϕ_α), the resulting function of four real coordinate variables is smooth; due to the smoothness of the coordinate transformation maps in (1.2a) and to the chain rule in the calculus of several variables, this statement remains true upon making the transition, by means of (1.2b), to any other chart (M_β,ϕ_β). Similarly, a curve $\gamma = \{x(t) | a \le t \le b\}$ passing through the point $x \in \mathbf{M}$ is said to be smooth if, in any chart whose domain contains some neighborhood $\mathcal{N}_x$ of x, the coordinates $x^\mu(t)$, $\mu = 0,1,2,3$, are smooth functions of the parameter $t \in [a,b]$.

Once the above definitions are given, a *vector* tangent to a smooth curve γ at a point $x \in \gamma$ can be defined by choosing a parametrization $x(t)$ of γ such that $x(0) = x$, and then mathematically identifying such a vector with the map

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[3] This definition corresponds to a C^∞ manifold. It can be relaxed to that of various C^n manifolds (Hawking and Ellis, 1973), but that does not contribute substantially new results to CGR.

$$\boldsymbol{X} : f \mapsto df(x(t))/dt\big|_{t=0} \in \mathbf{R}^1 , \tag{1.3}$$

which assigns a real number to each smooth real-valued function f defined on a neighborhood of $x \in \mathbf{M}$. The linear space $T_x\mathbf{M}$, consisting of all the tangent vectors $\boldsymbol{X}$ determined by this coordinate-independent construction for *all* smooth curves passing through a given $x \in \mathbf{M}$, is then called the *tangent space* of $\mathbf{M}$ above the point x.

The *cotangent space* $T_x{}^*\mathbf{M}$ above the same x can be then defined as the algebraic dual (Berberian, 1992) of $T_x\mathbf{M}$, i.e., as the linear space consisting of all real-valued linear functionals $\boldsymbol{\omega}$ over $T_x\mathbf{M}$. By a *covector* at x one then simply means any linear functional $\boldsymbol{\omega} \in T_x{}^*\mathbf{M}$ from this cotangent space. Furthermore, if for each smooth real-valued function f defined on some neighborhood of x one introduces the following linear functional,

$$\boldsymbol{df} : \boldsymbol{X} \mapsto \boldsymbol{X}f \in \mathbf{R}^1 , \qquad \boldsymbol{X} \in T_x\mathbf{M} , \tag{1.4}$$

then it is readily seen that the cotangent space $T_x{}^*\mathbf{M}$ defined in this manner coincides with the linear space of all of the above functionals that correspond to functions f which are smooth in a neighborhood of x.

The algebraic tensor product of r copies of $T_x\mathbf{M}$ and s copies of $T_x{}^*\mathbf{M}$, namely the linear space defined as[4]

$$T_x^{r,s}\mathbf{M} = \left[\bigotimes_{m=1}^{r}\left(T_x\mathbf{M}\right)_m\right] \otimes \left[\bigotimes_{n=1}^{s}\left(T_x^*\mathbf{M}\right)_n\right] , \tag{1.5}$$

is called the space of *tensors of type* (r,s) above $x \in \mathbf{M}$. Any equation that holds amongst tensor-valued functions on $\mathbf{M}$ (i.e., tensor fields) defined in this coordinate-independent manner obviously "holds good for all systems of coordinates." Moreover, the covariant transformation rules "with respect to any substitutions whatever" of coordinates, to which Einstein referred in his original formulation of the principle of general covariance, can be shown to be mere consequences of the above coordinate-independent definitions.

To see that, let us introduce in the tangent space $T_x\mathbf{M}$ above $x \in \mathbf{M}$ the set of vectors

$$\left\{\boldsymbol{\partial}_\mu \mid \mu = 0,1,2,3\right\} , \qquad \boldsymbol{\partial}_\mu = \partial/\partial x^\mu := \left(\partial/\partial y^\mu\right)_{y=x} , \tag{1.6}$$

that are tangent to the four coordinate lines $\{y \in M_\alpha \mid y^\mu \in \mathbf{R}, y^\nu = x^\nu, \nu \neq \mu\}$ of some chart (M_α, ϕ_α) around x. Upon labelling the points $x(t)$ of the curve γ in (1.3) with the coordinates supplied by this chart, and then using the chain rule for the differentiation of $f(x(t))$ with respect to t, we can expand any vector $\boldsymbol{X} \in T_x\mathbf{M}$ in this vector basis as follows[5]:

$$\boldsymbol{X} = X^\mu \boldsymbol{\partial}_\mu \in T_x\mathbf{M} , \qquad X^\mu = \left(dx^\mu/dt\right)_{t=0} \tag{1.7}$$

Consequently, the set in (1.6) constitutes a vector basis in $T_x\mathbf{M}$, called a *holonomic linear*

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[4] In modern multilinear algebra this construction is carried out without using any bases in the constituent vector and covector spaces – cf., e.g., Chapter 13 of (Berberian, 1992).

[5] Throughout this monograph we shall use the Einstein convention of summation over all values assumed by each pair of *repeated* contravariant and covariant (i.e., upper and lower) indices.

frame. This shows that the tangent space $T_x\mathbf{M}$ is 4-dimensional. We also note that if γ is the worldline of a massive particle, and t is its proper time in CGR, then the vector $\boldsymbol{X}$ in (1.7) represents the 4-velocity $\boldsymbol{v}$ of that particle, so that the above components of that vector coincide with 4-velocity components with respect to that linear frame.

Let us now consider the coordinates y^μ used in (1.6), and adopt for any fixed $\mu = 0,...,3$ the special choice $f(y) = y^\mu, y \in M_\alpha$, for the function in (1.4). We then see that, in accordance with (1.6) and (1.7), the linear map in (1.4) assumes the form

$$\boldsymbol{dx}^\mu : \boldsymbol{X} \mapsto \left(\boldsymbol{X}y^\mu\right)_{y=x} = X^\mu \quad , \qquad \boldsymbol{X} \in T_x\mathbf{M} \quad . \tag{1.8}$$

From this we can immediately deduce that any $\boldsymbol{\omega} \in T_x{}^*\mathbf{M}$ can be expanded as follows:

$$\boldsymbol{\omega} = \omega_\mu \, \boldsymbol{dx}^\mu \quad , \qquad \omega_\mu = \boldsymbol{\omega}\left(\boldsymbol{\partial}_\mu\right) \quad , \qquad \boldsymbol{dx}^\mu(\boldsymbol{\partial}_\nu) = \delta^\mu_\nu \quad . \tag{1.9}$$

Hence $\{\boldsymbol{dx}^\mu \mid \mu = 0,...,3\}$ constitutes in $T_x{}^*\mathbf{M}$ a covector basis that is dual to the vector basis in (1.6), and is called a *holonomic linear coframe* in $T_x{}^*\mathbf{M}$. From (1.7) and (1.9) we then deduce, by expanding the same vector or covector in the vector and covector bases, respectively, provided by different coordinate charts, that with a change of coordinate chart, such as that provided by the coordinate transformation map in (1.2), we have

$$\boldsymbol{\partial}'_{\mu'} = A_{\mu'}{}^{\mu} \boldsymbol{\partial}_\mu \quad , \qquad \boldsymbol{\partial}'_{\mu'} = \partial / \partial x'^{\mu'} \quad , \qquad A_{\mu'}{}^{\mu} = \left(\partial y^\mu / \partial y'^{\mu'}\right)_{y=x} \quad , \tag{1.10a}$$

$$dx'^{\nu'} = A^{\nu'}{}_{\nu} \, dx^\nu, \qquad A^{\nu'}{}_{\nu} = \left(\partial y'^{\nu'} / \partial y^\nu\right)_{y=x} \tag{1.10b}$$

Furthermore, from the duality relation in (1.9), and from its counterpart in the bases for the primed coordinates, we immediately obtain:

$$A_\mu{}^\lambda A^\nu{}_\lambda = A_\lambda{}^\nu A^\lambda{}_\mu = \delta^\nu_\mu \tag{1.11}$$

According to the rules of multilinear algebra, any (r,s) tensor $\boldsymbol{T}$ from the tensor space depicted in (1.5) can be expanded as follows in terms of a basis constructed from the vector basis elements in (1.6), as well as from their dual covector basis elements:

$$\boldsymbol{T} = T^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s} \boldsymbol{\partial}_{\mu_1} \otimes \cdots \otimes \boldsymbol{\partial}_{\mu_r} \otimes \boldsymbol{dx}^{\nu_1} \otimes \cdots \otimes \boldsymbol{dx}^{\nu_s} \tag{1.12}$$

Since vectors and covectors emerge as special cases of $(1,0)$ and $(0,1)$ tensors, respectively, it immediately follows from (1.10) and (1.11) that the components of the tensor $\boldsymbol{T}$ in (1.12) transform according to the following well-known rule:

$$T'^{\mu'_1 \cdots \mu'_r}_{\nu'_1 \cdots \nu'_s} = A^{\mu'_1}_{\mu_1} \cdots A^{\mu'_r}_{\mu_r} A_{\nu'_1}{}^{\nu_1} \cdots A_{\nu'_s}{}^{\nu_s} \, T^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s} \tag{1.13}$$

In this manner the transformation laws for tensor components, which in the coordinate-dependent formulation of differential geometry originally used by Einstein (1916) had to be postulated, emerge now as straightforward consequences of the coordinate-independent definition of tensors. Thus, indeed, Einstein's original formulation of the principle of general covariance does not possess any physical content beyond the fact that he chose differential geometry as the mathematical medium for formulating CGR.

2.2. General Covariance and Classical Frame Bundles

As described in Sec. 1.1, the ambivalence surrounding the physical meaning of the original formulation of the the principle of general covariance has led to its being only casually mentioned, or even ignored, in some of the best known and most influential textbooks on general relativity. However, investigations into the Cauchy initial-value problem in CGR eventually led to the realization that solutions to that problem fall naturally into equivalence classes provided by Lorentzian manifolds related by diffeomorphisms (cf., e.g., Hawking and Ellis, 1973, p. 227). Thus, two CGR models formulated over the same manifold $\mathbf{M}$, or even over distinct manifolds $\mathbf{M}$ and $\mathbf{M}'$, turned out to be *physically* equivalent if there is a diffeomorphism[6] $\psi : \mathbf{M} \to \mathbf{M}'$, such that the Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$, the family $\{\boldsymbol{T}_\alpha\}$ of all (matter and radiation) tensor fields, and the family $\{x_\beta(t)\}$ of all possible test-particle worldlines in $\mathbf{M}$ are related by that diffeomorphism to the corresponding ones in $\mathbf{M}'$ as follows:

$$\boldsymbol{g}^{\mathrm{L}} \to \psi_* \boldsymbol{g}^{\mathrm{L}} , \quad \{\boldsymbol{T}_\alpha\} \to \{\psi_* \boldsymbol{T}_\alpha\} . \quad \{x_\beta(t)\} \to \{\psi(x_\beta(t))\} . \tag{2.1}$$

In the above first two mappings $\psi_* \boldsymbol{g}^{\mathrm{L}}$ and $\psi_* \boldsymbol{T}_\alpha$ denote the push-forwards under the diffeomorphism ψ, which we shall now define in a general context.

In the case of any smooth map ϕ from a manifold $\mathbf{M}$ into any other manifold $\mathbf{M}'$, the *push-forward* $\phi_* \boldsymbol{X} \in T_{\phi(x)}\mathbf{M}'$ of a vector $\boldsymbol{X} \in T_x\mathbf{M}$ can be defined, due to (1.3), by the following map acting on smooth functions f defined on a neighborhood of $\phi(x) \in \mathbf{M}'$,

$$\phi_* \boldsymbol{X} : f \mapsto df\big(\phi(x(t))\big)/dt\big|_{t=0} , \qquad x = x(0) . \tag{2.2}$$

Thus, if the vector $\boldsymbol{X}$ is visualized, in heuristic terms, as an "infinitesimal" oriented line segment joining the point $x \in \mathbf{M}$ to an "infinitesimally close" point $x + dx \in \mathbf{M}$, then its push-forward by ϕ can be visualized as the oriented line segment joining the point $\phi(x) \in \mathbf{M}'$ to $\phi(x + dx) \in \mathbf{M}'$ – i.e., as the outcome of "pushing forward" by means of the mapping ϕ all the points of such an "infinitesimal" line segment.

Similarly, the *pull-back* $\phi^* \boldsymbol{\omega}$ of a covector $\boldsymbol{\omega}$ in $T^*\mathbf{M}'$ is defined as the covector in $T^*\mathbf{M}$ determined, in accordance with the definition of covectors as linear functionals given in Sec. 2.1, by the following map:

$$\phi^* \boldsymbol{\omega} : \boldsymbol{X} \mapsto \boldsymbol{\omega}(\phi_* \boldsymbol{X}) \in \mathbf{R}^1 , \qquad \boldsymbol{\omega} \in T^*_{\phi(x)}\mathbf{M}' , \quad \phi^* \boldsymbol{\omega} \in T^*_x\mathbf{M} \tag{2.3}$$

In the case where ϕ is a diffeomorphism, so that its inverse ϕ^{-1} exists as a smooth map from $\mathbf{M}'$ onto $\mathbf{M}$, then the pull-back of a vector $\boldsymbol{X}$ in $T\mathbf{M}'$, as well as the push-forward of a covector $\boldsymbol{\omega}$ in $T^*\mathbf{M}$, can be also defined as follows:

$$\phi^* \boldsymbol{X} := (\phi^{-1})_* \boldsymbol{X} , \qquad \phi_* \boldsymbol{\omega} := (\phi^{-1})^* \boldsymbol{\omega} \tag{2.4}$$

Furthermore, in that case the definition of pull-backs and push-forwards can be extended to arbitrary tensors $\boldsymbol{T}$ over the two respective manifolds, either by using (1.5) and applying

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[6] A *diffeomorphism* $\psi : \mathbf{M} \to \mathbf{M}'$ is a one-to-one map between the two manifolds $\mathbf{M}$ and $\mathbf{M}'$, which is smooth, together with its inverse $\psi^{-1} : \mathbf{M}' \to \mathbf{M}$.

multilinear algebra techniques (Abraham and Marsden, 1978, p. 58), or by applying the (2.2)-(2.4) to the vectors and covectors in the expansion (1.12) for all charts in an atlas, and then verifying that the outcome is chart-independent on account of (1.13).

In recent times, some general relativists[7] have been identifying the principle of general covariance with the diffeomorphism invariance of CGR models. On the other hand, it still seems to be not generally realized that in 1952 Einstein *implicitly* clarified his ultimate views on general covariance in general relativity in the following passage, which appeared in the appendix to the fifteenth (expanded and final) edition of his well-known popular exposition of CGR: "On the basis of general relativity ... space as opposed to 'what fills space' ... has no separate existence ... If we imagine the gravitational field, *i.e.* the functions g_{ik}, to be removed, there does not remain a space of the type [of Minkowski space] but absolutely *nothing*, not even a 'topological space.' For the functions g_{ik} describe not only the field, but at the same time also the topological and metrical structural properties of the manifold. ... There is no such thing as empty space, *i.e.* space without a field. Space-time does not claim existence on its own, but only as a structural quality of the field." (Einstein, 1961, p. 155.)

A search into the historical origins of the above statement has recently led J. Stachel to the following reformulation of the epistemological ideas contained in the quotation in the fibre-theoretical language of modern differential geometry: "Although the points of a [classical spacetime] manifold are *mathematically* individuated by a coordinate system, they are not *physically* individuated before one has a metric tensor field on that manifold. ... [Thus], the points of the base manifold do *not* represent physical events, which are instead represented by mappings from a point on the cross section of the bundle into a point of the base manifold." (Ashtekar and Stachel, 1991, pp. 35-36.)

Stachel did not specify the exact nature of these bundles, but as we shall demonstrate, the *physical* idea underlying Einstein's final conceptualization of general covariance hinges on the concept of classical frame *above* a manifold, rather than on the concept of chart *within* a manifold. This implies that it is the cross-sections of frame bundles that are of paramount importance to the *physical* significance of general relativistic covariance. Consequently, as we shall see in this section, classical frame bundles provide the natural vehicle for embedding the diffeomorphism invariance of CGR models directly into the mathematical formalism, and thus satisfying *one* of the requirements of general covariance; moreover, as we shall see in the next section, the use of classical frame bundles also uncovers the presence of Lorentz-frame gauge invariance in CGR, thus bringing to the surface the *other* essential aspect of general covariance.

To develop the basic ideas of fibre-bundle theory in a physically transparent manner, and to the extent required by the formulation of CGR, let us consider the set

$$T\mathbf{M} = \bigcup_{x\in\mathbf{M}} T_x\mathbf{M} = \bigcup_{x\in\mathbf{M}} \{\boldsymbol{X} | \boldsymbol{X} \in T_x\mathbf{M}\} \quad , \tag{2.5a}$$

consisting of all vectors tangent to a given manifold $\mathbf{M}$. This set assumes the structure of a manifold if, for each chart (M_α,ϕ_α) assigned to $\mathbf{M}$ by (1.1a), we assign a corresponding

[7] Cf., e.g., the review articles by C. Rovelli, L. Smolin and others in (Ashtekar and Stachel, 1991).

chart (TM_α,ψ_α) to $T\mathbf{M}$, whereby TM_α is obtained by taking the union of all tangent spaces above points $x \in M_\alpha$, whereas the map ψ_α is given by

$$\psi_\alpha : \boldsymbol{X} \mapsto (x^0,x^1,x^2,x^3,X^0,X^1,X^2,X^3) \in O_\alpha \times \mathbf{R}^4 , \qquad \boldsymbol{X} \in T_x\mathbf{M} \tag{2.5b}$$

The corresponding sets of all cotangent vectors, and, in general, of (r,s)-tensors,

$$T^*\mathbf{M} = \bigcup_{x\in\mathbf{M}} T_x^*\mathbf{M} , \qquad T^{r,s}\mathbf{M} = \bigcup_{x\in\mathbf{M}} T_x^{r,s}\mathbf{M} , \tag{2.6}$$

can be transformed into manifolds in a similar manner. These manifolds are then called, respectively, the *tangent bundle* $T\mathbf{M}$, the *cotangent bundle* $T^*\mathbf{M}$, and the *(r,s)-tensor bundle* $T^{r,s}\mathbf{M}$ over the given 4-dimensional differential manifold $\mathbf{M}$.

Each tangent space $T_x\mathbf{M}$, cotangent space $T_x{}^*\mathbf{M}$, or, in general, tensor space (1.5), is called a *fibre* above $x \in \mathbf{M}$ when considered as a submanifold in the respective bundle. Since each one of these fibres is a linear space, all the above bundles are special cases of vector bundles. By counting the number of coordinates assigned to its points by the coordinate charts, it becomes obvious that, as a manifold, $T^{r,s}\mathbf{M}$ has the dimension $4 + 4^{r+s}$; therefore, as special cases corresponding to $(1,0)$ and $(0,1)$ tensor bundles, respectively, $T\mathbf{M}$ and $T^*\mathbf{M}$ are 8-dimensional manifolds.

After introducing the general concept of fibre bundle in the next section, we shall see that all the above vector bundles are instances of bundles associated with the principal bundle of all linear frames over the manifold $\mathbf{M}$. As a mathematical object, a *linear frame* $\boldsymbol{u}(x)$ above $x \in \mathbf{M}$ is an ordered set of four linearly independent vectors $\boldsymbol{e}_i(x)$, $i = 0,1,2,3$, that belong to $T_x\mathbf{M}$, so that in some chart (M_α,ϕ_α) containing a neighborhood of x we have

$$\boldsymbol{u}(x) = (\boldsymbol{e}_0(x),\boldsymbol{e}_1(x),\boldsymbol{e}_2(x),\boldsymbol{e}_3(x)) , \qquad \boldsymbol{e}_i(x) = \lambda_i{}^\mu(x)\partial_\mu \in T_x\mathbf{M} . \tag{2.7}$$

Upon introducing coordinates for all vectors in each linear frame within the union $GL\mathbf{M}$ of all such frames in the same manner as in (2.5), we obtain a 20-dimensional manifold, called the *general linear frame bundle*[8] over $\mathbf{M}$. A fibre of this bundle consists of all linear frames above a given point $x \in \mathbf{M}$; therefore, it does not constitute a linear space, so that the general linear frame bundle $GL\mathbf{M}$ is *not* a vector bundle. On the other hand, on account of (1.7) and the fact that the linear frame in (2.7) constitutes a vector basis in the tangent space $T_x\mathbf{M}$, we can write

$$\boldsymbol{X} = X^\mu\partial_\mu = X^i\boldsymbol{e}_i \in T_x\mathbf{M} , \qquad X^\mu = X^i\lambda_i{}^\mu(x) , \tag{2.8}$$

so that the effect of any operations executed on linear frames in $GL\mathbf{M}$ can be transferred to the vectors in the tangent bundle $T\mathbf{M}$.

In view of (1.12), in order to extend this feature to tensor bundles $T^{r,s}\mathbf{M}$ in general, we have to introduce for each linear frame $\boldsymbol{u}(x)$ in (2.7) its dual frame in $T_x{}^*\mathbf{M}$:

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[8] Kobayashi and Nomizu (1963), Spivak (1979), as well as other authors, denote the (general) linear frame bundle $GL\mathbf{M}$ by $L(\mathbf{M})$. However, later in this section we shall introduce a subbundle of this general linear frame bundle, which consists of frames that are orthonormal in a Lorentzian metric. In CGR such frames represent local Lorentz frames, so that we shall reserve the symbol $L\mathbf{M}$ for bundles consisting of them.

$$\boldsymbol{u}^*(x) = (\boldsymbol{\theta}^0(x), \boldsymbol{\theta}^1(x), \boldsymbol{\theta}^2(x), \boldsymbol{\theta}^3(x)) \ , \qquad \boldsymbol{\theta}^i(x) = \lambda^i{}_\mu(x)\boldsymbol{dx}^\mu \in T_x^*\mathbf{M} \ . \tag{2.9}$$

The *general linear coframe bundle* $GL^*\mathbf{M}$ can be then defined by means of the union of all such coframes in the same general manner in which $GL\mathbf{M}$ was defined. Moreover, according to the algebraic definition of duality, we must have

$$\boldsymbol{\theta}^j(\boldsymbol{e}_i) = \delta_i^j \ , \qquad \boldsymbol{dx}^\nu(\boldsymbol{\partial}_\mu) = \delta_\mu^\nu \ , \qquad i,j,\mu,\nu = 0,1,2,3 \ . \tag{2.10}$$

Consequently, the coefficients relating the (in general non-holonomic) linear frames and coframes in (2.8) and (2.9) to their respective holonomic counterparts, given by tangent and cotangent vectors to coordinate lines, have to satisfy the following relations:

$$\lambda_i{}^\mu(x)\lambda^j{}_\mu(x) = \delta_i^j \ , \qquad \lambda^i{}_\mu(x)\lambda_i{}^\nu(x) = \delta_\mu^\nu \ . \tag{2.11}$$

Thus, these two principal bundles are closely interrelated, so that by (2.11) any operation performed on a linear frame can be related to a corresponding operation performed on its dual coframe, and vice versa. On account of (2.8)-(2.11), we can generalize (1.12) into the relation

$$\boldsymbol{T} = T^{i_1 \cdots i_r}_{j_1 \cdots j_s}\, \boldsymbol{e}_{i_1} \otimes \cdots \otimes \boldsymbol{e}_{i_r} \otimes \boldsymbol{\theta}^{j_1} \otimes \cdots \otimes \boldsymbol{\theta}^{j_s} \ , \tag{2.12}$$

so that such operations can be transferred to the elements of the tensor bundles $T^{r,s}\mathbf{M}$. Hence, in Sec. 2.3 these bundles will be seen to be associated to $GL\mathbf{M}$ or to $GL^*\mathbf{M}$.

On the other hand, according to the earlier quotation from (Einstein, 1961, p. 155), these fibre theoretical constructs cannot be related to a general relativistic classical spacetime in the absence of a Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ on $\mathbf{M}$. Mathematically, such a metric is determined by a symmetric *(0,2)* tensor field over $\mathbf{M}$, which assigns to each point $x \in \mathbf{M}$ a symmetric tensor

$$\boldsymbol{g}^{\mathrm{L}}(x) = g_{\mu\nu}(x)\boldsymbol{dx}^\mu \otimes \boldsymbol{dx}^\nu \in T_x^{0,2}\mathbf{M} \ , \qquad g_{\mu\nu}(x) = g_{\nu\mu}(x) \ , \tag{2.13a}$$

of signature (+1,–1,–1,–1). This last statement means that at each $x \in \mathbf{M}$ there are *orthonormal* linear coframes $\boldsymbol{u}^*(x)$ (often called *cotetrads*) for which the coefficients in (2.7) for the corresponding linear frames $\boldsymbol{u}(x)$ (often called *tetrads*) satisfy the relations,

$$g_{\mu\nu}(x)\lambda_i{}^\mu(x)\lambda_j{}^\nu(x) \ = \ \eta_{ij} \ = \ \mathrm{diag}(+1,-1,-1,-1) \ , \tag{2.13b}$$

where the 4×4 matrix $\boldsymbol{\eta}$ with elements η_{ij} has the same form as the one for the special relativitistic Minkowski metric in $\mathbf{R}^4$, i.e., $\eta_{ij} = 0$ for all $i \neq j$; whereas the entries $\eta_{00} = +1$ and $\eta_{11} = \eta_{22} = \eta_{33} = -1$ appear along its diagonal. In turn, this implies that in any given chart (M_α, ϕ_α) one can choose a set of solutions for the coefficients in (2.11) and (2.13b) that is smooth over M_α, so that such solutions supply a family of four orthonormal covector fields $\{\boldsymbol{\theta}^i(x)\}$, $x \in M_\alpha$, usually called in physics literature a *vierbein* or a *cotetrad field* over M_α – with the corresponding family of orthonormal vector fields $\{\boldsymbol{e}_i(x)\}$, $x \in M_\alpha$, being called a tetrad field over M_α.

The presence of a Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ on $\mathbf{M}$ enables the *reduction* of the general linear frame bundle $GL\mathbf{M}$ to the *Lorentzian frame bundle* $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ over $\mathbf{M}$. This reduction

can be performed by selecting from the frame bundle $GL\mathbf{M}$ only those linear frames $\boldsymbol{u} = \{\boldsymbol{e}_0,\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ which are orthonormal in that metric, and for which $\boldsymbol{e}_0$ is timelike[9]. In turn, by selecting from the general linear coframe bundle $GL^*\mathbf{M}$ all the coframes dual to orthonormal frames in $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ we arrive at the *Lorentzian coframe bundle* $L^*\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ over $\mathbf{M}$, in terms of whose elements the Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ can be expanded as follows:

$$\boldsymbol{g}^{\mathrm{L}} = \eta_{ij}\,\boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j\,, \qquad \boldsymbol{u}^* = (\boldsymbol{\theta}^0,\boldsymbol{\theta}^1,\boldsymbol{\theta}^2,\boldsymbol{\theta}^3) \in L^*\mathbf{M}(\boldsymbol{g}^{\mathrm{L}}) \tag{2.14}$$

The above reduction procedure can be imparted with an operational meaning in a physical setting in which each linear frame $\boldsymbol{u}(x) = \{\boldsymbol{e}_0(x),\boldsymbol{e}_1(x),\boldsymbol{e}_2(x),\boldsymbol{e}_3(x)\} \in GL\mathbf{M}$ is envisaged as being physically realized by a *classical local frame* consisting of three "rigid rods," labeled with the symbols $\boldsymbol{e}_1(x),\boldsymbol{e}_2(x),\boldsymbol{e}_3(x)$ and arranged in the spatial configuration of a 3-frame, and by a set of "standard clocks," labeled with the symbol $\boldsymbol{e}_0(x)$, and attached to various locations around the origin of that spatial 3-frame. However, according to the earlier quotation dealing with Einstein's final thoughts on the epistemic nature of the concept of spacetime in CGR, neither $\mathbf{M}$ nor $GL\mathbf{M}$ possess a general relativistic significance unless a Lorentzian metric field $\boldsymbol{g}^{\mathrm{L}}$ is specified in $\mathbf{M}$. Indeed, the *existence* of such a metric is a feature of the physical reality around us, rather than being a matter of mere choice left to the discretion of an observer of a given physical occurrence.

In the presence of such a metric, one can envisage a *metrization* of $GL\mathbf{M}$, which produces the above described reduction of the general linear bundle $GL\mathbf{M}$ to the Lorentzian frame bundle $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ in a physically rather than mathematically implemented manner. In the *physically existing* metric in the spacetime described by $\mathbf{M}$, this metrization is achieved by identifying each linear frame $\{\boldsymbol{e}_0(x),\boldsymbol{e}_1(x),\boldsymbol{e}_2(x),\boldsymbol{e}_3(x)\} \in GL\mathbf{M}$ whose component $\boldsymbol{e}_0(x)$ is timelike with a classical local frame of reference, whose four axes are metrized, so that they can serve as metric standards for spatio-temporal measurements. Thus, according to Einstein's (1905, 1916) conceptualization, carried out at the macrolevel, this would be achieved by labeling sets consisting of three "rigid rods" that are joined together with triples $\{\boldsymbol{e}_1(x),\boldsymbol{e}_2(x),\boldsymbol{e}_3(x)\}$, and attaching to them "standard clocks" labeled by the vectors $\boldsymbol{e}_0(x)$. One can then *physically* reduce the bundle $GL\mathbf{M}$ by means of operational verifications aimed at establishing which of the physical representatives of $GL\mathbf{M}$ are actually local orthonormal frames. Thus, in accordance with Einstein's (1916) operational procedures, light signals are to be used to check which ones of the various ordered sets of "rigid rods," labeled by the triples $\{\boldsymbol{e}_1(x),\boldsymbol{e}_2(x),\boldsymbol{e}_3(x)\}$, actually consist of rods that are of unit length, and which amongst these latter sets consist of unit rods which are at right angles to each other; furthermore, one has to subsequently verify, also by means of light signals, which ones of the neighboring "standard clocks" attached to a given set $\{\boldsymbol{e}_1(x),\boldsymbol{e}_2(x),\boldsymbol{e}_3(x)\}$ of "rigid rods," and mathematically labeled by the vector $\boldsymbol{e}_0(x)$, are actually synchronized.

On the other hand, if some relabeling of all frames, test bodies and fields is carried out, this *physical* verification cannot distinguish between various categories of choices for *mathematical labels* $\{\boldsymbol{e}_0(x),\boldsymbol{e}_1(x),\boldsymbol{e}_2(x),\boldsymbol{e}_3(x)\}$, if such relabelings are related, in accor-

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[9] A vector $\boldsymbol{X}$ tangent to a manifold $\mathbf{M}$ with Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ is timelike, null, or spacelike if and only if $\boldsymbol{g}^{\mathrm{L}}(\boldsymbol{X},\boldsymbol{X}) > 0$, $\boldsymbol{g}^{\mathrm{L}}(\boldsymbol{X},\boldsymbol{X}) = 0$, or $\boldsymbol{g}^{\mathrm{L}}(\boldsymbol{X},\boldsymbol{X}) < 0$, respectively. Therefore, a smooth curve is called timelike, null, or spacelike if and only if all vectors tangent to it are timelike, null, or spacelike, respectively.

dance with (2.1), by means of diffeomorphisms which give rise to various equivalent Lorentzian metric representatives. This freedom in relabeling then gives rise to the earlier mentioned diffeomorphism invariance of CGR models. This kind of invariance is therefore seen to be quite similar to the freedom in the choice of coordinate labels, which underlie Einstein's (1916) original formulation of the principle of general covariance.

As a result of the above described physical realization of linear frames $\boldsymbol{u} \in GL\mathbf{M}$ by means of "rigid rods" and "standard clocks," the principal bundle $GL\mathbf{M}$ emerges as a *classical frame bundle*. On the other hand, as acknowledged by Einstein on various occasions, cited in a review article by Stachel (1980), the concept of "rigid rod" is, strictly speaking, inconsistent with the very basic relativistic tenet that stipulates the impossibility of transmitting signals with supraluminal velocities.

This last problem can be dealt with by an alternative operational definition in which the spatial frame labeled by a triple $\{\boldsymbol{e}_1(x),\boldsymbol{e}_2(x),\boldsymbol{e}_3(x)\}$ is provided by a neutral classical test particle that marks the origin O of a local classical frame, and by six other identical test particles A_α, $\alpha = \pm1,\pm2,\pm3$, which are in its immediate vicinity, and mark its positive and negative axes. It has to be then operationally verified that such physical setup, taken in combination with "standard clocks" in the immediate vicinity of all these test particles, and at relative rest to the spatial axes determined by them, constitutes a local Lorentz frame. This verification can consist of checks, under identical *local* conditions, of whether all the test particles A_α, $\alpha = \pm1,\pm2,\pm3$, are indeed at a unit distance from O, and whether the axes joining O to each A_α, $\alpha = \pm1,\pm2,\pm3$, are at right angles to each other. Such operational procedures can be carried out, at least in principle, by observing the recoil of light signals sent between the test particles constituting such a classical frame, whose constituents are *assumed* to behave deterministically, so that absolutely precise compensations for the momenta transferred by such recoils are deemed to be in principle feasible.

Naturally, pragmatically dictated approximations (cf., e.g., de Sabbata and Melnikov, 1988) have to be taken for granted in all of the above operational descriptions, since none of the above procedures can be carried out instantaneously. However, at least in principle, a limiting procedure can be envisaged whereby the spatial separations between all the constituents of classical local frame are allowed to tend to zero, so that all the required approximations become increasingly negligible, and vanish altogether in that limit.

In this manner, a physical realization of $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ as a *classical Lorentz frame bundle* can be in principle envisaged as a selection procedure carried out on the classical frames of reference represented by elements of $GL\mathbf{M}$, which operationally singles out the ones which are orthonormal in an *existing* Lorentzian metric. Of course, the underlying assumption in this procedure is that the uncertainty principle can be altogether neglected. In Chapter 4 we shall dispense with this assumption, and thus arrive at a concept of *quantum* Lorentz frame bundle in which that most fundamental principle is taken into account.

2.3. Moving Frames in Principal Frame Bundles

While discussing in Sec. 1.1 the historical circumstances surrounding the development of CGR, we saw that Einstein's motivation for introducing the principle of general covariance lay in his desire to satisfy the following universal general relativistic postulate:

"The laws of physics must be of such a nature that they apply to systems of reference in any kind of motion." (Einstein, 1916, p. 113.)

In Sec. 2.1 we showed, however, that his original formulation (Einstein, 1916, p. 117) of the principle of general covariance was automatically satisfied once he adopted Riemannian geometry as his mathematical framework for CGR. The idea of diffeomorphism invariance provides a more subtle reflection of this general relativistic postulate, whose physical significance emerged more fully from the formulation in Sec. 2.2 of the concept of classical frame bundle. However, the clearest reflection of the idea of a "system of reference in any kind of motion" can be found in the concept of "moving frame" in a principal frame bundle, introduced by Cartan (1923-25) with general relativity in mind.

This idea involves group-theoretical aspects displayed by such bundles, which emerge with their greatest clarity from the modern general formulation of the interrelated concepts of principal and associated fibre bundles. We therefore turn now to the general formulation of principal bundles, and we will be dealing with associated bundles in the next section.

The type[10] of *principal bundle* encountered in CGR, as well as in classical theories of gauge fields, consists of a manifold $\mathbf{P}$, called its *total space*, of a second manifold $\mathbf{M}$, called its *base space*, of a *projection* Π that maps $\mathbf{P}$ smoothly onto $\mathbf{M}$, and of a *structure group* $\mathbf{G}$ whose elements g act on all the elements $\boldsymbol{u}$ of the total space $\mathbf{P}$ from the right[11] in such a manner that: 1) the map $(\boldsymbol{u},g) \mapsto \boldsymbol{u} \cdot g$ determining this action is a smooth map from $\mathbf{P}\times\mathbf{G}$ into $\mathbf{P}$; 2) $(\boldsymbol{u} \cdot g')\cdot g'' = \boldsymbol{u} \cdot (g'g'')$ for all $\boldsymbol{u} \in \mathbf{P}$ and all $g', g'' \in \mathbf{G}$; 3) each submanifold of $\mathbf{P}$ consisting of all $\boldsymbol{u} \in \mathbf{P}$ mapped by Π into $x \in \mathbf{M}$ (denoted by $\Pi^{-1}(x)$ and called the *fibre above* x), is left invariant by $\mathbf{G}$, i.e., each element $\boldsymbol{u} \in \Pi^{-1}(x)$ is mapped into an element $\boldsymbol{u} \cdot g \in \Pi^{-1}(x)$ belonging to the same fibre; 4) the action of the structure group $\mathbf{G}$ on each fibre $\Pi^{-1}(x)$ is transitive and free, i.e., all the elements within each given fibre $\Pi^{-1}(x)$ are obtained by the action of all the elements of $\mathbf{G}$ on any fixed element within that fibre $\Pi^{-1}(x)$.

Let us illustrate the mathematical ingredients $(\mathbf{P}, \Pi, \mathbf{M}, \mathbf{G})$ of a principal bundle with the case of a general linear frame bundle $GL\mathbf{M}$ defined in the preceding section: its total space $\mathbf{P}$ consists of all the linear frames over its base space $\mathbf{M}$, and its projection Π maps each one of the frames $\boldsymbol{u}(x)$, defined by (2.7) above some base point $x \in \mathbf{M}$, into x itself; consequently, the fibre $\Pi^{-1}(x)$ consists of all the linear frames above x. We observe that according to (2.7), for a given chart around $x \in \mathbf{M}$, to each element $\boldsymbol{u} \in \Pi^{-1}(x)$ are assigned the sixteen coordinates $\lambda_i{}^{\mu}(x)$, $i,\mu = 0,...,3$; in view of the linear independence of the four elements of that frame, considered as vectors in the tangent space $T_x\mathbf{M}$, the sixteen numbers have to constitute an invertible 4×4 real matrix. The set $\mathrm{GL}(4,\mathbf{R})$ of all such matrices constitutes a Lie group $\mathrm{GL}(4,\mathbf{R})$, known a the *general linear group* of 4×4 real (and invertible) matrices.

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[10] This general type is known as a C^{∞}-bundle. The structure group G of such bundles is required to be a Lie group. As a Lie group G is also a differential manifold, so that in the present definition P×G denotes a product manifold. Cf. (Isham, 1989), or (Nakahara, 1990), or (Spivak, 1979) for more details.

[11] In case that, as in the examples treated below, g is a matrix that acts by matrix multiplication on the elements of $\mathbf{R}^n$, for its action from the right those elements have to be viewed as one-row matrices; whereas for its action from the left those elements have to be viewed as one-column matrices. Hence, one mode of action can be related to the other by taking the transpose of the matrices in question.

Clearly, the group multiplication in $\mathrm{GL}(4,\mathbf{R})$ is given by matrix multiplication; whereas the manifold structure of $\mathrm{GL}(4,\mathbf{R})$ results from charts which assign to each matrix $A \in \mathrm{GL}(4,\mathbf{R})$ a vector in the Euclidean space $\mathbf{R}^{16}$, which consists of its sixteen matrix elements arranged in a single row, representing a 16-tuple. It is therefore obvious that there is a one-to-one correspondence between all $\boldsymbol{u} \in \Pi^{-1}(x)$ and all the elements of $\mathrm{GL}(4,\mathbf{R})$, which provides a map between the manifolds $\Pi^{-1}(x)$ and $\mathrm{GL}(4,\mathbf{R})$, and which is smooth in both directions, i.e., so that it is a diffeomorphism. We shall therefore see in the next section that, regarded as a manifold, the Lie group $\mathrm{GL}(4,\mathbf{R})$ plays the role of typical fibre of the principal bundle $GL\mathbf{M}$. Furthermore, each element A of the general linear group $\mathrm{GL}(4,\mathbf{R})$ acts from the right on any linear frame $\boldsymbol{u} \in GL\mathbf{M}$ by assigning to it the following new frame $\boldsymbol{u}' \in GL\mathbf{M}$ within the same fibre:

$$\boldsymbol{u} \mapsto \boldsymbol{u}' = (\boldsymbol{e}'_0, \boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3) = \boldsymbol{u} \cdot A \ , \qquad \boldsymbol{e}'_i = \boldsymbol{e}_j A^j{}_i \ , \tag{3.1a}$$

$$\boldsymbol{u}, \boldsymbol{u}' \in \Pi^{-1}(x) \subset GL\mathbf{M} \ , \qquad A = \left\| A^i{}_j \right\| \in \mathrm{GL}(4,\mathbf{R}) \ ; \tag{3.1b}$$

Hence, as a Lie group $\mathrm{GL}(4,\mathbf{R})$ also plays the role of structure group of the principal frame bundle $GL\mathbf{M}$.

In a similar manner, the general linear coframe bundle $GL^*\mathbf{M}$ can be made into a principal bundle with the same structure group:

$$\boldsymbol{u}^* \mapsto \boldsymbol{u}^{*\prime} = (\boldsymbol{\theta}'^0, \boldsymbol{\theta}'^1, \boldsymbol{\theta}'^2, \boldsymbol{\theta}'^3) = \boldsymbol{u}^* \cdot A \ , \qquad \boldsymbol{\theta}'^i = \boldsymbol{\theta}^j A_j{}^i \ , \tag{3.2a}$$

$$\boldsymbol{u}^*, \boldsymbol{u}^{*\prime} \in \Pi^{-1}(x) \subset GL^*\mathbf{M} \ , \qquad A = \left\| A_j{}^i \right\| \in \mathrm{GL}(4,\mathbf{R}) \ . \tag{3.2b}$$

Two principal bundles for which there is a diffeomorphism between their total spaces that maps fibres onto fibres, and which possess diffeomorphic base manifolds and isomorphic structure groups, are said to be *equivalent principal bundles*. Thus, as principal bundles, $GL\mathbf{M}$ and $GL^*\mathbf{M}$ are seen to be equivalent.

When the principal bundle $GL\mathbf{M}$ is physically realized as a classical frame bundle, then the transformation in (3.1) obviously represents a change of local frame of reference. However, as pointed out in the quotation of Einstein (1961) in the preceding section, a physical meaning can be assigned to the general linear frame bundle $GL\mathbf{M}$ only in the presence of a Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$, which then mediates the reduction of $GL\mathbf{M}$ to the Lorentzian frame bundle $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ of frames that are orthonormal in that metric, so that they constitute local Lorentzian frames. This latter bundle becomes a principal bundle with total space $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$, with base space $\mathbf{M}$, and with a projection Π obtained by allowing the projection in $GL\mathbf{M}$ to act only on orthonormal frames with timelike $\boldsymbol{e}_0$. Hence, any fibre $\Pi^{-1}(x)$ of $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ will consist only of local Lorentz frames over $x \in \mathbf{M}$. This implies that the changes of such frames,

$$\boldsymbol{u} \cdot \Lambda = \boldsymbol{u}' = (\boldsymbol{e}'_0, \boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3) \ , \qquad \boldsymbol{e}'_i = \boldsymbol{e}_j \Lambda^j{}_i \ , \qquad \boldsymbol{u}, \boldsymbol{u}' \in \Pi^{-1}(x) \subset L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}}), \tag{3.3}$$

will be effected only by those 4×4 real matrices $\Lambda \in \mathrm{GL}(4,\mathbf{R})$ that belong to the *pseudo-orthogonal group* $\mathrm{O}(3,1)$, namely the Lie subgroup of $\mathrm{GL}(4,\mathbf{R})$ which preserves the

Minkowski inner product in $\mathbf{R}^4$, and which is known in physics as the Lorentz group. Since the preservation of the Minkowski inner product takes place if and only if,

$$\Lambda_i{}^k \eta_{kl} \Lambda_j{}^l = \eta_{ij} \ , \qquad \Lambda = \left\| \Lambda^i{}_j \right\| \in \mathrm{O}(3,1) \ , \tag{3.4a}$$

this pseudo-orthogonal group of 4×4 real matrices can be also defined by

$$\mathrm{O}(3,1) = \left\{ \Lambda \in \mathrm{GL}(4,\mathbf{R}) \,\middle|\, \Lambda^{\mathrm{T}} \boldsymbol{\eta} \Lambda = \boldsymbol{\eta} \right\} \ , \qquad \boldsymbol{\eta} = \left\| \eta_{ij} \right\| \ , \tag{3.4b}$$

in terms of the 4×4 matrix $\boldsymbol{\eta}$ providing the Minkowski metric of special relativity. Clearly, $\mathrm{O}(3,1)$ constitutes the structure group of the principal bundle $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ of local Lorentz frames.

In the case of the Lorentzian coframe bundle $L^*\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ the transformations in (3.2) assume the special form,

$$\boldsymbol{u}^* \mapsto \boldsymbol{u}^{*\prime} = (\theta'^0, \theta'^1, \theta'^2, \theta'^3) = \boldsymbol{u}^* \cdot \left(\Lambda^{\mathrm{T}} \right)^{-1} \ , \qquad \theta'^i = \Lambda^i{}_j \, \theta^j \ , \tag{3.5a}$$

$$\boldsymbol{u}, \boldsymbol{u}^* \in L^*\mathbf{M}(\boldsymbol{g}^{\mathrm{L}}) \ , \qquad \left(\Lambda^{\mathrm{T}} \right)^{-1} = \left\| \Lambda_i{}^j \right\| = \boldsymbol{\eta} \Lambda \, \boldsymbol{\eta} \in \mathrm{O}(3,1) \ . \tag{3.5b}$$

In view of the fact that, by (3.4b), we have

$$\Lambda^i{}_k \eta^{kl} \Lambda^j{}_l = \eta^{ij} \ , \qquad \eta^{ij} = \eta_{ij} \ , \qquad \Lambda = \left\| \Lambda^i{}_j \right\| \in \mathrm{O}(3,1) \ , \tag{3.6}$$

we see that, as a principal bundle, the bundle $L^*\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ of Lorentz coframes also has $\mathrm{O}(3,1)$ as structure group, and that it is actually equivalent to $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$.

The transformation laws in (3.3) and (3.5) turn out to mediate the transitions between "moving frames" in Lorentz frame bundles. We shall see that such "moving frames" bring us a step closer to realizing Einstein's earlier cited dictum that: "*The laws of physics must be of such a nature that they apply to systems of reference in any kind of motion.*"

Indeed, Cartan (1923-35) had originally executed that step by introducing in principal frame bundles, such as $GL\mathbf{M}$ and $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$, the concept of a *moving frame* obtained by assigning, in a smooth manner, to each element x from some open domain $\mathbf{M}^{\boldsymbol{s}}$ in the base manifold $\mathbf{M}$ a unique element $\boldsymbol{s}(x)$ from the fibre $\Pi^{-1}(x)$ above x. For example, a moving frame in $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ (often called in physics a *vierbein*) is given by the map

$$\boldsymbol{s} \ : \ x \mapsto \boldsymbol{s}(x) \in \Pi^{-1}(x) \subset L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}}) \ , \qquad x \in \mathbf{M}^{\boldsymbol{s}} \subset \mathbf{M} \ , \tag{3.7}$$

where, in any chart whose domain intersects $\mathbf{M}^{\boldsymbol{s}}$, the coordinates of $\boldsymbol{s}(x)$ have to change smoothly as x varies over $\mathbf{M}^{\boldsymbol{s}}$. Thus indeed, one can regard "moving frames" (of which vierbeins are special cases) as providing a mathematical basis for the formulation of the concept of "systems of reference in any kind of motion."

Although we shall use it frequently, nowadays the term "moving frame" is rarely used in mathematics, having been superseded by the more general term of "local section of a fibre bundle."

To define this very general term, let us consider an arbitrary C^∞-*fibre bundle*, namely

a mathematical structure $(\mathbf{E}, \pi, \mathbf{M}, \mathbf{F})$ which consists of: a manifold $\mathbf{E}$, called its *total space*; a submanifold $\mathbf{M}$ of $\mathbf{E}$ called its *base space*; a *projection* map π from $\mathbf{E}$ onto $\mathbf{M}$, required to be smooth; a *typical* (or *standard*) *fibre* $\mathbf{F}$, diffeomorphic to each one of the fibres $\pi^{-1}(x)$. Clearly, principal fibre bundles are special cases of fibre bundles, and in their case the typical fibre $\mathbf{F}$ coincides with their structure group $\mathbf{G}$.

Any *section* $\boldsymbol{s}$ of any such fibre bundle $(\mathbf{E}, \pi, \mathbf{M}, \mathbf{F})$ is defined by a smooth map,

$$\boldsymbol{s} \; : \; x \; \mapsto \; \boldsymbol{s}(x) \in \pi^{-1}(x) \subset \mathbf{E} \; , \qquad x \in \mathbf{M}^{\boldsymbol{s}} \subset \mathbf{M} \; , \tag{3.8}$$

of an open set $\mathbf{M}^{\boldsymbol{s}}$ in $\mathbf{M}$, called its domain of definition, into the total space $\mathbf{E}$ of that fibre bundle. If $\mathbf{M}^{\boldsymbol{s}}$ coincides with $\mathbf{M}$ then $\boldsymbol{s}$ is called a *global section* or *cross-section*. For reasons explained in Sec. 2.4, a fibre bundle which possesses a cross-section is called *trivial*.

In view of the usual connotation of this term, it has to be emphasized that its usage in this context is of a purely technical nature, since in actuality many "trivial" bundles turn out to be quite interesting from the physical as well as the mathematical point of view. In fact, it turns out that a non-compact[12] Lorentzian manifold $\mathbf{M}$ allows a spin structure (which is essential for the treatment of spin-1/2 fields) if and only if $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ possesses a global section (Geroch, 1968); hence all Lorentz frame bundles $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ that are of interest in CGR are trivial in the above technical sense.

The concept of global section of a classical frame bundle generalizes the concept of frame of reference in special relativity. To see that, let us consider a manifold $\mathbf{M} \cong \mathbf{R}^4$ provided by the Minkowski space $(\mathbf{R}^4, \boldsymbol{\eta})$ of special relativity, which carries the Minkowski metric $\boldsymbol{\eta}$. In that case the Lorentz frame bundles $L\mathbf{M}(\boldsymbol{\eta})$ is obviously a trivial bundle, and as such it possesses various cross-sections. A special class of such cross-sections of $L\mathbf{M}(\boldsymbol{\eta})$ can be identified with global Lorentz frames $\mathcal{L}$ in the Minkowski space $(\mathbf{R}^4, \boldsymbol{\eta})$ by the following construction: the Minkowski coordinates x^i in each global Lorentz frame $\mathcal{L} = \{\boldsymbol{e}_i(O) | \; i = 0,1,2,3\}$, with origin at $O \in \mathbf{M} \cong \mathbf{R}^4$, are used in the construction of coordinate charts for the tangent bundle $T\mathbf{M}$, in the manner described in Sec. 2.2; on the other hand, since in this case $\mathbf{M}$ is not only a manifold but also a vector space $\mathbf{R}^4$, all the points in the tangent space $T_x\mathbf{M}$ above each $x \in \mathbf{M}$ can be identified with corresponding points in $\mathbf{M}$. This enables the identification of the Minkowski frame $\mathcal{L}$ with the global section

$$\boldsymbol{s}(\mathcal{L}) \; : \; x \; \mapsto \; \left\{\boldsymbol{e}_i(x) \in T_x\mathbf{M} \middle| \; i = 0,1,2,3\right\} \in L\mathbf{M}(\boldsymbol{\eta}) \; , \qquad x = x^i \boldsymbol{e}_i(O) \in \mathbf{M}, \tag{3.9}$$

of $L\mathbf{M}(\boldsymbol{\eta})$, in which $\{\boldsymbol{e}_i(x) | \; i = 0,1,2,3\}$ is obtained by parallel translating from $O \in \mathbf{M} \cong \mathbf{R}^4$ to $x \in \mathbf{M}$ the elements of $\mathcal{L} = \{\boldsymbol{e}_i(O) | \; i = 0,1,2,3\}$. This is obviously feasible since they can be simultaneously regarded as linear frame elements in the tangent space $T_O\mathbf{M}$, and as vectors in $\mathbf{R}^4$. As we shall see in Sec. 2.6 and in later chapters, the resulting identification of each global Lorentz frame $\mathcal{L}$ with the cross-section $\boldsymbol{s}(\mathcal{L})$ of $L\mathbf{M}(\boldsymbol{\eta})$ in (3.9) can be used to extrapolate certain special relativistic results to the general relativistic regime.

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[12] A topological space, defined as in Footnote 1, is called *compact* if every family of open sets which covers that space, in the sense that their union equals that space, has a finite subcovering, i.e., a finite subfamily of that family also cover that space. Compact manifolds are usually deemed physically unacceptable as spacetime manifolds, since they allow for closed timelike worldlines, thus leading to violations of causality.

2.4. Gauge Invariance in Associated Bundles

The choice of a (local or global) section of a principal frame bundle is, in physical theories, tantamount to a choice of *gauge*, in the sense first introduced in 1918 by H. Weyl (1923, 1929) for electromagnetic fields, and later extrapolated by Yang and Mills (1954) to non-Abelian gauge fields on Minkowski space. The various ramifications of this important idea emerge most clearly from the general formulation of the concept of bundles associated to a principal bundle. Such a general formulation also mediates its extrapolation to CGR.

A variety of definitions of associated bundles can be found in literature (Choquet-Bruhat *et al.*, 1987; Isham, 1989; Nakahara, 1990; Spivak, 1979), but in the ultimate analysis, they are all equivalent. In this section[13] we shall adopt the one which assigns prominence to bundle coordinates, since it applies most readily to those cases where a manifold structure is defined in a associated bundle prior to its bundle structure, such as is the case with the tensor bundles introduced in Sec. 2.1.

The introduction of bundle coordinates in a fibre bundle $(\mathbf{E}, \pi, \mathbf{M}, \mathbf{F})$, generically defined as in the previous section, can be achieved by adapting to fibre bundles the method of introducing coordinates for manifolds, presented at the beginning of Sec. 2.1. Thus, we require that the base manifold $\mathbf{M}$ of this bundle can be covered with a family of open subsets $\mathbf{M}_\alpha$, which are submanifolds of $\mathbf{M}$ (but are not necessarily related to the sets M_α used in Sec. 2.1 in defining an atlas of charts). For each $\mathbf{M}_\alpha$ we construct, in the standard manner (Isham, 1989, p. 21; Nakahara, 1990, p. 139), the manifold which is the product of $\mathbf{M}_\alpha$ and the typical fibre $\mathbf{F}$ of the bundle, and assume that there is a diffeomorphism,

$$\phi_\alpha : u \mapsto (x,f) \in \mathbf{M}_\alpha \times \mathbf{F}, \qquad u \in \pi^{-1}(\mathbf{M}_\alpha) = \bigcup_{x \in \mathbf{M}_\alpha} \pi^{-1}(x) \subset \mathbf{E}, \tag{4.1}$$

called a *local trivialization map*, which assigns to each element u that belongs to a fibre of above $\mathbf{M}_\alpha$ the *fibre coordinates* (x,f). As a consequence, the restriction of ϕ_α to each fibre $\pi^{-1}(x)$ over some $x \in \mathbf{M}_\alpha$ determines a diffeomorphism between that fibre and the typical fibre $\mathbf{F}$. Hence, in turn, for each $x \in \mathbf{M}_\alpha \cap \mathbf{M}_\beta$, the map

$$\phi_\beta \circ \phi_\alpha^{-1} : (\mathbf{M}_\alpha \cap \mathbf{M}_\beta) \times \mathbf{F} \to (\mathbf{M}_\alpha \cap \mathbf{M}_\beta) \times \mathbf{F}, \tag{4.2}$$

gives rise to a diffeomorphism

$$\mathbf{g}_{\beta\alpha}(x) : \mathbf{F} \to \mathbf{F}, \qquad x \in \mathbf{M}_\alpha \cap \mathbf{M}_\beta, \tag{4.3}$$

of the typical fibre $\mathbf{F}$ onto itself, known by the name of a *transition function*, or a *structure function*, of the given fibre bundle $(\mathbf{E}, \pi, \mathbf{M}, \mathbf{F})$.

Let us illustrate the introduction of bundle coordinates in the case of the tangent bundle $T\mathbf{M}$ defined in Sec. 2.2. From (2.5b) we infer that its typical fibre $\mathbf{F}$ can be taken as equal to $\mathbf{R}^4$, so that if we take $\mathbf{M}_\alpha$ actually equal to M_α, then (4.1) assumes the form

$$\phi_\alpha : \boldsymbol{X} \mapsto (x,X) \in \mathbf{M}_\alpha \times \mathbf{R}^4, \quad \boldsymbol{X} \in \pi^{-1}(\mathbf{M}_\alpha) \subset T\mathbf{M}, \quad X = (X^0, X^1, X^2, X^3). \tag{4.4a}$$

We observe that the above fibre coordinates do not coincide with the manifold coordinates

[13] Cf. Sec. 4.1 in Chapter 4 for an alternative definition, which is based on the concept of G-product.

in (2.5b), since they do not supply any specific manifold coordinates for any point $x \in \mathbf{M}_\alpha$, but only to the tangent vectors $\boldsymbol{X}$. If we similarly set

$$\boldsymbol{\phi}_\beta : \boldsymbol{X} \mapsto (x, X') \in \mathbf{M}_\beta \times \mathbf{R}^4, \ \boldsymbol{X} \in \pi^{-1}(\mathbf{M}_\beta) \subset T\mathbf{M}, \ X' = (X'^0, X'^1, X'^2, X'^3), \quad (4.4b)$$

and treat the bundle coordinate 4-tuples from $\mathbf{R}^4$ as one-column matrices, then we deduce from (1.10) that these tangent bundle coordinates transform as follows:

$$\mathbf{g}_{\beta\alpha}(x) : \ X \ \mapsto \ X' = (A^{-1})^{\mathrm{T}} X , \qquad A \in \mathrm{GL}(4, \mathbf{R}) . \quad (4.5)$$

It is easy to check that, in general, the family of all transition functions of a fibre bundle $(\mathbf{E}, \pi, \mathbf{M}, \mathbf{F})$ satisfies the following *compatibility relations*:

$$\mathbf{g}_{\alpha\beta}(x) = \left(\mathbf{g}_{\beta\alpha}(x)\right)^{-1} , \qquad x \in \mathbf{M}_\alpha \cap \mathbf{M}_\beta , \quad (4.6a)$$

$$\mathbf{g}_{\alpha\beta}(x)\mathbf{g}_{\beta\gamma}(x) = \mathbf{g}_{\alpha\gamma}(x) , \qquad x \in \mathbf{M}_\alpha \cap \mathbf{M}_\beta \cap \mathbf{M}_\gamma \quad (4.6b)$$

In addition, it is obvious that $\mathbf{g}_{\alpha\alpha}(x)$ equals, for each $x \in \mathbf{M}$, the identity map of $\mathbf{F}$ onto itself. Hence, the family of all transition functions displays a group structure at each fixed point $x \in \mathbf{M}$.

In the case of the tangent bundle $T\mathbf{M}$ this group structure is determined in (4.5) as being provided by the adjoint representation $U : A \mapsto (A^{-1})^{\mathrm{T}}$ of the general linear group $\mathrm{GL}(4, \mathbf{R})$. Hence, if we view each matrix $(A^{-1})^{\mathrm{T}}$ in this representation as giving rise to a transformation of $\mathbf{R}^4$ obtained by multiplying with $(A^{-1})^{\mathrm{T}}$ from the left each one-column matrix that represents an element of $\mathbf{R}^4$, then we conclude that the tangent bundle $T\mathbf{M}$ is associated to the general linear frame bundle $GL\mathbf{M}$ in the sense of the following definition:

A fibre bundle $(\mathbf{E}, \pi, \mathbf{M}, \mathbf{F})$ is *associated* to a principal bundle $(\mathbf{P}, \Pi, \mathbf{M}, \mathbf{G})$ by a representation U of the structure group $\mathbf{G}$, consisting of diffeomorphisms $U(g) : \mathbf{F} \to \mathbf{F}$, $g \in \mathbf{G}$, that act from the left[14] on the typical fibre $\mathbf{F}$, if for any family of local trivializations $\boldsymbol{\Phi}_\alpha$ of the principal bundle, given by the maps

$$\boldsymbol{\Phi}_\alpha : \boldsymbol{u} \mapsto (x, g) \in \mathbf{M}_\alpha \times \mathbf{G} , \qquad \boldsymbol{u} \in \Pi^{-1}(\mathbf{M}_\alpha) \subset \mathbf{P} , \quad (4.7)$$

there is an associated family of local trivialization maps (4.1) of $(\mathbf{E}, \pi, \mathbf{M}, \mathbf{F})$ such that

$$\mathbf{g}_{\beta\alpha}(x) = U(g_{\beta\alpha}(x)) , \qquad g_{\beta\alpha}(x) \in \mathbf{G} , \qquad x \in \mathbf{M}_\alpha \cap \mathbf{M}_\beta , \quad (4.8)$$

for all intersecting submanifolds $\mathbf{M}_\alpha$ and $\mathbf{M}_\beta$, where $g_{\beta\alpha}(x)$ is the structure group element obtained for $x \in \mathbf{M}_\alpha \cap \mathbf{M}_\beta$ by restricting to $\{x\} \times \mathbf{G}$ the coordinate transition map

$$\boldsymbol{\Phi}_\beta \circ \boldsymbol{\Phi}_\alpha^{-1} : \ (\mathbf{M}_\alpha \cap \mathbf{M}_\beta) \times \mathbf{G} \ \to \ (\mathbf{M}_\alpha \cap \mathbf{M}_\beta) \times \mathbf{G} \quad (4.9)$$

As we mentioned earlier, an immediate example of the above concept can be obtained by viewing the tangent bundle $T\mathbf{M}$ as a fibre bundle associated to the general linear frame

[14] For group representations by means of diffeomorphisms, *left action* gives rise to group multiplication in an order which is the reversal of that for *right action*, so that actions from the right and from the left satisfy two distinct composition laws, given respectively by $U(g) \circ U(g') = U(g'g)$ and $U(g) \circ U(g') = U(gg')$.

bundle $GL\mathbf{M}$. To see that, one first has to establish that $GL\mathbf{M}$ can play the role of principal bundle in the context of the above definition. Hence, consider any two local trivializations Φ_α and Φ_β of $GL\mathbf{M}$. For example, let us construct Φ_α from the coefficients in (2.7), by arranging them into an invertible matrix $\lambda(x) \in \mathrm{GL}(4,\mathbf{R})$ at each point $x \in \mathbf{M}_\alpha$, so that we arrive at the following local trivialization map:

$$\Phi_\alpha : \boldsymbol{u} \mapsto (x,\lambda) \in \mathbf{M}_\alpha \times \mathrm{GL}(4,\mathbf{R}) , \quad \boldsymbol{u} \in \Pi^{-1}(\mathbf{M}_\alpha) , \quad \lambda = \left\| \lambda_i^{\ \mu} \right\| \in \mathrm{GL}(4,\mathbf{R}) . \tag{4.10}$$

Furthermore, let us consider another local trivialization Φ_β obtained by a similar construction, based on the coefficients in the expansion of vector frame elements in the chart (1.1b):

$$\Phi_\beta : \boldsymbol{u} \mapsto (x,\lambda') \in \mathbf{M}_\beta \times \mathrm{GL}(4,\mathbf{R}) , \qquad \lambda' = \left\| \lambda'_i{}^{\mu} \right\| \in \mathrm{GL}(4,\mathbf{R}) , \tag{4.11a}$$

$$\boldsymbol{u} = (\boldsymbol{e}_0,\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3) \in \Pi^{-1}(\mathbf{M}_\beta) \subset GL\mathbf{M} , \qquad \boldsymbol{e}_i = \lambda'_i{}^{\mu'}(x)\partial'_{\mu'} \in T_x\mathbf{M} \tag{4.11b}$$

Then, for any $x \in \mathbf{M}_\alpha \cap \mathbf{M}_\beta$, the so constructed matrices from $\mathrm{GL}(4,\mathbf{R})$ constitute the fibre coordinates of a given linear frame $\boldsymbol{u} \in \Pi^{-1}(x)$. We therefore see that $\mathrm{GL}(4,\mathbf{R})$ represents the typical fibre of $GL\mathbf{M}$, so that $GL\mathbf{M}$ emerges as the principal bundle for which the transition functions between the two sets of fibre coordinates constructed in (4.10) and (4.11) is given by the maps

$$g_{\beta\alpha}(x) : \lambda \mapsto \lambda' = \lambda A^{-1} , \quad \partial'_{\mu'} = A_{\mu'}{}^{\mu}\partial_\mu , \quad A = \left\| A_{\mu'}{}^{\mu} \right\| \in \mathrm{GL}(4,\mathbf{R}) . \tag{4.12}$$

In view of (4.4) and (4.5), we conclude that $T\mathbf{M}$ is indeed associated to $GL\mathbf{M}$.

According to the above definition of associated bundles, every principal bundle is associated to itself. On the other hand, if two principal fibre bundles are associated to each other, then they share the same structure group, so that they are deemed to be equivalent in the sense defined in the preceding section. We saw this to be the case with the general linear frame bundle $GL\mathbf{M}$ and the general linear coframe bundle $GL^*\mathbf{M}$. In general, any two fibre bundles associated with the same principal bundle, and therefore sharing the same base manifold $\mathbf{M}$ and structure group $\mathbf{G}$, are called *equivalent* if and only if, for any given family of local trivialization maps Φ_α and Φ_β of their common principal bundle, there are diffeomorphisms $\lambda_\alpha(x)$ and $\lambda_\beta(x)$ between their respective fibres above each $x \in \mathbf{M}_\alpha \cap \mathbf{M}_\beta$ in terms of which their transition functions are related as follows:

$$\mathbf{g}'_{\alpha\beta}(x) = \lambda_\alpha^{-1}(x) \circ \mathbf{g}_{\alpha\beta}(x) \circ \lambda_\beta(x) , \qquad x \in \mathbf{M}_\alpha \cap \mathbf{M}_\beta \tag{4.13}$$

For example, if we use (1.8) and (1.9) for the construction of the cotangent bundle $T^*\mathbf{M}$, we see that its typical fibre $\mathbf{F}$ can be taken equal to $\mathbf{R}^4$, as was the case with tangent bundle $T\mathbf{M}$, and that $T^*\mathbf{M}$ is associated to $GL\mathbf{M}$ by the fundamental representation $U : A \mapsto A$ of $\mathrm{GL}(4, \mathbf{R})$; then the manner in which the diffeomorphisms $\lambda_\alpha(x)$ and $\lambda_\beta(x)$ in (4.13) can be constructed becomes obvious, so that $T^*\mathbf{M}$ and $T\mathbf{M}$ are seen to be equivalent bundles.

More generally, by using (1.10)-(1.12) we can construct local trivializations[15] of any

[15] Note that from the physics point of view, a local trivialization of $T^{r,s}\mathbf{M}$ represents a *local tensor field*; whereas a global trivialization (which is feasible if and only if this bundle is trivial) represents a *global tensor field*.

tensor bundle $T^{r,s}\mathbf{M}$ which make it evident that $T^{r,s}\mathbf{M}$ has $\mathbf{R}^{r+s}$ as typical fibre, and that it is associated to $GL\mathbf{M}$ by the following tensor representation of GL(4, **R**),

$$U : A \mapsto (A^{-1})^{\mathrm{T}} \otimes \cdots \otimes (A^{-1})^{\mathrm{T}} \otimes A \otimes \cdots \otimes A \quad , \tag{4.14}$$

in which $(A^{-1})^{\mathrm{T}}$ occurs r-times, and A occurs s-times; hence, two such tensor bundles are equivalent if and only if they share the same value for $r+s$.

When a principal bundle $(\mathbf{P}, \Pi, \mathbf{M}, \mathbf{G})$ plays a physical role, then each one of its sections, defined as in (3.8), is referred to as a choice of *gauge*. A gauge is called *global* if its domain $\mathbf{M}^s$ equals the based manifold $\mathbf{M}$; otherwise, it is called *local*. Since any section of $(\mathbf{P}, \Pi, \mathbf{M}, \mathbf{G})$ obviously gives rise to a local trivialization over $\mathbf{M}^s$, the existence of a global gauge is tantamount to the existence of a global trivialization, so that the principal bundle in question is said to be trivial, and therefore so are all its associated bundles.

We have seen in the preceding section that, in the case of the principal frame bundle $GL\mathbf{M}$, the choice of a section coincides with the choice of a moving frame in Cartan's sense of the word. However, according to Einstein's epistemic statement quoted in Sec. 2.2, $GL\mathbf{M}$ cannot possess any physical significance in the absence of a Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ over $\mathbf{M}$. On the other hand, the existence of such a metric enables the reduction of the general linear bundle $GL\mathbf{M}$ to the Lorentz frame bundle $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$, which contains only those frames $\{\boldsymbol{e}_0(x),\boldsymbol{e}_1(x),\boldsymbol{e}_2(x),\boldsymbol{e}_3(x)\} \in GL\mathbf{M}$ for which $\boldsymbol{e}_0(x)$ is timelike, and which are orthonormal in the given metric $\boldsymbol{g}^{\mathrm{L}}$. In view of (2.12), such generically nonholonomic Lorentz frames and coframes can be used in the course of constructing local trivialization of a tensor bundle $T^{r,s}\mathbf{M}$, in place of the holonomic frames in (1.12). Hence, we arrive at the conclusion that each tensor bundle $T^{r,s}\mathbf{M}$ (and, in particular, the tangent bundle $T\mathbf{M}$ and the cotangent bundle $T^*\mathbf{M}$) can be associated with the principal bundle $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$.

On account of (3.3) and (3.4), the Lorentz frame bundle $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ has O(3,1) as its structure group, so that the matrices A in (4.14) can be restricted to being Lorentz matrices Λ. Consequently, by choosing a local trivialization of $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ we can say that we have made a choice of *vierbein* (or *Lorentz-frame*) *gauge of classical frames* in a CGR model based on a particular Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ over $\mathbf{M}$.

2.5. Connections and Gauge Transformations

The concept of local Lorentz frame is essential to the precise mathematical formulation of the concept of free fall in CGR, which, as we shall see in Sec. 2.6, is in turn essential to the formulation of the principle of strong equivalence[16]. Indeed, motion in free fall can be defined in Newtonian terms as motion under the influence of no other forces than the force exerted by a gravitational field. However, in a geometrized physical theory, such as CGR, the concept of "force" loses its meaning, and is replaced by purely geometric concepts emerging from that of "connection," such as the Levi-Civita connection that in CGR

[16] It has been conjectured (Schiff, 1960), as well as proven (Lightman and Lee, 1973) at least for certain types of test bodies, that the strong principle of equivalence actually follows from its weak counterpart. The latter asserts the equality of gravitational and inertial mass, and has survived batteries of tests, confirming it to one part in a trillion in measurements based on comparisons of the gravitational acceleration of macroscopic aluminum and platinum test bodies of equal inertial mass (Braginsky and Panov, 1971).

determines parallel transport in a curved spacetime manifold. In turn, parallel transport determines the free fall motion to which the strong principle of equivalence of general relativity is applied, and has therefore played a central role in the development of CGR.

To geometrize the concept of free fall, Einstein (1916) postulated in his original formulation of CGR that the motions of all (classical and neutral) massive point particles in free fall within a classical spacetime manifold $\mathbf{M}$ follow timelike geodesics, so that their 4-velocity vectors $\boldsymbol{v} \in \boldsymbol{T}\mathbf{M}$ are parallel transported along their worldlines γ. In subsequent years, Einstein and Grommer (1927), as well as many other theoreticians, have attempted to derive this geodesic postulate from Einstein's general relativistic equations of motion, by treating a massive point particle as a limiting case of a continuous distribution of mass in mutual interaction with gravitational fields. A review article by P. Havas (Howard and Stachel, 1989) describes the rich history of these ingenious and varied attempts, which started already in the early 1920s. However, a *rigorous* derivation of the geodesic postulate from Einstein's equations is still pending.[17]

On the other hand, the geodesic postulate for massive particles, together with its counterpart for light-rays, whose free-fall motions are assumed to follow null geodesics, is routinely used for deriving all the best-known predictions of CGR presented in standard textbooks, and confirmed by observations, namely the perihelion precessions of planetary motions, the cosmological and the gravitational red-shifts, the time delay of radar signals, the bending of light rays in the gravitational fields of massive bodies, etc. (cf., e.g., Hawking and Ellis, 1973; Misner *et al.*, 1973; Straumann, 1984; Wald, 1984). Thus, at least at the macroscopic level, the empirical confirmation of the geodesic postulate appears to be beyond all reproach.

At the time Einstein adopted Riemannian geometry as basic to his formulation of CGR, the concept of connection, heuristically viewed as governing "infinitesimal" parallel transport, was deemed to emerge from that of metric. However, the work of Cartan (1923-1935) clearly showed that the concepts of connection and metric are actually independent. That pioneering work paved the way for contemporary mathematical formulations of the concept of connection.

In the modern mathematical approach to the theory of connections (Kobayashi and Nomizu 1963; Spivak, 1979), the concept of connection is first introduced in principal bundles. It is then shown to determine parallel transport in those bundles. Once parallel transport is thus defined in a principal bundle, it is then extended to all the bundles associated to that principal bundle, and it is used there to define covariant differentiation.

At first sight, this approach to connections and parallel transport might appear rather abstract. However, it turns out that it presents distinct advantages even at the physical level, since it is very readily applicable (Isham, 1989; Nakahara, 1990) to theories of gauge fields – as will become very evident in Chapter 7. On the other hand, covariant differentiation in associated bundles can be also introduced directly, namely by using the concept of Koszul connection (Spivak, 1979), defined in terms of covariant differentiation operators satisfying the later derived relations in (5.14). When such an approach is adopted in CGR

[17] Indeed, as pointed out in a review paper by J. Ehlers, claims as to the success of any of these attempts "are based on formal, so far not rigorously justifiable approximation methods." (Ehlers, 1987, p. 65.) Furthermore, such methods cannot deal with the geodesic postulate for light rays, which are supposed to follow null paths.

(Hawking and Ellis, 1973, Sec. 2.5; Straumann, 1984, Sec. 5.1), then those relations are adopted as postulates for a so-called "affine" or "linear" connection ∇ Hence, those readers who find, at a first reading, the subsequent presentation of connections on principal bundles too abstract, can postpone the more careful study of this formulation of the concept of "connection" until they reach Chapter 7, where this approach first becomes truly essential. In the meantime, they can take the four equations in (5.14), and other later introduced basic properties of covariant differentiation operator forms ∇, for granted.

To define connections and parallel transport in principal bundles, let us consider a principal bundle $(\mathbf{P}, \Pi, \mathbf{M}, \mathbf{G})$ over a 4-dimensional manifold $\mathbf{M}$ (although a base manifold $\mathbf{M}$ of any finite dimension could be as easily treated), and let us assume that its structure group $\mathbf{G}$ is a Lie group[18] of dimension n. According to the definition of principal bundles given in Sec. 2.3, each fibre $\Pi^{-1}(x)$ in $\mathbf{P}$ can be obtained by the action of all the elements of $\mathbf{G}$ on any given element of that fibre. Since $\Pi^{-1}(x)$ is a submanifold of the total space $\mathbf{P}$ of that principal bundle, this implies that each fibre $\Pi^{-1}(x)$ is an n-dimensional manifold. Consequently, the space $T_u\Pi^{-1}(x)$ tangent to $\Pi^{-1}(x)$ at a point $\boldsymbol{u} \in \Pi^{-1}(x)$ is an n-dimensional linear space, called in mathematics literature the *vertical space* at $\boldsymbol{u} \in \mathbf{P}$, and usually denoted by $V_u\mathbf{P}$. Moreover, since the base manifold $\mathbf{M}$ is 4-dimensional, $\mathbf{P}$ must be an $(n+4)$-dimensional manifold, so that the space $T_u\mathbf{P}$ tangent to $\mathbf{P}$ at any given $\boldsymbol{u} \in \Pi^{-1}(x)$ is an $(n+4)$-dimensional linear space, which contains $V_u\mathbf{P}$ as a subspace.

A connection on the principal bundle $(\mathbf{P}, \Pi, \mathbf{M}, \mathbf{G})$ is meant to govern the parallel transport of frames along smooth paths γ that lie within its base manifold $\mathbf{M}$. Hence, it is intuitively clear that such transport should take place in $\mathbf{P}$ only along those directions that are specified by vectors $\mathbf{X} \in T_u\mathbf{P}$ which *do not* lie within the vertical space $V_u\mathbf{P}$. On the other hand, since the vectors $\boldsymbol{X}$ tangent to *all* smooth paths γ passing through a given point $x \in \mathbf{M}$ span the entire 4-dimensional linear space $T_x\mathbf{M}$ tangent to $\mathbf{M}$ at that point, the corresponding vectors $\mathbf{X} \in T_u\mathbf{P}$ in whose direction parallel transport *can* take place within the total space $\mathbf{P}$ have to span a 4-dimensional subspace of $T_u\mathbf{P}$. This latter subspace is called in mathematics literature the *horizontal subspace* of $T_u\mathbf{P}$ for that connection, and it is usually denoted by $H_u\mathbf{P}$.

In essence, a connection on the principal bundle $(\mathbf{P}, \Pi, \mathbf{M}, \mathbf{G})$ can be provided by the smooth assignment to each $\boldsymbol{u} \in \mathbf{P}$ of a horizontal space $H_u\mathbf{P}$ in such a manner that we have

$$T_u\mathbf{P} = H_u\mathbf{P} \oplus V_u\mathbf{P} \ , \qquad \boldsymbol{u} \in \mathbf{P} \ , \tag{5.1}$$

and that the following key property holds: if $(R_g)_*$ denotes the push-forward on the vectors in $H_u\mathbf{P}$, defined in accordance with (2.2)-(2.4) by the right action of $\mathbf{G}$ in $\mathbf{P}$,

$$R_g \ : \ \boldsymbol{u} \ \mapsto \ \boldsymbol{u} \cdot g \ , \qquad g \in \mathbf{G} \ , \tag{5.2a}$$

then all horizontal spaces are also "pushed forward" along each fibre in such a manner that

$$(R_g)_* H_u\mathbf{P} = H_{u \cdot g}\mathbf{P} \ , \qquad \boldsymbol{u} \in \mathbf{P} \ , \ g \in \mathbf{G} \ , \tag{5.2b}$$

[18] The reader is assumed to be familiar with the basics of the theory of Lie groups and Lie algebras, at least to the extent that this theory is explained in (Isham, 1989), (Nakahara, 1990), or other similar mathematical textbooks aimed at physicists. A rigorous and more detailed presentation can be found in (Spivak, 1979).

i.e., so that, while $g \in \mathbf{G}$ maps each $\boldsymbol{u} \in \Pi^{-1}(x)$ into $\boldsymbol{u} \cdot g \in \Pi^{-1}(x)$, it also "pushes" along the fibre $\Pi^{-1}(x)$ each "horizontal" vector in $T_{\boldsymbol{u}}\mathbf{P}$ into a "horizontal" vector in $T_{\boldsymbol{u}\cdot g}\mathbf{P}$.

There is an infinity of ways in which horizontal spaces can be assigned to a principal bundle in such a manner that the above conditions are satisfied. Hence, there is an infinity of connections that can be introduced in a principal bundle $(\mathbf{P}, \Pi, \mathbf{M}, \mathbf{G})$. Each one of these connections can be characterized by a *connection form* $\boldsymbol{\omega}$, determined at each $\boldsymbol{u} \in \mathbf{P}$ by a map $\boldsymbol{\omega}_{\boldsymbol{u}} : T_{\boldsymbol{u}}\mathbf{P} \to L(\mathbf{G})$, which assigns to each vector $\mathbf{X} \in T_{\boldsymbol{u}}\mathbf{P}$ an element $\boldsymbol{\omega}(\mathbf{X})$ of the Lie algebra $L(\mathbf{G})$ of the structure group $\mathbf{G}$ in such a manner that all horizontal vectors $\mathbf{X} \in H_{\boldsymbol{u}}\mathbf{P}$ are mapped into the zero vector of that algebra, whereas the vertical vectors $\mathbf{X} \in V_{\boldsymbol{u}}\mathbf{P}$ are mapped linearly into the generically non-zero vectors of a fundamental field[19].

The Lie algebra $L(\mathbf{G})$ of any Lie group $\mathbf{G}$ can be identified, when viewed merely as a linear space, with the space $T_{\boldsymbol{e}}\mathbf{G}$ tangent to $\mathbf{G}$ at the unit element $\boldsymbol{e}$ of the group $\mathbf{G}$. Hence, we can expand the values $\boldsymbol{\omega}(\mathbf{X})$ assumed by the connection form $\boldsymbol{\omega}$ in any vector basis of $T_{\boldsymbol{e}}\mathbf{G}$, which in view of the fact that $\mathbf{G}$ is an n-dimensional Lie group, and that $L(\mathbf{G})$ is n-dimensional Lie algebra, must consist of n linearly independent vectors in $T_{\boldsymbol{e}}\mathbf{G}$:

$$\boldsymbol{\omega}(\mathbf{X}) = \boldsymbol{\omega}^{a}(\mathbf{X})\mathcal{Y}_{a} , \qquad \mathcal{Y}_{a} \in T_{\boldsymbol{e}}\mathbf{G} , \qquad a = 1,\ldots,n = dim\,\mathbf{G} . \tag{5.3}$$

In view of the linearity of the map $\boldsymbol{\omega}_{\boldsymbol{u}} : T_{\boldsymbol{u}}\mathbf{P} \to L(\mathbf{G})$, the coefficients in this expansion give rise to the following n linear functionals on $T_{\boldsymbol{u}}\mathbf{P}$,

$$\boldsymbol{\omega}^{a} : \mathbf{X} \mapsto \mathrm{X}^{a} = \boldsymbol{\omega}^{a}(\mathbf{X}) \in \mathbf{R}^{1}, \qquad \mathbf{X} \in T_{\boldsymbol{u}}\mathbf{P} , \tag{5.4}$$

which we shall call the *Cartan connection forms* associated with the chosen basis in the Lie algebra of the structure group $\mathbf{G}$. Indeed, if

$$\boldsymbol{s} : x \mapsto \boldsymbol{u} \in \Pi^{-1}(x) , \qquad x \in \mathbf{M}^{\boldsymbol{s}} \subset \mathbf{M} , \tag{5.5}$$

is a section of the principal bundle $(\mathbf{P}, \Pi, \mathbf{M}, \mathbf{G})$, i.e., a moving frame in Cartan's (1935) sense in the case where that principal bundle consists of frames, then the push-forward

$$\mathbf{X} = \boldsymbol{s}_{*}\boldsymbol{X} \in T_{\boldsymbol{s}(x)}\mathbf{P} , \qquad \boldsymbol{X} \in T_{x}\mathbf{M} , \tag{5.6}$$

of any vector $\boldsymbol{X}$ tangent to $\mathbf{M}$ is a vector tangent to $\mathbf{P}$, so that the Cartan connection forms on $T\mathbf{P}$ determine the following *connection one-forms* on $T\mathbf{M}$:

$$\boldsymbol{\omega}^{a}_{\boldsymbol{s}} : \boldsymbol{X} \mapsto \boldsymbol{\omega}^{a}(\boldsymbol{s}_{*}\boldsymbol{X}) \in \mathbf{R}^{1} , \qquad \boldsymbol{X} \in T\mathbf{M} , \tag{5.7}$$

In turn, these one-forms determine the covariant differentiation operators

$$\nabla_{\boldsymbol{X}} = \partial_{\boldsymbol{X}} + \boldsymbol{\omega}^{a}_{\boldsymbol{s}}(\boldsymbol{X})\mathbf{A}_{a;\boldsymbol{s}(x)} , \qquad \boldsymbol{X} \in T_{x}\mathbf{M} , \quad x \in \mathbf{M}^{\boldsymbol{s}} , \tag{5.8}$$

in the direction of each such $\boldsymbol{X}$, by a procedure whose main steps we shall now explain to the extent necessary for the understanding of the significance of the remaining terms in

[19] Cf. Secs. 2.4 and 2.5 of (Prugovečki, 1992) for the construction of that field, and for a more detailed survey of the theory of connections described in this section. Details of all the proofs and constructions given in this section can be found in such textbooks as (Choquet-Bruhat *et al.*, 1987; Isham, 1989; Spivak, 1979), which also contain the required basic definitions of and constructions of connection forms, horizontal lifts, etc.

(5.8) (cf. the references in Footnotes 17 and 18 for further details), and whose precise mathematical meaning will eventually emerge from (5.16a) and (5.17).

After defining a connection by assigning to the tangent space $T_{\boldsymbol{u}}\mathbf{P}$ above each $\boldsymbol{u} \in \mathbf{P}$ a horizontal subspace $H_{\boldsymbol{u}}\mathbf{P}$, one can define parallel transport in $\mathbf{P}$ along any smooth curve $\gamma = \{ x(t) | \, a \leq t \leq b \}$, with initial point $x' = x(a) \in \mathbf{M}$ and final point $x'' = x(b) \in \mathbf{M}$, by using a "lifting" procedure based on the fact that each vector $\boldsymbol{X}(t)$ tangent to γ at $x(t)$ possesses a unique horizontal lift $\boldsymbol{X}^*(t) \in H_{\boldsymbol{u}}\mathbf{P}$ at each $\boldsymbol{u} \in \boldsymbol{\Pi}^{-1}(x(t))$. Due to this fact, if we choose *any* point $\boldsymbol{u}_{x'} \in \boldsymbol{\Pi}^{-1}(x')$ in the fibre above the initial point $x' \in \mathbf{M}$ of γ, it can be established that there is a unique curve $\gamma^* = \{ \boldsymbol{u}(t) | \, a \leq t \leq b \}$ within the total space $\mathbf{P}$ of the principal bundle, called a *horizontal lift of* γ, which starts at $\boldsymbol{u}_{x'}$, and whose tangent at each $t \in [a,b]$ is a horizontal lift $\boldsymbol{X}^*(t) \in H_{\boldsymbol{u}}\mathbf{P}$ of $\boldsymbol{X}(t) \in T_{x(t)}\mathbf{M}$.

The end-point value $\boldsymbol{u}_{x''} = \boldsymbol{u}(b)$ assumed by that horizontal lift γ^* of γ is called the *parallel transport* of $\boldsymbol{u}_{x'}$ from x' to x'' along the smooth curve γ. By applying this "lifting" procedure to derive *all* the horizontal lifts of γ whose initial points $\boldsymbol{u}_{x'}$ belong to the fibre $\boldsymbol{\Pi}^{-1}(x')$, we obtain from the values $\boldsymbol{u}_{x''}$ at their end-points the diffeomorphism

$$\boldsymbol{\tau}_{\gamma}(x'',x') \; : \; \boldsymbol{\Pi}^{-1}(x') \; \rightarrow \; \boldsymbol{\Pi}^{-1}(x'') \; , \tag{5.9}$$

between the fibres of the principal bundle $(\mathbf{P}, \boldsymbol{\Pi}, \mathbf{M}, \mathbf{G})$ situated above the initial and final points of the chosen curve γ. This diffeomorphism provides, within the total space of that principal bundle, a complete description of the outcome of the operation of parallel transport along γ governed by the connection determined by (5.1). The corresponding parallel transport operation from x' to x'' along the same curve γ,

$$\tau_{\gamma}(x'',x') \; : \; \boldsymbol{\pi}^{-1}(x') \; \rightarrow \; \boldsymbol{\pi}^{-1}(x'') \; , \tag{5.10}$$

taking place within any fibre bundle $(\mathbf{E}, \boldsymbol{\pi}, \mathbf{M}, \mathbf{F}, \mathbf{G})$ which is associated to the principal bundle $(\mathbf{P}, \boldsymbol{\Pi}, \mathbf{M}, \mathbf{G})$, is then completely determined by the fact that to any family of local trivializations (4.1) of that principal bundle is associated a corresponding family of local trivializations (4.4) of that associated bundle, and by the fact that (4.8) holds true.

This is especially easy to see in those cases which are of exclusive interest in this and following chapters, namely where $(\mathbf{E}, \boldsymbol{\pi}, \mathbf{M}, \mathbf{F}, \mathbf{G})$ is a vector bundle (so that its typical fibre $\mathbf{F}$ is a vector space), and $(\mathbf{P}, \boldsymbol{\Pi}, \mathbf{M}, \mathbf{G})$ is a principal frame bundle from whose elements $\boldsymbol{u} \in \boldsymbol{\Pi}^{-1}(x')$ vector bases can be built within the fibres $\boldsymbol{\pi}^{-1}(x)$ of $\mathbf{E}$ – such as in the case with a tensor bundle $T^{r,s}\mathbf{M}$ associated to $GL\mathbf{M}$. Indeed, in such a setting the parallel transport[20] described in (5.10) can be performed by expanding the vectors within $\boldsymbol{\pi}^{-1}(x')$ in a basis constructed from the elements of $\boldsymbol{u}_{x'}$, so that the parallel transport of those vectors can be carried out by parallel transporting the elements of that basis while keeping the coefficients in that expansion constant. For example, if we take the general linear frame bundle $GL\mathbf{M}$ as our choice for principal frame bundle, so that (5.9) becomes

$$\boldsymbol{\tau}_{\gamma}(x'',x') \; : \; \{\boldsymbol{e}'_i\} \; \mapsto \; \{\boldsymbol{e}''_i\} \in \; \boldsymbol{\Pi}^{-1}(x'') \; , \qquad \{\boldsymbol{e}'_i\} \in \boldsymbol{\Pi}^{-1}(x') \subset GL\mathbf{M} \; , \tag{5.11}$$

[20] Note that the parallel transport of a frame automatically determines the parallel transport of its coframe since, by definition, the parallel transported coframe is the dual of the parallel transported frame.

and if we consider the tangent bundle $T\mathbf{M}$ as associated to $GL\mathbf{M}$, then (5.10) is given by

$$\tau_\gamma(x'',x') : \quad \boldsymbol{X} = X^i \boldsymbol{e}_i' \;\mapsto\; X^i \boldsymbol{e}_i'' \in T_{x''}\mathbf{M} \;, \qquad \boldsymbol{X} \in T_{x'}\mathbf{M} \;, \tag{5.12}$$

for *any* choice of linear frame in $T_{x'}\mathbf{M}$.

With the parallel transport within an associated vector bundle $(\mathbf{E}, \pi, \mathbf{M}, \mathbf{F}, \mathbf{G})$ thus defined, we can define the covariant derivative of any section Ψ of that associated bundle along any vector $\boldsymbol{X} \in T_x\mathbf{M}$ tangent to $\mathbf{M}$ at some chosen point $x \in \mathbf{M}$, where Ψ assumes the value $\Psi_x \in \pi^{-1}(x)$. This can be achieved by choosing any smooth curve γ that passes through x, and whose tangent at x equals the given vector $\boldsymbol{X}$, and then choosing any parametrization of that curve which assumes the value 0 at x, so that as we take the following limit,

$$\nabla_{\boldsymbol{X}} \Psi_x = \lim_{t\to 0} \frac{1}{t}\left[\tau_\gamma(x,x(t))\Psi_{x(t)} - \Psi_x\right] \;, \qquad x = x(0) \in \mathbf{M} \;, \tag{5.13}$$

$x(t)$ approaches x. The above limiting procedure produces an element of the fibre $\pi^{-1}(x)$ above x, called the *covariant derivative* of Ψ along the tangent vector $\boldsymbol{X}$ above $x \in \mathbf{M}$.

It is to be noted that, in physical theories employing fibre bundles, a section Ψ of a vector bundle is usually referred to as a *vector field*. It is interesting to observe that if the base manifold $\mathbf{M}$ of such a bundle were a vector space (such as the Minkowski space), and if the parallel transport between any two points in $\mathbf{M}$ were independent of the path (as is the case in flat manifolds – cf. Sec. 2.6), then the above covariant derivative would coincide with the directional derivative defined in calculus, since then parallel transport could be viewed as parallel translation within $\mathbf{M}$. It is the dependence of parallel transport upon the chosen path in case of "non-flat" connections that makes the difference between the ordinary directional differentiation of vector fields and their covariant differentiation.

It is now a standard task, whose solution is presented in many textbooks (Isham, 1989, p. 173; Nakahara, 1990, p. 347; Spivak, 1979, p. 368), to prove that the covariant derivative defined by (5.13) possesses the following fundamental properties at all points $x \in \mathbf{M}$ within the common domain of definition of any smooth function f, of any two vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ assuming values in $T\mathbf{M}$, and of any three vector fields Ψ, Ψ' and Ψ'' assuming values in the total space $\mathbf{E}$ of some associated bundle (such as a tensor bundle $T^{r,s}\mathbf{M}$, which might be associated to $GL\mathbf{M}$, or to some Lorentz frame bundle $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$):

$$\nabla_{\boldsymbol{X}+\boldsymbol{Y}} \Psi = \nabla_{\boldsymbol{X}} \Psi + \nabla_{\boldsymbol{Y}} \Psi \;, \tag{5.14a}$$

$$\nabla_{\boldsymbol{X}}(\Psi' + \Psi'') = \nabla_{\boldsymbol{X}} \Psi' + \nabla_{\boldsymbol{X}} \Psi'' \;, \tag{5.14b}$$

$$\nabla_{f\boldsymbol{X}} \Psi = f\, \nabla_{\boldsymbol{X}} \Psi \;, \tag{5.14c}$$

$$\nabla_{\boldsymbol{X}}(f\Psi) = f\, \nabla_{\boldsymbol{X}} \Psi + \boldsymbol{X}(f)\, \Psi \;. \tag{5.14d}$$

On the other hand, as mentioned earlier in this section, the above relations can be adopted in tensor bundles $T^{r,s}\mathbf{M}$ as postulates. In conjunction with the Leibniz rule for tensor products of tensor fields, and with the commutativity of $\nabla_{\boldsymbol{X}}$ with contractions of such fields, these postulates define a *covariant differentiation operator-form* $\nabla : \boldsymbol{X} \mapsto \nabla_{\boldsymbol{X}}$. In

physics literature this ∇ is often called a "connection" on $\mathbf{E}$ – or an "affine connection," or a "linear connection"; whereas, in mathematics literature it is sometimes called a "Koszul connection" – cf. (Spivak, 1979). Such a connection given directly on a vector bundle $\mathbf{E}$ can be defined as a map which takes any pair (Ψ, X) of fields Ψ and X from $\mathbf{E}$ and $T\mathbf{M}$, respectively, into vector fields $\nabla_X \Psi$ in $\mathbf{E}$ (Hawking and Ellis, 1973; Straumann, 1984). When dealing, from Chapter 4 onwards, with "quantum connections" on quantum bundles which are Hilbert or pseudo-Hilbert bundles (and therefore vector bundles, albeit with infinite dimensional fibres), we shall often follow this custom.

The computation of covariant derivatives proceeds in practice via the formula (5.8). To derive this formula, let us use the connection one-forms in (5.7) in the computation of the covariant derivative (5.13) of a vector field Ψ. Since $(\mathbf{P}, \Pi, \mathbf{M}, \mathbf{G})$ was assumed to be a principal frame bundle, its elements $\boldsymbol{u} \in \mathbf{P}$ supply associated frames $\{\mathbf{V}_\rho(\boldsymbol{u})\}$ within all the fibres $\pi^{-1}(x)$ of $\mathbf{E}$ – such as the tensorial frames in (2.12), associated with the vector frame in (2.7) that belongs to $GL\mathbf{M}$, or to $L\mathbf{M}(g^{\mathrm{L}})$, so that in both those cases ρ is actually a multi-component index. Each one of the values $\Psi_x \in \pi^{-1}(x)$ assumed by the vector field Ψ can be then expanded in a corresponding associated frame $\{\mathbf{V}_\rho(\boldsymbol{u})\}$,

$$\Psi_x = \Psi^\rho_{x;\boldsymbol{u}} \mathbf{V}_\rho(\boldsymbol{u}) \in \pi^{-1}(x) , \qquad \boldsymbol{u} \in \Pi^{-1}(x) , \tag{5.15a}$$

and the same can be done with the parallel transported vector in (5.13):

$$\tau_\gamma(x, x(t))\Psi_{x(t)} = \Psi^\rho_{x(t);\boldsymbol{u}(t)} \mathbf{V}_\rho\big(\tau_\gamma(x, x(t))\boldsymbol{u}(t)\big) , \qquad \boldsymbol{s}: \ x(t) \mapsto \boldsymbol{u}(t) . \tag{5.15b}$$

It then follows from (5.3)-(5.7) that

$$\lim_{t\to 0} \frac{1}{t}\Big[\mathbf{V}_\rho\big(\tau_\gamma(x, x(t))\boldsymbol{u}(t)\big) - \mathbf{V}_\rho(\boldsymbol{u})\Big] = \boldsymbol{\omega}^\alpha_{\boldsymbol{s}}(X)\mathbf{A}_{\alpha;\boldsymbol{u}}\mathbf{V}_\rho(\boldsymbol{u}) , \qquad \boldsymbol{u} = \boldsymbol{u}(0) , \tag{5.16a}$$

where $\mathbf{A}_{\alpha;\,\boldsymbol{u}}$ are the infinitesimal generators[21] of the representation U_x of $\mathbf{G}$ acting within the fibre $\pi^{-1}(x)$ of the vector bundle $(\mathbf{E}, \pi, \mathbf{M}, \mathbf{F}, \mathbf{G})$. This representation is related by the restriction $\phi_\alpha(x) : u \mapsto f$ of $\boldsymbol{\phi}_\alpha : u \mapsto (x, f)$ in (4.1) to the representation U of $\mathbf{G}$ that acts within the typical fibre $\mathbf{F}$ of that vector bundle as follows:

$$U_x = \phi^{-1}_\alpha(x) \circ U \circ \phi_\alpha(x) : \ \pi^{-1}(x) \to \pi^{-1}(x) , \qquad U: \ \mathbf{F} \to \mathbf{F} . \tag{5.16b}$$

Hence, these generators implicitly depend on the choice of section $\boldsymbol{s}$ in (5.5) (cf. Sec. 2.6 for specific examples). Upon combining (5.16) with (5.14) by means of the same type of procedure as in deriving in ordinary calculus the formula for the derivative of the product of two functions, we obtain the formula in (5.8), and establish in the process that

$$\partial_X \Psi_x = \Big[X^\mu \partial_\mu\big(\Psi^\rho_{x;\boldsymbol{u}}\big)\Big] \mathbf{V}_\rho(\boldsymbol{u}) \in \pi^{-1}(x) . \tag{5.17}$$

A corresponding formula for the covariant differentiation operator form ∇ (namely connection, in physics terminology) can be written in the following form,

---

[21] Note that the generators of a representation U_x of G coincide with the elements of the Lie algebra $L(U_x)$ when that representation is itself viewed as being a Lie group.

$$\nabla = \boldsymbol{d} + \boldsymbol{A}^{\boldsymbol{s}} \ , \tag{5.18a}$$

for any vector bundle $\mathbf{E}$. The simplicity of the above equation underlines the fact that the obtained formulae for covariant differentiation are valid at all points $\boldsymbol{x} \in \mathbf{M}$ in the domain of definition of a vector field $\boldsymbol{\Psi}$, as well as for all vectors $\boldsymbol{X} \in \boldsymbol{T}\mathbf{M}$ tangent to $\mathbf{M}$ at those points. In the above expression $\boldsymbol{d}$ and $\boldsymbol{A}^{\boldsymbol{s}}$ are regarded as operator-valued forms on the tangent bundle $\boldsymbol{T}\mathbf{M}$, and are given by the following mappings:

$$\boldsymbol{d} = \boldsymbol{dx}^{\mu}\partial_{\mu} \ : \ \boldsymbol{X} \mapsto \boldsymbol{dx}^{\mu}(\boldsymbol{X})\partial_{\mu} = X^{\mu}\partial_{\mu} \ , \tag{5.18b}$$

$$\boldsymbol{A}^{\boldsymbol{s}} = \boldsymbol{\omega}_{\boldsymbol{s}}^{\alpha}\mathbf{A}_{\alpha;\boldsymbol{s}} \ : \ \boldsymbol{X} \mapsto \boldsymbol{\omega}_{\boldsymbol{s}}^{\alpha}(\boldsymbol{X})\mathbf{A}_{\alpha;\boldsymbol{s}} \ . \tag{5.18c}$$

In physics literature, the operator-valued form $\boldsymbol{A}^{\boldsymbol{s}}$ is called the *gauge potential* corresponding to the choice (5.5) of section $\boldsymbol{s}$ of the principal frame bundle $(\mathbf{P}, \boldsymbol{\Pi}, \mathbf{M}, \mathbf{G})$; that section $\boldsymbol{s}$ is therefore called a *local gauge choice* – in full accordance with the use in Sec. 2.4 of the term "local gauge" for a choice of a local trivializations $\boldsymbol{\Phi}_{\alpha}$ given by (4.7). Hence, the transition from one gauge potential to another

$$\boldsymbol{A}^{\boldsymbol{s}} \ \mapsto \ \boldsymbol{A}^{\boldsymbol{s}'} \ , \qquad \boldsymbol{s}' \ : \ x \mapsto \boldsymbol{u}' \in \boldsymbol{\Pi}^{-1}(x) \ , \qquad x \in \mathbf{M}^{\boldsymbol{s}'} \ , \tag{5.19}$$

resulting from a change of gauge choice (i.e., from a choice of section of the principal frame bundle) is referred to as a *gauge transformation.*

In view of the fact that the action of the structure group $\mathbf{G}$ on each fibre $\boldsymbol{\Pi}^{-1}(x)$, of the principal frame bundle $(\mathbf{P}, \boldsymbol{\Pi}, \mathbf{M}, \mathbf{G})$ is transitive, any two elements within that fibre can be related by the action of an element of $\mathbf{G}$, which maps one into the other. Hence, any two such local gauge choices $\boldsymbol{s}$ and $\boldsymbol{s}'$ are related by means of a map

$$g \ : \ x \ \mapsto g(x) \in \mathbf{G} \ , \qquad \boldsymbol{s}'(x) = \boldsymbol{s}(x) \cdot g(x) \ , \qquad x \ \in \mathbf{M}^{\boldsymbol{s}} \cap \mathbf{M}^{\boldsymbol{s}'} \ , \tag{5.20}$$

defined within the common domain $\mathbf{M}^{\boldsymbol{s}} \cap \mathbf{M}^{\boldsymbol{s}'}$ of the two gauge choices – which obviously coincides with the common domain of definition of the two gauge potentials in (5.19). From (5.2) and (5.16)-(5.18) then follows the well-known formula[22]

$$\boldsymbol{A}^{\boldsymbol{s}'} = U(g^{-1})\boldsymbol{d}U(g) + U(g^{-1})\boldsymbol{A}^{\boldsymbol{s}}U(g) \ , \qquad \boldsymbol{d}U(g) = \left(\partial U / \partial g^{\alpha}\right)\boldsymbol{dg}^{\alpha} \ , \tag{5.21}$$

relating any two gauge potentials of a given connection within their common domain of definition. In fact, for a given family of local trivializations (4.4) satisfying the compatibility relations in (4.6), and of corresponding gauge choices represented by sections of the principal bundle, the coefficients of a connection can be constructed by piecing together with the help of the gauge transformations in (5.20) the gauge potentials in (5.21).

In view of the role played in a physics context, such as the theory of Yang-Mills fields or CGR, by the structure group $\mathbf{G}$ in (5.20), $\mathbf{G}$ is called by some authors (Choquet-

[22] These relations are usually derived in the context of classical Yang-Mills theories for which $U(g) = g$ (cf. (Choquet-Bruhat *et al.*, 1987, pp. 402-407), or (Isham, 1989, p. 161), or (Nakahara, 1990, p. 335), but those derivations hold equally well for the kind of faithful representations of the structure group (i.e., gauge group of the first kind) $\mathbf{G}$ that will be encountered in the later chapters of this monograph.

Bruhat *et al.*, 1987; Drechsler and Mayer, 1977) the "gauge group" in the corresponding principal bundle $(\mathbf{P}, \Pi, \mathbf{M}, \mathbf{G})$; however, other authors (Bleecker, 1981; Isham, 1989) use that same term for a certain subgroup of the group of all automorphisms of that principal bundle. As we shall see in Chapter 7, the two definitions are closely related, but they are by no means equivalent. Hence, when we encounter the structure group $\mathbf{G}$ in a physics context, we shall refer to it as a *gauge group of the first kind*, and call the aforementioned subgroup of automorphisms a *gauge group of the second kind*.

2.6. Levi-Civita Connections and the Strong Equivalence Principle

We saw in Sec. 2.2 that in his later years Einstein explicitly formulated the following fundamental general relativistic tenet: "There is no such thing as empty space, *i.e.* space without a [metric] field. Space-time does not claim existence on its own, but only as a structural quality of the [metric] field." (Einstein, 1961, p. 155.)

Amongst other things, this implies that the purely mathematical concept of connection on the general linear frame bundle $GL\mathbf{M}$ is of no direct physical relevance. It is only the presence of a Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ on the base manifold $\mathbf{M}$ of that principal bundle, which mathematically gives rise to the *Lorentzian manifold* $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$, that imparts the linear frames $\boldsymbol{u}$ in $GL\mathbf{M}$ with metric qualities, so that one can interpret them as classical local frames of reference, in the manner described in Sec. 2.2. It is therefore clear that no connection on $GL\mathbf{M}$ which does not preserve those metric qualities under its parallel transport can be of physical relevance. In other words, in order to be a candidate for a *physical* connection, the very first requirement made of a connection on $GL\mathbf{M}$, with connection form ω, is that it be compatible with the Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ on $\mathbf{M}$ provided by some CGR physical model[23] of spacetime structure.

In a purely mathematical context, a connection on $GL\mathbf{M}$ is said to be *compatible* with a (in general Riemannian or pseudo-Riemannian) metric field $\boldsymbol{g}$ on $\mathbf{M}$ if *all* the covariant derivatives of that metric field are zero. In other words, if ∇ denotes the covariant differentiation operator forms to which that connection gives rise, in accordance to (5.3)-(5.8), on the various tensor bundles $T^{r,s}\mathbf{M}$ associated to $GL\mathbf{M}$, then $\nabla \boldsymbol{g} \equiv 0$. There is, however, a mathematically equivalent but physically more significant and more direct way of defining such compatibility: a connection on $GL\mathbf{M}$ is compatible with a metric $\boldsymbol{g}$ given on $\mathbf{M}$ if and only if any linear frame which is orthonormal in that metric $\boldsymbol{g}$ remains orthonormal after being parallel transported in any manner possible for that connection. This means that, after the reduction of $GL\mathbf{M}$ to the principal bundle $L\mathbf{M}(\boldsymbol{g})$ of frames that are orthonormal in $\boldsymbol{g}$ is performed, all the operators for parallel transport, such as those in (5.9), leave $L\mathbf{M}(\boldsymbol{g})$ invariant if they are generated by that connection.

On the other hand, we saw in Sec. 2.2 that, at a purely physical level, the presence of a Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ mediates the metrization of local classical frames represented by the elements of the principal bundle $GL\mathbf{M}$. Such a metrization produces operationally the

[23] Due to the complexity of the *physically existing* spacetime, none of the theoretical structures encountered in CGR textbooks or articles can be claimed to provide anything more than *models* which represent only very rough approximations of the *actual* spacetime structures that they are meant to describe macroscopically. For example, the models based on Robertson-Walker metrics that supply the standard Big Bang model of our universe require all matter and radiation to be spatially uniformly distributed, despite the actual presence of various galactic structures.

reduction of that bundle to the principal bundle $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$, by the process of selection of local Lorentz frames from amongst all conceivable local classical frames over a spacetime manifold. Hence, it is advantageous physically as well as mathematically to formulate a connection compatible with the metric $\boldsymbol{g}^{\mathrm{L}}$ directly on the bundle $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$.

On physical grounds, *we shall assume everywhere in the sequel* that the Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ is connected and space-time orientable (Straumann, 1984; Wald, 1984), so that we can cover $\mathbf{M}$ with future-oriented and spatially right-handed moving frames. Since the physical parallel transport of a classical Lorentz frame obviously does not reverse the orientation of its spatial axes, or the direction in which its standard clocks are running, this means that we can restrict our attention to the subbundle $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ of $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ consisting of those orthonormal frames $\{\boldsymbol{e}_i \mid i = 0,1,2,3\}$ whose $\boldsymbol{e}_0$-vectors all point in the future direction, and whose spatial triples $\{\boldsymbol{e}_a \mid a = 1,2,3\}$ are all right-handed. The structure group of $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ equals the subgroup $\mathrm{SO}_0(3,1)$ of the *special orthogonal group* $\mathrm{SO}(3,1)$ of all 4×4 real matrices which preserve the Minkowski metric in $\mathbf{R}^4$ and have a determinant equal to one, so that it is a subgroup of the orthogonal group $\mathrm{O}(3,1)$ in (3.4). In physical terms the group $\mathrm{SO}_0(3,1)$ represents the *restricted* Lorentz group (also known as the *proper orthochronous* Lorentz group – cf. Ohnuki, 1976, p. 5; or simply the *proper* Lorentz group – cf. Kim and Noz, 1986, p. 52), and it is the largest subgroup of $\mathrm{SO}(3,1)$ which does not incorporate any space and/or time reflections (cf. Cornwell, 1984, p. 66).

The Lie group $\mathrm{SO}_0(3,1)$ is cast into the role of gauge group of the first kind upon adopting $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ as a principal frame bundle. Its Lie algebra $so(3,1)$ constitutes a six-dimensional vector space, identifiable with the tangent space to this Lie group $\mathrm{SO}_0(3,1)$ at its unit element. From the physics point of view, it is most convenient to express the covariant derivatives for any connection compatible with the metric $\boldsymbol{g}^{\mathrm{L}}$ on $\mathbf{M}$ in a basis for $so(3,1)$ consisting of the six generators of Lorentz "rotations" in each of the (ij)-planes, so that (5.3) assumes the following form:

$$\boldsymbol{\omega}(\mathbf{X}) = \boldsymbol{\omega}^{ij}(\mathbf{X})\mathcal{Y}_{ij}\,, \qquad \mathcal{Y}_{ij} \in T_e\mathrm{SO}_0(3,1)\,, \qquad j > i = 0,1,2,3 \tag{6.1a}$$

It has, however, become customary to use the Minkowski metric components η_{ij} to raise and lower indices, and to exploit the antisymmetry of those generators under the exchange of (ij)-indices, in order to re-write (6.1a) in the form (Drechsler, 1984)

$$\boldsymbol{\omega}(\mathbf{X}) = \tfrac{1}{2}\boldsymbol{\omega}_{ij}(\mathbf{X})\mathcal{Y}^{ij}\,, \qquad \boldsymbol{\omega}_{ji} = -\boldsymbol{\omega}_{ij}\,. \tag{6.1b}$$

Let us consider now any vector bundle $(\mathbf{E}, \boldsymbol{\pi}, \mathbf{M}, \mathbf{F})$ associated (in the sense defined in Sec. 2.4) to the restricted Lorentz frame bundle $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ by some representation U of $\mathrm{SO}_0(3,1)$ in the typical fibre $\mathbf{F}$ of that vector bundle. By definition, a vierbein gauge (cf. Sec. 2.4) can be fixed in $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ by making a choice of a section $\boldsymbol{s}$ of $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$. In view of (6.1), the covariant differentiation operators in (5.8) assume in such a gauge the form

$$\nabla_{\boldsymbol{X}} = \partial_{\boldsymbol{X}} + \tfrac{1}{2}\boldsymbol{\omega}^{\boldsymbol{s}}_{ij}(\boldsymbol{X})\hat{\mathbf{M}}^{ij}_{\boldsymbol{s}}\,, \qquad \hat{\mathbf{M}}^{ji}_{\boldsymbol{s}} = -\hat{\mathbf{M}}^{ij}_{\boldsymbol{s}}\,, \tag{6.2}$$

where $\hat{\mathbf{M}}^{ji}_{\boldsymbol{s}}$ are the infinitesimal generators of the Lorentz transformations describing the "infinitesimal rotations" of the (ij)-axes of the local Lorentz frame $\boldsymbol{u} = \boldsymbol{s}(\boldsymbol{x})$ above the point

$x \in \mathbf{M}$ at which the covariant derivative is computed. In particular, the above formula (6.2) can be used in any tensor bundle $T^{r,s}\mathbf{M}$ associated to $L\mathbf{M}_0(\boldsymbol{g}^L)$ but, as we shall see in Chapter 4, its significance actually transcends such bundles with finite-dimensional fibres.

The connections on $GL\mathbf{M}$ compatible with a metric $\boldsymbol{g}$ on $\mathbf{M}$ constitute a very large class, whose elements are sometimes referred to as *Riemann-Cartan connections* (cf. Hehl *et al.*, 1976; Drechsler, 1984). To single out the connection used by Einstein in formulating CGR, we have to impose the additional condition that the connection on $GL\mathbf{M}$ should be torsion-free, namely that its torsion operator,

$$\mathbf{T}(\boldsymbol{X},\boldsymbol{Y}) = \nabla_{\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\boldsymbol{Y}}\boldsymbol{X} - [\boldsymbol{X},\boldsymbol{Y}] \ , \tag{6.3a}$$

should be equal to zero for any two vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\mathbf{M}$. This requirement, which is obviously equivalent to the relation

$$\nabla_{\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\boldsymbol{Y}}\boldsymbol{X} = [\boldsymbol{X},\boldsymbol{Y}] := \boldsymbol{XY} - \boldsymbol{YX} \ , \tag{6.3b}$$

implies,[24] in conjunction with the compatibility requirement $\nabla\boldsymbol{g} \equiv \mathbf{0}$, the Ricci identity

$$\boldsymbol{X}\big(\boldsymbol{g}(\boldsymbol{Y},\boldsymbol{Z})\big) = \boldsymbol{g}(\nabla_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}) + \boldsymbol{g}(\boldsymbol{Y},\nabla_{\boldsymbol{X}}\boldsymbol{Z}) \ , \tag{6.4}$$

for any three vector fields $\boldsymbol{X},\boldsymbol{Y}$ and $\boldsymbol{Z}$ assuming values in $T\mathbf{M}$, and defined within some common domain in $\mathbf{M}$. Hence, upon algebraically combining (6.4) with the two equations obtained by cyclically permuting in $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ in that equation, we arrive at the following very important result:

$$\begin{aligned}\boldsymbol{g}(\nabla_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}) = {} & \tfrac{1}{2}\big[\boldsymbol{X}(\boldsymbol{g}(\boldsymbol{Y},\boldsymbol{Z})) + \boldsymbol{Y}(\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Z})) - \boldsymbol{Z}(\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y}))\big] \\ & + \tfrac{1}{2}\big[\boldsymbol{g}(\boldsymbol{Z},[\boldsymbol{X},\boldsymbol{Y}]) - \boldsymbol{g}(\boldsymbol{X},[\boldsymbol{Y},\boldsymbol{Z}]) - \boldsymbol{g}(\boldsymbol{Y},[\boldsymbol{X},\boldsymbol{Z}])\big] \ .\end{aligned} \tag{6.5}$$

The above result implies the well-known *fundamental lemma of (pseudo-)Riemannian geometry*, namely that on any given pseudo-Riemannian manifold $(\mathbf{M},\boldsymbol{g})$ there is a *unique* torsion-free connection on the linear frame bundle $GL\mathbf{M}$ which is compatible with a given metric $\boldsymbol{g}$ on $\mathbf{M}$ – called the *Levi-Civita connection* or the *Riemannian connection* on $(\mathbf{M},\boldsymbol{g})$.

We can reconfirm this very fundamental result by inserting $\boldsymbol{X} = \boldsymbol{e}_i$, $\boldsymbol{Y} = \boldsymbol{e}_j$ and $\boldsymbol{Z} = \boldsymbol{e}_k$ into (6.5), where $\boldsymbol{u} = (\boldsymbol{e}_0,\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3)$ are the linear frames belonging to the chosen moving frame (5.5) in $GL\mathbf{M}$. We then obtain expressions which uniquely relate the metric components g_{ij} to the connection coefficients in that chosen moving frame in $GL\mathbf{M}$ (also known as *Christoffel symbols* in the case where all linear frames in that moving frame are holonomic, i.e., with elements as in (1.6)):

$$\Gamma^i{}_{jk} = \boldsymbol{\theta}^i\big(\nabla_{\boldsymbol{e}_j}\boldsymbol{e}_k\big) = g^{il}\,\boldsymbol{g}(\boldsymbol{e}_l,\nabla_{\boldsymbol{e}_j}\boldsymbol{e}_k) \ , \quad g_{jk} = \boldsymbol{g}(\boldsymbol{e}_j,\boldsymbol{e}_k) \ , \quad g^{ij}g_{jk} = \delta^i{}_k \tag{6.6}$$

In view of the additional fact that, due to (2.10), (6.6) and the general relation (5.14b),

$$\nabla_{\boldsymbol{e}_i}\boldsymbol{e}_j = \Gamma^k{}_{ij}\,\boldsymbol{e}_k \ , \qquad \nabla_{\boldsymbol{e}_i}\boldsymbol{\theta}^j = -\Gamma^j{}_{ik}\boldsymbol{\theta}^k \ , \tag{6.7}$$

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[24] Cf. (Choquet-Bruhat *et al.*, 1987), p. 308; or (Spivak, 1979), p. 256; or (Straumann, 1984), p. 57. All these references also contain a detailed derivation of (6.5) and of the fundamental lemma of Riemannian geometry.

so that we can deduce that the connection coefficients in (6.6) uniquely determine all the corresponding gauge potentials in (5.18). This means that the Levi-Civita connection is completely determined in $GL\mathbf{M}$ by the metric field $\boldsymbol{g}$ since, in case that the chosen moving frame is a *vierbein* (i.e., consists exclusively of local Lorentz frames), we can infer from (6.2) and (6.6) that

$$\boldsymbol{\omega}_{ik}(\boldsymbol{e}_j) = \Gamma_{ijk} = \eta_{il}\Gamma^{l}{}_{jk} \ . \tag{6.8}$$

The connection one-forms in a generic moving frame in $GL\mathbf{M}$ can be then obtained by the executing of the gauge transformations in (5.20), leading to such generic moving frames.

The Levi-Civita connection form on the (restricted) Lorentz frame bundle $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ is thus established by (6.6)-(6.8) for any given Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$. In those cases where that manifold provides a classical model for spacetime, we can make contact with general relativistic physics via Einstein's *geodesic postulate*, which asserts (cf. Sec. 2.5) that any neutral point particle (of non-zero mass) follows a timelike[25] geodesics while in free fall within a classical spacetime manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$; whereas, any light-ray (or "photon," conceptualized as a classical particle of zero mass) follows in free fall a null geodesic.

By definition, in a manifold with a connection defined on it, a geodesic is any smooth curve γ having the property that if any of its tangents are parallel transported everywhere along γ, such a procedure results only in vectors that are also tangent to γ. In the context of the geodesic postulate, this means that if $\boldsymbol{v}$ denotes a 4-velocity field whose elements along a worldline $\gamma = \{\, x(\tau) \mid \tau \in \mathbf{I} \,\}$ of a massive particle in free fall represent the 4-velocities of that particle as τ ranges over some interval $\mathbf{I}$ in $\mathbf{R}^1$, then there is a choice of parameter τ (representing its *proper time*) for which in any coordinate chart (M_α,ϕ_α) given by (1.1a), whose domain M_α contains some segment of γ, we have (cf., e.g., Wald, 1984, p. 41)

$$\nabla_{\boldsymbol{v}}\boldsymbol{v} = 0 \ , \qquad \boldsymbol{v} = \dot{x}^\mu \partial_\mu \ , \qquad \dot{x}^\mu := dx^\mu/d\tau \ , \qquad x \in \gamma \ . \tag{6.9}$$

If we then specialize (5.8) to the Levi-Civita connection, whose coefficients are given by (6.6) in any moving frame within $GL\mathbf{M}$, we obtain from (6.9) the geodesic equations for that segment in their best-known form,

$$\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \ , \qquad \ddot{x}^\lambda := d^2x^\lambda/d\tau^2 \ , \quad \Gamma^\lambda_{\mu\nu} = g^{\lambda\kappa}\boldsymbol{g}(\partial_\kappa, \nabla_{\partial_\mu}\partial_\nu), \tag{6.10a}$$

when that moving frame is the holonomic frame (1.6) for the coordinate chart (M_α,ϕ_α). On the other hand, if we use a vierbein gauge, we have to expand the 4-velocity vector with respect to the Lorentz frames of the $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$-section (3.7) corresponding that gauge in order to obtain from (6.9) the system of equations

$$\dot{v}^i + \Gamma^i{}_{jk}v^jv^k = 0 \ , \qquad \dot{v}^i = dv^i/d\tau \ , \quad v^i = \lambda^i{}_\mu\,\dot{x}^\mu \ , \quad \Gamma^i{}_{jk} = \boldsymbol{\theta}^i(\nabla_{\boldsymbol{e}_j}\boldsymbol{e}_k) \ , \tag{6.10b}$$

for each geodesic segment which lies within the common domain of the nonholonomic and holonomic moving frames used in (6.10b).

Generically, there are *affine parameter* choices of τ for which the geodesic equation

[25] By definition, a smooth curve in a Lorentzian manifold is timelike, null or spacelike if all its tangent vectors $\boldsymbol{X}$ are timelike, null, or spacelike, respectively – cf. Footnote 9.

assumes the form (6.9) even for null or spacelike geodesics (cf., e.g., Choquet-Bruhat *et al.*, 1982, p. 303). Hence, according to the geodesic postulate this equation holds also for light-rays, or zero-mass classical "particles" – except that for such physical entities there is no concept of "proper time," and τ no longer has any direct physical meaning.

We shall show now that when the geodesic postulate is adapted to moving frames, then it becomes closely related to Einstein's original formulation of the strong equivalence principle – which was discussed in Sec. 1.1, and for which an in-depth historical analysis has been provided by Norton (1989). To arrive at such an adaptation, we consider a moving frame *adapted* to a smooth curve γ in $\mathbf{M}$ in the sense of being given by a section $\boldsymbol{s}$ of the general linear frame bundle $GL\mathbf{M}$ for which the Levi-Civita connection coefficients $\Gamma^{i}{}_{jk}$ in (6.6) vanish at all points $x \in \gamma$ which lie within the domain $\mathbf{M}^{\boldsymbol{s}}$ of that section. A special instance of such moving frames are the vierbeins that are adapted to γ, so that they are represented by sections $\boldsymbol{s}$ of the Lorentz frame bundle $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ for which $\Gamma^{i}{}_{jk} = 0$ at all $x \in \gamma \cap \mathbf{M}^{\boldsymbol{s}}$. Such vierbeins can be obtained by restricting an arbitrary section $\boldsymbol{s}'$ of $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ to a 3-dimensional hypersurface intersecting γ at a single point x', constructing a smooth flow of curves which intersect that hypersurface at other points in a neighborhood of x' and contains γ, and then parallel transporting all the frames at those points along the curves in that flow. Since the Levi-Civita connection is compatible with the metric $\boldsymbol{g}^{\mathrm{L}}$, this kind of parallel transport will preserve the mutual orthonormality of the axes in all these frames, so that the above required section $\boldsymbol{s}$ of $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ will result from this procedure.

Let us consider the case of a moving frame $\boldsymbol{s}$ adapted to a timelike geodesic γ. We shall call such a moving frame an *inertial moving frame* for the geodesic γ, if in the subset

$$\boldsymbol{s}_\gamma = \{(\boldsymbol{e}_0(x),\boldsymbol{e}_1(x),\boldsymbol{e}_2(x),\boldsymbol{e}_3(x))\,\big|\; x \in \gamma \cap \mathbf{M}^{\boldsymbol{s}}\} \subset \boldsymbol{s} \quad , \tag{6.11}$$

of frames lying above points in the geodesic γ, all the frame elements $\boldsymbol{e}_0(x)$ coincide with the tangent vectors to that geodesic for a suitable choice of affine parameter τ, i.e., with the 4-velocity vectors of a particle whose free-fall worldline is γ. In particular, if all the frames in $\boldsymbol{s}_\gamma$ are orthonormal, we shall call $\boldsymbol{s}$ an *inertial Lorentz moving frame* along γ.

Indeed, if a metrization of $GL\mathbf{M}$ is carried out in the manner described in Sec. 2.2, so that all the spatial classical frames that are labeled by triples $\{\boldsymbol{e}_1(x),\boldsymbol{e}_2(x),\boldsymbol{e}_3(x)\}$ with $x \in \gamma \cap \mathbf{M}^{\boldsymbol{s}}$ are constructed from classical test particles, then, by the geodesic postulate, the particle that marks the origin O of each such local classical frame will follow γ if it is allowed to remain in free fall. Hence, with the passage of its proper time, that frame, viewed as a physically existing object within which the remaining test particles A_α, $\alpha = \pm1,\pm2,\pm3$, are maintained in their original metric relations with respect to each other, can be sequentially identified with the elements of the set $\boldsymbol{s}_\gamma$ in (6.11). In fact, by its very construction, all the elements of that set display the following properties:

$$g_{ij}(x) = \boldsymbol{g}(\boldsymbol{e}_i(x),\boldsymbol{e}_j(x)) = \eta_{ij} \quad , \qquad \boldsymbol{e}_0(x) = \boldsymbol{v} \quad , \qquad x \in \gamma \cap \mathbf{M}^{\boldsymbol{s}} \quad , \tag{6.12a}$$

$$\Gamma^{i}{}_{jk}(x) = g^{il}\,\boldsymbol{g}(\boldsymbol{e}_l(x),\nabla_{\boldsymbol{e}_j(x)}\boldsymbol{e}_k(x)) = 0 \quad , \qquad x \in \gamma \cap \mathbf{M}^{\boldsymbol{s}} \quad . \tag{6.12b}$$

These properties mathematically characterize a classical Lorentz frame in free-fall. It has to be stressed, however, that these relations are in general satisfied only by the elements of the

set $\boldsymbol{s}_\gamma$, and cannot be extended to the entire moving frame $\boldsymbol{s}$ when gravity is actually present within the spacetime region corresponding to $\mathbf{M}^{\boldsymbol{s}}$.

We can now formulate the strong equivalence principle as follows: *In any inertial Lorentz moving frame along a timelike geodesic γ, all the nongravitational laws of physics, expressed in tensor coordinates with respect to that inertial moving frame, should at each point along γ coincide with their special relativistic counterparts expressed in tensor coordinates with respect to global Lorentz frames in Minkowski spacetime.*

Formulated in this mathematically and physically precise form, the strong equivalence principle reflects Einstein's (1916) conceptualization of free-fall physical conditions that are illustrated by his well-known *gedanken* experiments, imagined to be carried out by an observer isolated within an elevator which is in free fall. On the other hand, even those contemporary textbooks on CGR which pay to this most important principle the attention it deserves, formulate it in terms of coordinates rather than frames[26]. However, as discussed in Sec. 1.1, such "formulations of the principle of equivalence characteristically obscure [the] crucial distinction between first-order laws and second-order laws by blurring the distinction between 'infinitesimal' laws, holding at a single point, and local laws, holding on a neighborhood of a point." (Friedman, 1983, p. 202.)

In order to underline this crucial distinction, and give a mathematically precise formulation of the strong equivalence principle in coordinate form, let us introduce normal coordinates associated with an inertial Lorentz moving frame by means of the following construction: let us set $x^0(\tau) = \tau$ at all points along the timelike geodesic γ, and consider then at each point $x \in \gamma$ all the spacelike geodesics with tangents spanned by spatial triples $\{\boldsymbol{e}_1(x), \boldsymbol{e}_2(x), \boldsymbol{e}_3(x)\}$ of the corresponding Lorentz frames; if X^1, X^2 and X^3 are the components of those tangents, let us assign, within some neighborhood of $x(\tau) \in \gamma$, the coordinates (τ, X^1, X^2, X^3) to each point which lies on one of these geodesics at unit affine parametric distance from $x(\tau)$. These coordinates are defined by this construction within a tube around γ, and within that tube they are such $\partial_0 = \boldsymbol{e}_0(x)$ and $\partial_\alpha = \boldsymbol{e}_\alpha(x)$, $\alpha = 1,2,3$, at each $x \in \gamma$. Hence they represent inertial coordinates which are normal at x.

By using these inertial coordinates, we can now formulate the strong equivalence principle in coordinate form as follows: *In any Lorentz moving frame which is inertial for some timelike geodesic γ, all the nongravitational laws of physics, expressed in the normal coordinates associated with that inertial frame, should at each point along γ equal, up to first-order terms in those coordinates, their special relativistic counterparts expressed in the tensor coordinates associated with the respective Lorentz frames.*

This formulation of the strong equivalence principle might appear intricate, but it is precise, and it implicitly reveals the foundational reasons for the "ambiguities" (Straumann, 1984, Sec. 1.4.4) that manifest themselves in the passage from the special to the general relativistic regime. In purely mathematical terms, these ambiguities result from the fact that partial differentiation is commutative, whereas, covariant differentiation is not in the presence of non-zero curvature, to which gravity gives rise. In physical terms, they result, however, from ignoring the distinction between "infinitesimal" laws, which hold within the

[26] For example, the concept of "local Lorentz frame" defined in §8.6 of (Misner *et al.*, 1973) is actually formulated in terms of coordinates. In (Straumann, 1984) the same *coordinates* are suitably described as "inertial," but it is added that they are "interpreted as a local inertial system." (Straumann, 1984, p. 89.)

fibres of various tensor bundles, and local laws, which hold on neighborhoods of points in the base manifold of those bundles.

2.7. The Einstein Field Equations and Canonical Gravity

As is well-known, the formulation of the Einstein field equations in their final form, which in Planck natural units (in which $G = c = \hbar = 1$) reads

$$R_{ij} - \tfrac{1}{2} g_{ij} R = -8\pi T_{ij} \ , \tag{7.1}$$

marked the end of Einstein's protracted struggle to reconcile gravity with all the basic relativistic tenets. In the above form, these equations hold true in an arbitrary moving frame, corresponding to any section $\boldsymbol{s}$ of $GL\mathbf{M}$. In addition to the components of a Lorentzian metric tensor field $\boldsymbol{g}^{\mathrm{L}}$ in the chosen moving frame $\boldsymbol{s}$, they contain on their left-hand side the components of the *Ricci tensor field* $\boldsymbol{R}$ in that same frame, together with the corresponding *Riemann curvature scalar field* R,

$$\boldsymbol{R} = R_{ij}\boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j \ , \qquad R_{ij} = R^k{}_{ikj} \ , \qquad R = R^i{}_i = g^{ik}R_{ki} \ ; \tag{7.2}$$

whereas, on their right-hand side they contain the components of the *stress-energy tensor* $\boldsymbol{T}$ in the moving frame $\boldsymbol{s}$:

$$\boldsymbol{T} = T_{ij}\boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j \tag{7.3}$$

As Einstein fell in later years more and more under the spell of differential geometry, he would sometimes describe the left-hand side of the equations (7.1) as "beautiful," and the right-hand side as "ugly." The reason for this verdict was that the stress-energy tensor $\boldsymbol{T}$ contains contributions from all matter and nongravitational radiation fields, and as such it cannot be inferred from the geometry of spacetime, but rather from other physical theories, such as those hydrodynamics, electromagnetism, strong and weak interactions, etc. – none of which were formulated in geometric form during Einstein's lifetime.[27] On the other hand, the Einstein tensor, whose components constitute the left-hand side of the equations in (7.1), is expressed exclusively in terms of geometric quantities, related to the metric. Indeed, the Ricci tensor field in (7.2) is related to the *Riemann curvature tensor field*,

$$R^i{}_{jkl}\, \boldsymbol{e}_i \otimes \boldsymbol{\theta}^j \otimes \boldsymbol{\theta}^k \otimes \boldsymbol{\theta}^l \ , \qquad R^i{}_{jkl} = \boldsymbol{\theta}^i\big(\mathbf{R}(\boldsymbol{e}_k, \boldsymbol{e}_l)\boldsymbol{e}_j\big) \ , \tag{7.4a}$$

which is derivable from the adopted connection by means of the *curvature operator form*

$$\mathbf{R}(\boldsymbol{X},\boldsymbol{Y}) : \boldsymbol{Z} \mapsto \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{Z} - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z} \ , \tag{7.4b}$$

defined for arbitrary vector fields $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ assuming values in $T\mathbf{M}$. It then follows from (7.4b) and (6.7) that

$$R^i{}_{jkl} = \boldsymbol{e}_k(\Gamma^i{}_{lj}) - \boldsymbol{e}_l(\Gamma^i{}_{kj}) + \Gamma^i{}_{km}\Gamma^m{}_{lj} - \Gamma^i{}_{lm}\Gamma^m{}_{kj} - \boldsymbol{\theta}^i(\nabla_{[\boldsymbol{e}_k,\boldsymbol{e}_l]}\boldsymbol{e}_j) \ , \tag{7.5}$$

[27] As is well-known, soon after the formulation of the equations (7.1), Einstein started to devote all his energies to a quest after a unified field theory, which would integrate classical gravity and electromagnetism in a single geometric scheme, but he failed to achieve that goal.

and since in the case of Einstein's equations one deals with the Levi-Civita connection, whose connection coefficients are related to the metric field by (6.6), the Einstein tensor can be expressed exclusively in terms of the components of the metric tensor field $\boldsymbol{g}^{\mathrm{L}}$.

In fact, for holonomic moving frames, whose components are as in (1.6) and (1.8), the Lie (commutator) bracket appearing in the last term of (7.5) vanishes, so that the components of the Riemann curvature tensor assume the form which is most frequently used in standard CGR textbooks,

$$R^{\kappa}{}_{\lambda\mu\nu} = \Gamma^{\kappa}{}_{\lambda\nu,\mu} - \Gamma^{\kappa}{}_{\lambda\mu,\nu} + \Gamma^{\kappa}{}_{\mu\rho}\Gamma^{\rho}{}_{\lambda\nu} - \Gamma^{\kappa}{}_{\nu\rho}\Gamma^{\rho}{}_{\lambda\mu} \quad , \qquad \Gamma^{\kappa}{}_{\lambda\mu,\nu} = \partial\Gamma^{\kappa}{}_{\lambda\mu}\big/\partial x^{\nu} , \tag{7.6a}$$

where, on account of (6.5), the following relations hold for Christoffel symbols:

$$\Gamma^{\kappa}{}_{\mu\nu} = \tfrac{1}{2}g^{\kappa\lambda}\left(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}\right) , \qquad g_{\lambda\mu,\nu} = \partial g_{\lambda\mu}\big/\partial x^{\nu} \tag{7.6b}$$

On the other hand, the nonholonomic moving frames belonging to vierbein gauges are orthonormal, so that it is the terms in the first rather than in the second of the square brackets in (6.5) that vanish identically, whereas, in such moving frames the connection coefficients (known in CGR as *Ricci rotation coefficients*) are on account of (2.7)-(2.11) related to the Christoffel symbols in (7.6b) as follows:

$$\Gamma^{i}{}_{jk} = \lambda^{i}{}_{\mu}\lambda_{j}{}^{\nu}\left(\lambda_{k}{}^{\kappa}\Gamma^{\mu}{}_{\nu\kappa} + \partial_{\nu}\lambda_{k}{}^{\mu}\right) . \tag{7.7}$$

In either case, since the metric, Ricci and stress-energy tensors in (7.1) are all symmetric, the Einstein equations can be viewed as a nonlinear system consisting of a total of ten equations for the components of the metric field.

If one adds to the Einstein tensor the term λg_{ij} , containing the cosmological constant λ, then the equations (7.1) can be derived[28] (as is actually done in some textbooks) from the fact that such a modification of Einstein's tensor is the most general divergenceless tensor which is a function of only the metric tensor field, and of the first and second order derivatives of that field. However, Einstein inferred these equations from physical considerations based on his famous equation $E = mc^2$, in a manner which underlines the smooth transition from special to general relativity. Nevertheless, the constructive criticisms and suggestions of outstanding mathematicians, such as Hilbert and Levi-Civita, played a major factor in enabling him to arrive, after several unsuccessful attempts, at the final (and correct) form (7.1) of these equations (cf. Cattani and De Maria, 1989, p. 185). In fact, Einstein publicly presented these equations for the first time to the Berlin Academy on November 25, 1915, and that date turned out to be five days after, unbeknownst to him, Hilbert had already presented the same equations to the Göttingen Academy (cf. Norton, 1989, p. 150).

Shortly after these equations were publicly announced, Schwarzschild arrived at his well-known explicit solution describing the static and spherically-symmetric metric field around a pointlike source concentrated in a singularity, which nowadays is taken to

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[28] The following has been said of Einstein's views on such methods: "That Einstein did not resort to such manipulative tricks confirms the view that purely mathematical gimmickry played no significant part in the development of the relativity programme." (Zahar, 1989, p. 292.)

represent the simplest model of a black hole. Most of the other explicit and global solutions that can be found in typical textbooks on general relativity (Misner *et al.*, 1973; Straumann, 1984; Wald, 1984) followed within a couple of decades, including the one for the Friedmann-Lemaître models of our universe with (what came later to be known as) Robertson-Walker metrics, which eventually paved the way for the standard Big Bang cosmological model.

Although a globally defined Lorentzian manifold emerges from all these solutions, they clearly represent greatly over-simplified models of the physical situations they are meant to describe. Their textbook derivations start by adopting a global differential manifold $\mathbf{M}$, and then use various appealing symmetry arguments to introduce on $\mathbf{M}$ a class of Lorentzian metrics and stress-energy tensors, from which a solution of the Einstein field equations is then computed. On the other hand, while trying to draw "some lessons from general relativity," J. Stachel has presented an analysis of Einstein's own views on the formulation of general relativistic spacetime models, in which he arrives at the following conclusion[29]: "Even relativists have not fully adopted [Einstein's] point of view, 'no metric, no nothing'. ... They start out by introducing a global manifold ..., and then put such structures on this manifold as the metric tensor field. Is that the way any one of us actually goes about solving the field equations of general relativity? Of course not. One first solves them on a generic patch, and *then* one tries to maximally extend the local solution ... from that patch to a global manifold which is not known ahead of time."

Indeed, the epistemic feature that distinguishes general relativity most from all other physical theories lies in the interaction between the geometry of spacetime, which reflects the presence of gravity in the universe, and all other fields, which collectively represent all matter and nongravitational radiation. Hence, at the foundational level, it is incorrect to envisage that in the general relativistic conceptualization of physical reality any class of global Lorentzian manifolds, possessing some appealing symmetry features, is *a priori* given, and that the presence of nongravitational fields in the stress-energy tensor simply selects the "right metric" in the manner depicted in typical textbooks derivations of Schwarzschild solution, the standard Big Bang cosmological model, etc. Rather, the image of physical reality suggested by Einstein's general relativistic epistemology is that of *geometrodynamics* (Wheeler, 1962; Misner *et al.*, 1973) represented by a *perpetual* mutual interaction between geometry and nongravitational sources, whereby the two *constantly* affect each other.

This geometrodynamic view eventually crystalized from Dirac's (1950, 1958, 1959) pioneering work on Hamiltonian dynamics with constraints, as it led to the Arnowitt-Deser-Misner (1962) formulation of the initial-value problem in CGR – also referred to as the general relativistic Cauchy problem. That formulation, which has become known as the *ADM method* for canonical gravity, relies on the postulate that any Lorentzian manifold $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$ that can provide a physically acceptable model for spacetime has to be globally hyperbolic. At the mathematical level this condition can be stated in several equivalent ways (cf. Wald, 1984, p. 201), which are all tantamount to the requirement that $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$ should

[29] Cf. (Ashtekar and Stachel, 1991), p. 38. The quotation ends with the following observation, expressed in somewhat harsh but accurate terms: "So we are pulling a swindle when we are telling students ... that you first pick the manifold and then solve the field equations on it."

possess a Cauchy surface[30] Σ_0. Physically, the requirement of global hyperbolicity is equivalent to the so-called "strong cosmic censorship principle" (Penrose, 1969), which "forbids" the occurrence of "naked singularities." This means that the only singularities which might occur in spacetime as a result of gravitational collapse are black holes. Since any causal path that encounters such a singularity ends in it, such a postulate is equivalent to the physically sensible assumption that there is no breakdown in the predictability of behavior of matter and radiation in any acceptable model of spacetime.

In essence, the ADM method deals with the problem of reconstructing a globally hyperbolic Lorentzian manifold $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$ from data prescribed on an initial-data Cauchy surface Σ_0. It starts from the observation that the points of such an initial-data hypersurface can be always labeled by means of an atlas of coordinate charts[31], which are given by one-to-one maps of the general form $x \leftrightarrow (0, x^1, x^2, x^3)$ for all $x \in \Sigma_0$ in their domains. The end-result of this construction can be envisaged to constitute a foliation of a Lorentzian manifold $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$ into diffeomorphic maximal spacelike hypersurfaces Σ_t, whereby the entire differential manifold $\mathbf{M}$ is represented as a union

$$\mathbf{M} = \bigcup_{t \in \mathbf{I}} \Sigma_t \ , \qquad \Sigma_t = \left\{ x \leftrightarrow (\mathbf{x}, t) \middle| \ \mathbf{x} \leftrightarrow (x^1, x^2, x^3), \ x^0 = t \right\} \ , \tag{7.8}$$

over some interval $\mathbf{I}$ of values for the ADM parameter $t \in \mathbf{R}^1$. This parametrization of the hypersurfaces Σ_t, which we shall call *reference hypersurfaces*, is often envisaged as being related to the flow of a "global time." It has to be stressed, however, that in CGR there is no single "global time," but rather an infinity of "global times," that correspond to an infinity of physically as well as mathematically feasible and yet distinct foliations in (7.8). Furthermore, in general the values of the ADM parameter t can be changed at will by means of any smooth and monotonically increasing function of $t \in \mathbf{I}$, without changing in the least the physical conclusions that can be drawn by means of the ADM method. Hence, the values *generically* assumed by t have as little physical significance in canonical gravity as the coordinate charts generically have in CGR.

For each foliation (7.8) it proves convenient to express the metric in the form

$$\boldsymbol{g}^{\mathrm{L}} = \left(N^2 - N_a N_b \, \gamma^{ab} \right) \boldsymbol{dx}^0 \otimes \boldsymbol{dx}^0 + N_a \, \boldsymbol{dx}^0 \otimes \boldsymbol{dx}^a + N_a \, \boldsymbol{dx}^a \otimes \boldsymbol{dx}^0 - \boldsymbol{\gamma} \ , \tag{7.9a}$$

where N and N_a are called the ADM *lapse* and *shift* functions, respectively, whereas

$$\boldsymbol{\gamma}(x^0, \mathbf{x}) = \gamma_{ab}(x^0, \mathbf{x}) dx^a \otimes dx^b \ , \qquad a,b = 1,2,3 \ , \tag{7.9b}$$

determines a Riemannian 3-metric along each one of the three-dimensional reference hypersurfaces Σ_{x^0}. This 3-metric can be used in raising and lowering three-valued indices, so that, for example, we can set

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[30] In a Lorentzian manifold $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$ a *Cauchy surface* is defined as a spacelike slice of $\mathbf{M}$, represented by a 3-surface which is intersected by *any* causal curve (i.e., smooth curve whose tangent is at every point null or timelike) that can no further be causally extended either in the past, or in the future – cf., e.g., (Wald, 1984), p. 201.

[31] Cf. Sec. 2.1 for the definition of an atlas of charts. Coordinate-independent formulations of canonical gravity and the initial-value problem can be found in the review articles by Fisher and Marsden (1979), Isenberg and Nester (1980), as well as Choquet-Bruhat and York (1980).

$$N^a = \gamma^{ab} N_b \quad , \qquad \gamma_{ac}\gamma^{cb} = \delta_a{}^b \quad . \tag{7.10}$$

The ADM lapse and shift functions determine the differentials

$$L_0 = N(\mathbf{x},t)dt \quad , \qquad L_a = N_a(\mathbf{x},t)dt \quad , \qquad a = 1,2,3 \quad , \tag{7.11}$$

which can be viewed heuristically as representing the components of an "infinitesimal vector" between two "infinitesimally" close reference hypersurfaces Σ_t and Σ_{t+dt}. If we imagine a coherent flow of test particles in free fall, whose worldlines are geodesics orthogonal to the reference hypersurfaces, then L_0 represents, for a given value of the ADM parameter t, the "lapse" in the proper time τ of a test particle in free fall, whose worldline is the geodesic orthogonal to Σ_t that passes through the considered point $\mathbf{x}$, as that particle reaches Σ_{t+dt}. Similarly, L_a represents the a-th component of the "shift" 3-vector in Σ_{t+dt} which joins the point where that same worldline meets Σ_{t+dt} with the point where the coordinate line obtained by keeping the coordinates of $\mathbf{x}$ fixed, and varying t, meets Σ_{t+dt}.

A *coherent flow* of classical test particles can be now envisaged in purely operational terms, by locating a point test particle at each point in an initial-data Cauchy surface Σ_0, imparting to each one of these test particles a 4-velocity such that the resulting 4-velocity field is orthogonal to Σ_0, and then allowing all these test particles to move in free fall along the geodesics of the Levi-Civita connection in the considered spacetime. We can, therefore, operationally introduce *Gaussian normal coordinates* as follows: we first cover the initial-data Cauchy surface Σ_0 with charts that assign to all $x \in \Sigma_0$ one or more sets of coordinates; then, using the coherent flow whose flow lines are orthogonal to Σ_0, we assign corresponding coordinates to points $x \in \Sigma_\tau$ for various choices of $\tau \in \mathbf{R}^1$, where τ is the proper time of the classical test particles in that flow. It is noteworthy that such Gaussian normal coordinates can be always introduced, and that although generally they develop coordinate singularities as one follows them from the initial-data hypersurface Σ_0 to increasing values of τ, since the worldlines of some of the particles in the coherent flow might eventually intersect, the locii of these intersections constitute a manifold of lower dimensionality than that of $\mathbf{M}$ (cf. Lifshitz and Khalatnikov, 1963).

The ADM canonical form (7.9) of a Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ is especially simple in Gaussian normal coordinates which correspond to an ADM parameter t which coincides with the proper time τ of the particles in such coherent flows adapted to the foliation in (7.8). In that case the lapse function N is identically equal to one, whereas, all the shift functions N_a vanish identically, so that (7.9a) assumes the physically transparent form

$$\boldsymbol{g}^{\mathrm{L}} = d\tau \otimes d\tau - \gamma_{ab}\, \boldsymbol{dx}^a \otimes \boldsymbol{dx}^b \quad . \tag{7.12}$$

Since the timelike component of synchronous coordinates has an actual physical meaning, namely that of proper time of particles in a corresponding coherent flow, Gaussian normal coordinates are also known in CGR by the name of *synchronous coordinates* (cf. Misner *et al.*, 1973, p. 717; or Wald, 1984, p. 42).

Let $\boldsymbol{n}$ denote the field of future-oriented tangents to the worldlines of the classical test particles in a coherent flow adapted to a single hypersurface Σ_τ in the foliation (7.8). Along Σ_τ this field consists of unit timelike normals $\boldsymbol{n}^{(\tau)}$ to that hypersurface, in terms of which the *extrinsic curvature* of Σ_τ is given by the $(0,2)$-tensor field

$$\boldsymbol{K}^{(\tau)} = -n^{(\tau)}_{b|a}\, \boldsymbol{dx}^a \otimes \boldsymbol{dx}^b \ , \tag{7.13}$$

where the vertical bar indicates the covariant derivative in the direction of $\boldsymbol{\partial}_a = \partial/\partial x^a$ for the Levi-Civita connection corresponding to the Riemannian metric (7.9b) on Σ_τ. Since that connection can be obviously also obtained from the restriction to Σ_τ of the Levi-Civita connection corresponding to the Lorentzian metric (7.9a) on $\mathbf{M}$, the covariant derivative of any vector field $\boldsymbol{X}$ in $\mathbf{M}$ can be expressed as follows (cf. Misner *et al.*, 1973, p. 512):

$$\nabla_a \boldsymbol{X} = X^b{}_{|a}\, \boldsymbol{\partial}_b - K^{(\tau)}_{ab}\, X^b\, \boldsymbol{n}^{(\tau)} . \tag{7.14}$$

By a variational argument presented in detail in the original Arnowitt-Deser-Misner (1962) article, as well as in some textbooks (cf., e.g., Misner *et al.*, 1973), in the absence of nongravitational fields, the vacuum Einstein equations (i.e., the equations (7.1) with vanishing stress-energy tensor) can be recast in the form of equations for a system of nonlinear equations for the 3-metric $\boldsymbol{\gamma}$ and the extrinsic curvature $\boldsymbol{K}$ of the reference hypersurfaces Σ_t. The physically most transparent form of the resulting equations is obtained by introducing a tensor with components

$$\pi^{ab} = \gamma^{1/2}\left(\gamma^{ab}K_c{}^c - K^{ab}\right) \ , \qquad \gamma(\mathbf{x},t) = \det\|\gamma_{ab}\| \ , \tag{7.15}$$

canonically conjugated to those of the 3-metric on each one of the reference hypersurfaces, and by introducing a corresponding so-called *super-Hamiltonian* and *supermomentum*,

$$\mathcal{H}^0 = \gamma^{-1/2}\left[\tfrac{1}{2}(\pi^b{}_b)^2 - \pi^{ab}\pi_{ab}\right] - \gamma^{1/2}R^{(t)} \ , \tag{7.16a}$$

$$\mathcal{H}^a = -2\pi^{ab}{}_{|b} \ , \qquad a = 1,2,3 \ , \tag{7.16b}$$

where the Riemannian curvature scalar $R^{(t)}$ within the latter corresponds to the 3-metric $\boldsymbol{\gamma}$ along each reference hypersurface Σ_t.

Upon introducing then the "Hamiltonian"

$$H_0^{(t)} = \int_{x^0=t} \mathcal{H}\, d\mathbf{x} \ , \qquad \mathcal{H} = N\mathcal{H}^0 + N_a\mathcal{H}^a \ , \tag{7.17}$$

the ADM formalism can be formulated in terms of Poisson brackets, obtained by replacing partial derivatives with functional derivatives in the Hamiltonian classical mechanics definition (Fischer and Marsden, 1979, pp. 151-157),

$$\dot{\gamma}_{ab} = \left\{\gamma_{ab}, \mathcal{H}\right\} = \delta\mathcal{H}/\delta\pi^{ab} \ , \qquad \dot{\pi}^{ab} = \left\{\pi^{ab}, \mathcal{H}\right\} = -\delta\mathcal{H}/\delta\gamma_{ab} \ , \tag{7.18}$$

and interpreting the above "dot-differentiation" as differentiation with respect to the ADM parameter t. The above equations turn out to be equivalent to the six Einstein equations corresponding to the values $i,j = 1,2,3$, for the indices in (7.1), which are collectively referred to as the *Einstein geometrodynamic equations*. The remaining four equations, collectively referred to as the *Einstein constraint equations*, are equivalent to the subsidiary conditions

$$\mathcal{H}^0 = 0 \ , \qquad \mathcal{H}^a = 0 \ , \qquad a = 1,2,3 \ , \tag{7.19}$$

which have to be imposed *after* the Poisson brackets are computed.[32]

From the *canonical* point of view, however, the Lorentzian manifold $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$ is not given *a priori*, and the foliation in (7.8) has to regarded as a geometrodynamic *evolution* of 3-geometries, that have to be computed from a complete Cauchy set of initial data prescribed on some initially given 3-manifold Σ_0. In the generic case, these initial data are comprised of the initial-data manifold Σ_0 itself, of a given initial Riemannian metric $\boldsymbol{\gamma}_0$, of a given initial extrinsic curvature $\boldsymbol{K}_0$ for Σ_0, as well as of suitable initial data $\boldsymbol{\phi}_0$ for all the nongravitational fields, which we shall denote collectively by $\boldsymbol{\phi}$. These fields, which are briefly referred to as "matter" fields, enable the computation of the initial value $\boldsymbol{T}_0$ of the stress-energy tensor $\boldsymbol{T}$.

The basic idea as to how to formulate the CGR initial-value problem can be arrived at now by viewing the diffeomorphisms underlying the foliation in (7.8) as giving rise to identifications of each of the reference hypersurfaces Σ_t with the single initial-data manifold Σ_0. Hence, during the flow of "global time," the reference manifold Σ_0 does not change, but only the geometric data represented by 3-metric and extrinsic curvature change in conjunction with changes in the "matter" fields $\boldsymbol{\phi}$. If synchronous (i.e., Gaussian normal) coordinates are adopted, so that $N = 1$ and $N_a = 0$, then in accordance with (7.12), the resulting geometric part of the solution of the intial-value problem will be of the form

$$g_{00} = 1 \ , \qquad g_{a0} = g_{0a} = 0 \ , \qquad g_{ab} = -\gamma_{ab} \ , \qquad \mathrm{K}_{ab} = -\tfrac{1}{2}\dot{g}_{ab} \ . \tag{7.20}$$

In the presence of the nongravitational fields, the constraint equations resulting from (7.16) and (7.19) are modified by the appearance of the local *proper energy density* ρ and *proper 3-momentum components* j_a of nongravitational sources, represented in the above synchronous coordinates by (cf. Choquet-Bruhat and York, 1980, pp. 100 and 137)

$$\rho = T_{\mu\nu} n^\mu n^\nu = T_{00} \ , \qquad j_a = T_{a\nu} n^\nu = T_{a0} \ , \qquad a = 1,2,3 \ . \tag{7.21}$$

The Einstein geometrodynamic equations can be then generically solved only by numerical computational methods.[33] Such methods are provided by algorithmic schemes in which the computed classical spacetime manifold $\mathbf{M}$ is envisaged as being sliced into "thin" *segments* $\mathbf{S}_n$, $n = 0, \pm 1, \pm 2, \ldots$, corresponding to parameter values $t \in [t_{n-1}, t_n]$ in the geometrodynamic evolution of "matter" (i.e., matter fields as well as all nongravitational fields) and of the geometry on Σ_0. In such a context the initial data have to be presented in the form of differences, rather than of differentials, within the initial-data segment $\mathbf{S}_1$ corresponding to

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[32] These constraint equations are basically the Gauss-Codacci equations relating, via the Levi-Civita connection for the 3-metric in (7.9b), the extrinsic curvature to the Riemann curvature scalar for that metric – cf. (Misner *et al.*, 1973), Eqs. (21.75)-(21.76), or (Wald, 1984), Eqs. (10.2.23)-(10.2.24). A method that derives them by taking advantage of Gaussian normal coordinates can be found in (Choquet-Bruhat *et al.*, 1987), pp. 315-316.

[33] Cf. (Choquet-Bruhat and York, 1980), Secs. 1.2, 3.3 and 3.4 for a review, in which the equations containing the proper energy density and momentum in (7.21) appear as Eqs. (3.1) and (3.2). In such computational schemes it is more efficient to choose coordinate suitable charts in which $N \neq 1$ and $N_a \neq 0$, since the Gaussian normal coordinates leading to (7.20) eventually develop caustics, representing coordinate singularities. However, in principle the final outcome of such algorithmic procedures is independent of the choice of coordinates, and Gaussian normal coordinates make the physical meaning of the initial conditions more transparent.

the parameter values $t \in [t_0,t_1]$.

The initial data $\boldsymbol{\gamma}_0$, $\boldsymbol{K}_0$ and $\boldsymbol{\phi}_0$ have to obey along Σ_0 the constraint equations, and in that case they provide the starting point of the algorithm for the computation of $\boldsymbol{\gamma}_1$, $\boldsymbol{K}_1$ and $\boldsymbol{\phi}_1$ at t_1, by means of a suitable detailed form of the geometrodynamic equations – such as those in (7.18). The constraints then remain satisfied for all values of t, with violations occurring only from computational large margin of errors. Hence, in principle the process can be repeated, and $\boldsymbol{\gamma}_n$, $\boldsymbol{K}_n$ and $\boldsymbol{\phi}_n$ can be computed in all the other segments $\mathbf{S}_n$ by an iteration of this algorithmic procedure.

From the purely mathematical point of view, initial-value problems give rise to questions of existence and uniqueness of solutions. For the vacuum case proofs of great generality are known (Hawking and Ellis, 1973). For initial data $(\Sigma_0,\boldsymbol{\gamma}_0,\boldsymbol{K}_0)$ that satisfy the constraints, these proofs establish the existence of a unique "maximal Cauchy development" $(\Sigma_0,\boldsymbol{\gamma},\boldsymbol{K})$ to an entire range of values of t in such a manner that a globally hyperbolic Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ emerges, into which Σ_0 can be embedded as a Cauchy surface. As could be expected from Einstein's "hole" argument (which was recounted in Sec. 1.1), for diffeomorphically equivalent initial conditions the uniqueness of such Cauchy developments can be established only modulo diffeomorphisms between the resulting Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ and all other Lorentzian manifolds $(\mathbf{M}',\boldsymbol{g}'^{\mathrm{L}})$ isometric to $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ (cf. Hawking and Ellis, 1973, Sec. 7.1; Straumann, 1984, Sec. 2.7).

Similar theorems are much harder to prove in the presence of "matter," i.e., of non-gravitational sources, since then the results depend critically on the dynamical equation of the "matter" fields. Consequently, only particular categories of solutions have been treated thus far, and the entire area is still under active investigation. However, it is clear from what is already known that the uniqueness of solutions of the initial-value problem is a general feature of CGR. Hence, at the cosmological level, the description[34] of the universe that emerges from the CGR mathematical framework is a strictly deterministic one: if one takes Big Bang cosmology for granted, and assumes that general relativistic laws came into effect right after the formation of our universe, then despite the continuous geometrodynamic interaction between "matter" and geometry, which is at the heart of the general relativistic point of view, the entire "fate" of our universe was totally determined from that "instant" of "creation" onwards.

When it comes to the large-scale structure of our universe (Hawking and Ellis, 1973; Börner, 1988; Zichichi *et al.*, 1992), such a strict general relativistic determinism might be deemed acceptable. However, at the microlevel, it flies in the face of the entire philosophy of quantum mechanics which, crowned by Heisenberg's uncertainty principle, envisages a *fundamental* indeterminism in nature. Hence, one can ask: if the geometry of spacetime is constantly affected by the behavior of "matter," and the quantum states of that "matter" undergo "reductions" and other kinds of changes which are fundamentally statistical in nature, how is it possible that the geometry of spacetime is even at the *micro*level "classical" in the sense of being strictly deterministic in both its formulation and interpretation?

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[34] In cosmological models the proper time τ that corresponds to the synchronous coordinates in (7.12) associated with coherent flows representing the averaged motion of matter in our universe plays the role of a "cosmic time" measured by "fundamental observers" comoving with the expanding matter in the universe – cf. (de Sabbata and Gasperini, 1985), pp. 98-102.

There are two obvious but diametrically opposing ways to resolve this foundational inconsistency: 1) to follow in the footsteps of Einstein's philosophy, and assume that present-day quantum theory is merely a statistical approximation, necessitated only by our temporary lack of knowledge of some "hidden variables" that underlie a deeper and totally deterministic theory for the behavior of "matter" at the microlevel; 2) to assume that the classical geometry envisaged by Einstein in formulating general relativity is only an approximation of a more detailed geometry, which embodies quantum principles into its very mathematical structure and physical foundation, and which therefore allows for a probabilistic interpretation of its main theoretical constructs.

In choosing between these two diametrically opposite alternatives, one can only be guided by one's own *Weltanschauung*. As is well-known, Einstein remained passionately committed all his life to a deterministic philosophy, exemplified by his famous dictum that "God does not play dice" (Pais, 1982). This is a point of view that has to be respected by all those concerned with deeper philosophical issues, but which, in the ultimate analysis, entails an ontological outlook which, at the most fundamental physical level, clashes with the "freedom" of choice that one normally envisages in changing initial conditions during the course of performing experiments. Indeed, when it is extended to every single bit of energy and matter in the universe, such an outlook is compatible only with the assumption that there are no "experimenters," but only passive "observers" of phenomena, whose evolution had been fixed in its minutest detail at the "instant of creation." This certainly contradicts the entire philosophy of quantum mechanics (Bohr, 1955, 1963; Born, 1956; Heisenberg, 1958, 1971). Moreover, although the EPR paradox (Einstein *et al.*, 1935) is still the hub of ongoing disputes (van der Merwe *et al.*, 1988; Tarozzi and van der Merwe, 1988; Kafatos, 1989), the experiments of Aspect *et al.* (1981, 1982) have settled the basic issue of existence *vs* nonexistence of local hidden variables in favor of quantum mechanics.

All this suggests not only the tenability, but even the necessity of the opposite point of view, which envisages that our universe is indeterministic at the *micro*level. In such a conceptualization, the *apparently* deterministic features in the geometry of spacetime, which one undeniably observes at the *macro*level, are the *average* values of some of the basic observable quantities of an underlying *quantum* geometry. In actuality, however, those values are subject to statistical fluctuations of a quantum origin. Those fluctuations manifest themselves primarily at the microlevel, and prevail as we approach spatio-temporal separations of the order of Planck's length ($\approx 10^{-33}$ cm) and Planck's time ($\approx 10^{-44}$ sec) – both of which assume the value one in the dimensionless Planck natural units with $c = \hbar = G = 1$.

It is to the theoretical depiction and analysis of a quantum geometric realization of this fundamentally indeterministic alternative that the remainder of this monograph is devoted.

Chapter 3

Quantum Frames and Spacetime Localizability

The replacement of the concept of complete set of observables with that of *quantum* frame, incorporating a fundamental length ℓ, will mediate in Chapter 4 the extrapolation to QGR of the classical frame bundle approach to CGR presented in Chapter 2. In the nonrelativistic and special relativistic regimes, each quantum frame is *informationally* complete, in the sense that *any* quantum state of a system can be completely determined by measurements performed in relation to it. In the nonrelativistic context, conventional quantum mechanics is recovered in the sharp-point limit $\ell \to 0$; whereas, in the special relativistic context the same limit leads to divergences, which require infinite renormalizations. Upon formulating quantum propagation in relation to quantum frames by path integration methods, the implementation at the formal level of these infinite renormalizations converts ℓ-dependent quantum propagators into conventional Feynman propagators. Thus, the adoption of classical Lorentz frames and classical Minkowski geometry in a special relativistic *quantum* context emerges as the root cause of the divergencies that plague conventional theory.

3.1. The Uncertainty Principle and Representations of the Galilei Group

The canonical quantization procedure was originally formulated by Dirac (1925, 1930) for nonrelativistic quantum mechanics with n degrees of freedom. In that context it assigns to each classical observable, represented by a polynomial, or more generally by an analytic function $F(q,p)$ of the canonical position and momentum variables belonging to points $(q,p) \in \mathbf{R}^{2n}$ in the classical phase space $\Gamma = \mathbf{R}^{2n}$, a self-adjoint operator[1] $F(Q,P)$ on a Hilbert space $\mathcal{H}$ in such a manner that Poisson brackets are transformed into commutator brackets. In particular, in this quantization procedure the Poisson brackets

$$\left\{q^j,q^k\right\} = \left\{p_j,p_k\right\} = 0 \ , \qquad \left\{q^j,p_k\right\} = \delta^j{}_k \ , \qquad j,k = 1,\ldots,n \ , \tag{1.1}$$

are transformed into the well-known quantum canonical commutation brackets,

$$\left[Q^j,Q^k\right] = \left[P_j,P_k\right] = 0 \ , \qquad \left[Q^j,P_k\right] = i\delta^j{}_k \ , \qquad j,k = 1,\ldots,n \ , \tag{1.2}$$

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[1] A self-contained presentation of orthodox nonrelativistic quantum mechanics, which provides all the precise definitions and theorems required for a mathematically rigorous formulation of this discipline, and thereby implicitly reveals some of the root causes of the difficulties discussed in this chapter, can be found in (Prugovečki, 1981). Most of the quantum mechanics textbooks dealing with the physical aspects of this subject tend to ignore these key mathematical aspects, but a recent exception can be found in (Galindo and Pascual, 1991).

for position and momentum operators.

The above canonical commutation relations provide, in conjunction with Born's statistical interpretation of quantum mechanics, the best-known embodiment of Heisenberg's uncertainty principle, in the form of the following n inequalities:

$$\Delta q^j \cdot \Delta p_j \geq \hbar/2 \ , \qquad j = 1,...,n \ . \tag{1.3}$$

From the point of view of the nonrelativistic quantum frames which will be introduced in the next section, it is important to recall that the above "uncertainties" do not pertain to single measurements. Rather, they denote the standard deviations from the mean value of the measurement outcomes of position and momentum variables performed on an ensemble of identical quantum systems in the same quantum state; whereby, in principle, the position measurements should be envisaged as being performed, with absolute precision and in relation to a classical inertial frame of reference, on part of that ensemble; whereas, the momentum measurements should be envisaged as being performed, also with absolute precision, and in relation to the same classical inertial frame, on the remainder of that ensemble. On the other hand, Heisenberg's (1930b) well-known *gedanken* experiments deal with the uncertainties in the measurements performed on *single* systems. It is from this rarely commented upon dichotomy, and from the ensuing reconciliation of these *two* aspects of the uncertainty principle (Prugovečki, 1967), that the first indications as to the necessity of replacing classical reference frames with quantum counterparts emerges already in the nonrelativistic regime (Prugovečki, 1976, 1977; Ali and Prugovečki, 1977).

At the mathematical level, canonical quantization rules give rise to ordering problems as well as to mutually inconsistent results when Cartesian coordinates are replaced with classical canonical variables in some general curvilinear coordinates (Dirac, 1949). As a matter of fact, the following was pointed out by Dirac in his first papers on Hamiltonian dynamics with constraints: "The [canonical quantization procedure] of passing from the classical to the quantum theory is not well-defined, because whenever a classical quantity involves a product of two factors whose [Poisson bracket] does not vanish, there is an ambiguity in the order in which the two factors should appear in the corresponding quantum expression. In practice with simple examples one finds no difficulty in deciding what the order should be. With complicated examples it may be impossible to choose the order in each case so as to make all the quantum equations consistent, and then one would not know how to quantize the theory. The present-day methods of quantization are all of the nature of practical rules, whose application depends on considerations of simplicity." (Dirac, 1950, p. 145.)

These "ambiguities" represent one of the main impediments to extending the canonical quantization method to the general relativistic regime, where the principle of general covariance prohibits the singling out of any particular class of preferred coordinates. This impediment can be removed if one bases the quantization procedure on a purely group-theoretical foundation (Prugovečki, 1978c,d), related to changes of *inertial* frames. In such a formulation, it is *not the choice of coordinates, but rather the choice of reference frames that assumes paramount significance for the quantization procedure.* Indeed, as we saw in the preceding chapter, in the classical context such frames can be introduced even in the generic cases of curved spacetime manifolds; however, in those cases, their elements *cannot* be

represented by vectors or covectors that belong to the manifold itself, but rather to its tangent or cotangent spaces. This observation turns out to be of paramount significance for the extrapolation to the *general* relativistic regime of the group-theoretical method of quantization presented and explained in this chapter.

The first step in this extrapolation consists in formulating a concept of *global Galilei frame* of reference as a nonrelativistic counterpart of the concept of global Lorentz frame of reference used in classical special relativity. As a physical object, a global Galilei frame consists of an inertial frame of reference, visualizable as three (in principle infinitely extended) rigid rods joined together at right angles, and of a system of synchronized clocks stationary in relation to those rods. Of course, this definition appears to coincide with the one given by Einstein (1905) for a Lorentz frame, but the key difference lies in the additional *assumption* that signals of arbitrarily great (and, in a conceptual limit, infinitely high) velocities can be transmitted for the synchronization of all clocks, so that under a generic change of such frames the coordinates of a classical point particle transform as follows:

$$(b,\mathbf{a},\mathbf{v},R):\ (\mathbf{x},t)\ \mapsto\ (\mathbf{x}'=\mathbf{a}+\mathbf{v}t+R\mathbf{x},\, t'=t+b)\ ,\qquad \mathbf{x}\in\mathbf{R}^3\ ,\ \ t\in\mathbf{R}^1. \tag{1.4}$$

The above mapping of $\mathbf{R}^4$ onto itself represents a general *Galilean transformation* in a Newtonian spacetime. It corresponds to a time translation in the amount b in the initial value of time measurements, to a space translation in the amount $\mathbf{a}\in\mathbf{R}^3$, to a boost to the 3-velocity $\mathbf{v}\in\mathbf{R}^3$, and to a rotation $R\in\mathrm{SO}(3)$ – all performed on the original inertial frame.

At this purely operational level, a Galilean transformation represents the procedure of changing classically describable inertial frames of reference in a manner which preserves the fundamental laws of Newtonian classical mechanics, and of changing the zero-point of clocks attached to such frames – which in Newtonian physics keep track of "absolute" time. Upon performing successively any two general Galilean transformations, it can be verified by substituting the coordinates resulting from the first transformation into the coordinates of the second that the final outcome is again a Galilean transformation:

$$(b'',\mathbf{a}'',\mathbf{v}'',R''):\ (\mathbf{x},t)\ \mapsto\ (\mathbf{x}''=\mathbf{a}''+\mathbf{v}''t+R''\mathbf{x},\, t''=t+b'')\ , \tag{1.5a}$$

$$b''=b'+b\ ,\quad \mathbf{a}''=\mathbf{a}'+R'\mathbf{a}+b\mathbf{v}'\ ,\quad \mathbf{v}''=\mathbf{v}'+R'\mathbf{v}\ ,\quad R''=R'R\ . \tag{1.5b}$$

From this it immediately follows that the family of all Galilean transformations constitutes a group, known as the *Galilei group*, which obeys the following group multiplication law:

$$(b',\mathbf{a}',\mathbf{v}',R')\cdot(b,\mathbf{a},\mathbf{v},R)=(b'+b,\ \mathbf{a}'+R'\mathbf{a}+b\mathbf{v}',\ \mathbf{v}'+R'\mathbf{v},\ R'R) \tag{1.6}$$

The Galilei group is a locally compact[2] Lie group whose unitary irreducible representations have been exhaustively studied and classified (Lévi-Leblond, 1963, 1971), and turn out[3] to fall within two very distinct classes. The first class obeys the ordinary law of mul-

[2] A topological space is *locally compact* if each one of its points possesses compact neighborhoods (cf. Footnote 12 in Chapter 2). The definition of those concepts in the theory of locally compact topological groups and their representations which are not given in the sequel, since they are not absolutely essential for the considered applications to quantum mechanics, can be found in (Barut and Raczka, 1986).

[3] The theory of unitary irreducible representations of locally compact topological groups was developed primarily by Mackey (1951-53), who also applied it to quantum mechanics (Mackey, 1968). Barut and Raczka (1986) provide a self-contained and complete presentation of all basic concepts and results in this theory.

tiplication for elements of a group representation in a vector space, namely

$$U_{\rm cl}(b'',\mathbf{a}'',\mathbf{v}'',R'') = U_{\rm cl}(b',\mathbf{a}',\mathbf{v}',R')U_{\rm cl}(b,\mathbf{a},\mathbf{v},R)\,; \tag{1.7}$$

whereas, the second class consists only of *ray* (or *projective*) *representations*, for which that law is modified by the presence of a phase factor ω, so that it becomes

$$\hat{U}(b'',\mathbf{a}'',\mathbf{v}'',R'') = \exp(-i\omega)\,\hat{U}(b',\mathbf{a}',\mathbf{v}',R')\,\hat{U}(b,\mathbf{a},\mathbf{v},R)\ , \tag{1.8a}$$

$$\omega = m(\tfrac{1}{2}\mathbf{v}'^{\mathbf{2}}\,b + \mathbf{v}'\cdot R'\mathbf{a}) \in \mathbf{R}^1\ . \tag{1.8b}$$

The essential point is that these ray representations can be also viewed as providing ordinary vector representations of the central extension (cf. Barut and Raczka, 1986, p. 401) of the Galilei group, whose elements $(\theta,b,\mathbf{a},\mathbf{v},R)$ include a real phase θ, so that the phase factor in (1.8b) can be incorporated into their group multiplication law as follows:

$$\begin{aligned}&(\theta',b',\mathbf{a}',\mathbf{v}',R')\cdot(\theta,b,\mathbf{a},\mathbf{v},R)\\ &\quad= \left(\theta'+\theta+m\left(\tfrac{1}{2}\mathbf{v}'^{\mathbf{2}}\,b+\mathbf{v}'\cdot R'\mathbf{a}\right),\, b'+b,\, \mathbf{a}'+R'\mathbf{a}+b\mathbf{v}',\, \mathbf{v}'+R'\mathbf{v},\, R'R\right)\end{aligned} \tag{1.9}$$

The remarkable thing is that, upon setting above $b' = b = 0$ and $R' = R = I$, we obtain the group multiplication law for the well-known Weyl-Heisenberg group, whose Lie algebra gives rise to the canonical commutation relations in (1.2). Indeed, this group plays a crucial role in the derivation (cf. Prugovečki, 1981, p. 342) of von Neumann's theorem on the uniqueness, modulo unitary transformations, of all irreducible representations of these commutation relations for a finite number of degrees of freedom. On the other hand, we see from (1.6) and (1.9) that in case of three degrees of freedom this group emerges naturally as a subgroup of the central extension of the Galilei group, which in turn emerges naturally from the induced representation of the Galilei group, which can be derived (Lévy-Leblond, 1963, 1971) in a manner totally independent of the canonical quantization procedure.

As an example of this fact, let us consider the following ray representation

$$\begin{aligned}&\tilde{U}(b,\mathbf{a},\mathbf{v},R)\ :\ \tilde{\psi}(\mathbf{k},t)\ \mapsto\ \tilde{\psi}'(\mathbf{k},t)\\ &\quad= \exp\left\{i\left[\tfrac{1}{2}m\mathbf{v}^{\mathbf{2}}(t-b)-\mathbf{k}\cdot(\mathbf{a}+\mathbf{v}(t-b))\right]\right\}\tilde{\psi}(R^{-1}(\mathbf{k}-m\mathbf{v}),t-b)\ ,\end{aligned} \tag{1.10}$$

derived in the Hilbert space $L^2(\mathbf{R}^3)$ by the inducing method, and acting on all the complex-valued functions of $\mathbf{k} \in \mathbf{R}^3$ that are square-integrable on $\mathbf{R}^3$ with respect to the Lebesgue measure and satisfy[4] the free Schrödinger equation in the momentum representation. This representation coincides with the one ordinarily derived from the quantum mechanics of a single particle of spin zero and mass m, moving freely in three space dimensions (cf., e.g., Galindo and Pascual, 1991), by a transformation to the momentum representation.

[4] Strictly speaking, only for a dense set in $L^2(\mathbf{R}^3)$ (cf. Prugovečki, 1981, Sec. 3.2 in Chapter IV). Moreover, the elements of $L^2(\mathbf{R}^3)$ are not these functions themselves, but rather the equivalence classes consisting of all such functions that are equal almost everywhere in $\mathbf{R}^3$ with respect to the Lebesgue measure (*ibid*, Sec. 4.2 in Chapter II). Hence the value of these functions can be changed at will on a set of Lebesgue measure zero (such as on a finite or countably infinite set) without changing the quantum state vector represented by that wave function. This fact of apparently only mathematical interest will be shown in Sec. 3.2 to have significant physical implications.

Indeed, by taking the inverse Fourier transforms[5] of the $\mathbf{k}$-dependent functions in (1.10),

$$\hat{\psi}(\mathbf{x},t) \;=\; (2\pi)^{-3/2}\int_{\mathbf{R}^3} \exp(i\mathbf{x}\cdot\mathbf{k})\,\tilde{\psi}(\mathbf{k},t)d\mathbf{k} \quad , \tag{1.11}$$

we arrive at the configuration representation, which consists of the wave functions in the Hilbert space of $L^2(\mathbf{R}^3)$ with inner product

$$\left\langle \hat{\psi}_1 \middle| \hat{\psi}_2 \right\rangle \;=\; \int_{\mathbf{R}^3} \hat{\psi}_1^*(\mathbf{x},t)\hat{\psi}_2(\mathbf{x},t)\,d\mathbf{x} \quad . \tag{1.12}$$

These wave functions obey the familiar Schrödinger equation in configuration space,

$$i\partial_t\,\hat{\psi}(\mathbf{x},t) = (-1/2m)\Delta\hat{\psi}(\mathbf{x},t) \quad , \qquad \partial_t = \partial/\partial t \quad , \tag{1.13}$$

so that they represent the state vectors of such a particle in the Schrödinger picture, whose free time evolution is given by:

$$\hat{\psi}_t = \exp(-iH_0t)\hat{\psi} \quad , \qquad H_0 = (-1/2m)\Delta \quad , \qquad \hat{\psi}_t(\mathbf{x}) = \hat{\psi}(\mathbf{x},t) \quad . \tag{1.14}$$

The unitary Fourier-Plancherel transform of the representation in (1.10) can be easily computed by using (1.11), and it is then seen to assume under this transformation the form

$$\begin{aligned} &\hat{U}(b,\mathbf{a},\mathbf{v},R) \;:\; \hat{\psi}(\mathbf{x},t) \;\mapsto\; \hat{\psi}'(\mathbf{x},t) \\ &\quad = \exp\left[i\left(-\tfrac{1}{2}m\mathbf{v}^2(t-b)+m\mathbf{v}\cdot(\mathbf{x}-\mathbf{a})\right)\right]\hat{\psi}(R^{-1}[\mathbf{x}-\mathbf{a}-\mathbf{v}(t-b)],t-b) \quad . \end{aligned} \tag{1.15}$$

This result is in complete accordance with the fact, first pointed out by Bargmann (1954), that under all Galilean transformations involving non-zero velocity boosts, the free Schrödinger equation is *not* left invariant by any of the representations that satisfy (1.7), but it is left invariant by ray representations that satisfy (1.8) with appropriate choices of phase factors, as is the case with the one in (1.15). Thus, in the nonrelativistic regime, the choice between the two classes of representations amounts to a choice between classical and quantum spacetime symmetry features. Indeed, it can be shown (Prugovečki, 1978a,b) that classical as well as quantum statistical mechanics can be formulated within structurally similar mathematical frameworks, which brings forth their underlying common features. In that case the Galilei symmetry representations to which they conform constitute the single most fundamental mathematical feature that distinguishes them: in such formulations of statistical mechanics, the classical regime is characterized by vector representations of the Galilei group that obey (1.7); whereas, the quantum regime is characterized by ray representations of the Galilei group that obey (1.8) – cf. Chapter 3 in Prugovečki, 1984, for a review of these unified formulations of classical and quantum statistical mechanics.

Naturally, the ray representations of the Galilei group which obey the multiplication rule in (1.8) are not acceptable in nonrelativistic classical statistical mechanics, since they

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[5] The Lebesgue integral in (1.11) (which is equal to the corresponding Riemann integral whenever the latter exists) actually exists only for functions which are integrable as well as square-integrable in $\mathbf{k}\in\mathbf{R}^3$. Hence, in order to obtain a unitary operator on $L^2(\mathbf{R}^3)$ the Fourier transform has to be extended into the Fourier-Plancherel transform, which generally requires taking a limit-in-the-mean – cf. Sec. 4.3 in Chapter III of (Prugovečki, 1981).

do not leave invariant all the Newton's laws. On the other hand, the acceptability in quantum mechanics of ray representations in general is warranted by Wigner's theorem on such representations (cf., e.g., Barut and Raczka, 1986, p. 400). This theorem reflects the fact that in quantum mechanics it is not the probability amplitudes that are directly measurable, but rather it is *only* the corresponding probabilities that are observable.

For example, in the case of a normalized wave function (1.11) in configuration space, it is the probability measure

$$P^{\mathbf{Q}}_{\hat{\psi}_t}(B) = \int_B |\hat{\psi}(\mathbf{x},t)|^2 d\mathbf{x} \ , \qquad B \subset \mathbf{R}^3 \ , \tag{1.16a}$$

that, according to the orthodox interpretation of quantum mechanics, provides *all* the *probabilities* of measurement outcomes as to whether a quantum point particle, which is in a quantum state represented by the wave function (1.11), is located at the instant t at points $\mathbf{x} \in B$ in relation to a given classical inertial frame. Moreover, according to the same orthodox interpretation, applied to the same particle, which is at that very instant t in the same quantum state as that in (1.16a), the *probabilities* of momentum measurement outcomes with values $\mathbf{k} \in B$ in relation to the same classical inertial frame are given by

$$P^{\mathbf{P}}_{\hat{\psi}_t}(B) = \int_B |\tilde{\psi}(\mathbf{k},t)|^2 d\mathbf{k} \ , \qquad B \subset \mathbf{R}^3 \tag{1.16b}$$

According to the uncertainty principle, such measurements of position and momentum cannot be performed simultaneously on the *same* system. However, they can be performed, in the earlier discussed manner, on separate subensembles of an ensemble of such systems, if that entire ensemble is in the pure state represented by the density operator $\rho = |\psi\rangle\langle\psi|$, where ψ coincides with the state vector of each one of its elements.

This raises the question of whether such measurements are *informationally* complete in the sense that they might be capable of pinpointing, at least in principle, any quantum state represented by some density operator ρ. Indeed, position operators and momentum operators emerge at $t = 0$ as the infinitesimal generators of the ray representation in (1.15):

$$\hat{Q}^j = -\frac{i}{m}\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\hat{U}\left(0,0,\mathbf{v}_{(j)}(\varepsilon),I\right) - \mathbf{1}\right] \ , \qquad v^i_{(j)}(\varepsilon) = \varepsilon\, \delta^i_j \ , \tag{1.17a}$$

$$\hat{P}_j = i \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\hat{U}\left(0,\mathbf{a}_{(j)}(\varepsilon),0,I\right) - \mathbf{1}\right] \ , \qquad a^i_{(j)}(\varepsilon) = \varepsilon\, \delta^i_j \ . \tag{1.17b}$$

As is well-known, the set of position operators, as well as the set of momentum operators,

$$(\hat{Q}^j \hat{\psi})(\mathbf{x},t) = x^j \hat{\psi}(\mathbf{x},t) \ , \qquad (\hat{P}_j \hat{\psi})(\mathbf{x},t) = -i\, \partial \hat{\psi}(\mathbf{x},t)/\partial x^j \ , \qquad j = 1,2,3 \ , \tag{1.18}$$

give rise to complete sets of compatible observables. A general mathematical analysis (cf. Prugovečki, 1981, pp. 311-328) of this concept reveals that for any complete set of observables represented by operators which have a purely continuous spectrum, the simultaneous measurement of all the observables in that set cannot unambiguously pinpoint quantum states. This conclusion will be confirmed in Sec. 3.2 for both the position in (1.17a) and the momentum observables in (1.17b) by basically physical arguments, but in

the meantime, one can ask the following pertinent question: is it possible, at least in principle, to uniquely determine, modulo a constant phase factor, *any* state vector ψ of a quantum system by means of position measurements on a subensemble of an ensemble of quantum systems prepared in the same pure state, whereby all the probabilities in (1.16a) are *exactly* determined, and of momentum measurements on the remaining subensemble, whereby all the probabilities in (1.16b) are also *exactly* determined?

Perhaps surprisingly, we shall see[6] in the next section that the answer to this question is negative even in the highly idealized case of sequences of increasingly accurate position and momentum measurements on disjoint subensembles (so that the uncertainty principle is never violated) which approach in the limit the above required *absolute* precision: even if such measurements could be carried out with such *perfect* accuracy, there are quantum states which still would not be uniquely determined by such measurement procedures.

Thus, although the method of induced representations can supply, when applied to the Galilei group, a mathematical procedure that leads to all the *fundamental* observables ordinarily derived by the canonical method, it still leaves open the question of their informational completeness, which is crucial to any satisfactory solution of the problem of quantum state determination by means of measurements of spatio-temporal relationships.

3.2. Quantum Mechanics and Informational Completeness

In classical physics the concept of informational completeness is so deeply ingrained in the very concept of spatio-temporal frame of reference (such as a Galilean or a Lorentz frame) that it is never explicitly stipulated. Rather, it is taken for granted that measurements performed with respect to such frames can, at least in principle, unambiguously and completely determine *any* state of *any* classical system.

As repeatedly pointed out by Einstein in his key papers, in the ultimate analysis all measurements can be reduced to verifications of spatio-temporal relationships and coincidences. Stated in his own words, "the results of our measurings are nothing but verifications of meetings of the material points of our measuring instruments with other material points, coincidences between the hands of a clock and points on the clock dial, and observed point events happening at the same place at the same time. The introduction of a system of reference serves no other purpose than to facilitate the description of the totality of such coincidences." (Einstein, 1916, p. 117.) At the theoretical level, systems of reference not only "facilitate the description," but are essential to that description: without them the mathematical formulations of classical mechanics, classical electromagnetism, etc. would be impossible to achieve in their generally accepted forms.

All these epistemic observations remain basically true in the quantum regime, once certain modification are made. In fact, in his 1949 autobiographical notes Einstein made it clear that, in extending his ideas from the macroscopic to the microscopic world, "the material point . . . can hardly be conceived any more as the basic concept of [relativity] theory", and that "strictly speaking measuring rods and clocks would have to be represented as solutions of the basic equations (objects consisting of moving atomic configura-

[6] This fact appears to have been first noted by Reichenbach (1948), but it was independently rediscovered by Gale *et al.* (1968), Prugovečki (1977), Stulpe and Singer (1990), and other researchers.

tions), [and] not, as it were, as theoretically self-sufficient entities." (Cf. Schilpp, 1949, pp. 59-61.) On the other hand, as amply demonstrated by Heisenberg's (1930b) *gedanken* experiments, in the ultimate analysis measurements of quantum observables can be indeed reduced to verifications of spatio-temporal relationships and coincidences. This is very clearly illustrated by the time-of-flight measurements of velocities, the measurements of momenta of charged quantum particles in a homogeneous magnetic field, the Stern-Gerlach measurement of spin, etc.[7]

In Bohr's (1934, 1955, 1961) well-known analysis of such measurements, a fundamental dichotomy is displayed, however, by his treatment of a "system" versus that of an "apparatus": whereas, the former is always treated as a quantum entity, the later is viewed *as if* it were a classical entity, and therefore *not* as "consisting of moving atomic configurations," in the manner suggested by Einstein in the above quotation. The primary reason invoked by Bohr in defence of this *convention* was that "the account of the experimental arrangement and the results of observations must be expressed in unambiguous language with suitable application of the terminology of classical physics." (Bohr, 1961, p. 39.) Since, however, classical physics is obviously not the *only* source of "unambiguous language" for human communication, the philosophical reasons behind Bohr's adoption of such a convention have to run much deeper. They have been actually traced[8] to the influence of an existentialist philosophy during his early years, which ran counter to the primarily positivistic outlook which he adopted in his later years.

On the other hand, the often greatly over-simplified presentation of Bohr's philosophical outlook in many textbooks on quantum mechanics leaves the impression that the doctrine of treating the "apparatus" as a classical object is an insuperable component of the entire Copenhagen school. This perception has influenced other schools of thought, such as those subscribing to Everett's "many-worlds interpretation" of quantum mechanics (DeWitt and Graham, 1973). As a matter of fact, in critical examining that interpretation , J. S. Bell pointed out the following: "In dividing the world into pieces *A* and *B* Everett is indeed following an old convention of abstract quantum measurement theory, that the world does fall neatly into such pieces – instruments and systems. In my opinion this is an unfortunate convention. The real world is made of electrons and protons and so on, and as a result the boundaries of natural objects are fuzzy, and some particles in the boundary can only doubtfully be assigned to either object or environment. I think that fundamental physical theory should be so formulated that such artificial divisions are manifestly inessential." (Bell, 1987b, p. 96).

It has to be realized, however, that Bohr's "system" *vs* "apparatus" dichotomy, and all its implications for the quantum theory of measurement (Jammer, 1974; d'Espagnat, 1976), were not unconditionally advocated in the relevant writings of the other two principal representatives of the Copenhagen school, namely those of Born (1956) and of Heisenberg (1958). Indeed, as we pointed out in Sec. 1.3, Born had questioned the

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[7] Cf. (Prugovečki, 1981), pp. 5-9. The theory of quantum measurement is surveyed primarily in books on the philosophy of quantum mechanics (cf., e.g., d'Espagnat, 1976; Jammer, 1974), as well as in review papers that deal with the foundations of quantum mechanics (cf., e.g., Ballentine, 1970).

[8] Jammer (1966) appears to have been the first historian of science to point out the influence of Kierkegaard's existentialistic philosophy on Bohr. More recently, Folse (1985), Murdoch (1989) and Selleri (1990) have documented this influence via Bohr's father and via his mentor, the Danish philosopher Harald Høffding.

physical significance of the classical concept of "point in a continuum" (Born, 1955), and in fact had even introduced the idea of a quantum metric operator (Born, 1938, 1949). Similarly, while discussing the internal spatio-temporal structure of elementary particles, Heisenberg advocated ideas which were epistemically closely related to that of a quantum metric operator, as he pointed out the "possibility of which neither Kant nor the ancient philosophers could have thought: the word 'dividing' loses its meaning" (Heisenberg, 1976, p. 38). Indeed, since all measurements of physical quantities can be reduced, in the ultimate analysis, to measurements of spatio-temporal relationships, any conceptualization of spacetime which negates literal physical meaning to the *mental* procedure of indefinite subdivision of spacetime into smaller and smaller parts implies that there is a residual *irreducible* indeterminacy in the measurement of *all* physical quantities. Within the ontology of such a philosophical outlook, the classical treatment of an "apparatus" can be maintained in the quantum regime only as a convenient *approximation* at a *pragmatic* macroscopic level, but not as a fundamental epistemic principle.

The mathematical embodiment of this pragmatic approximation is normally injected into nonrelativistic quantum mechanics via the concept of a complete set **A** of compatible observables (Dirac, 1930), represented in a global Galilei frame of reference by a family $\{A_1,..., A_n\}$ of commuting self-adjoint operators $A_1,..., A_n$. According to the mathematically rigorous treatment of this concept, suggested by the work of von Neumann (1932), to each such family is associated a *unique* joint *spectral measure*

$$E^{\mathbf{A}} : B \mapsto E^{\mathbf{A}}(B) = E^{\mathbf{A}}(B)^* = E^{\mathbf{A}}(B)^2 \ , \tag{2.1}$$

which assigns projectors (i.e., orthogonal projection operators) to all Borel[9] sets B in $\mathbf{R}^n$. At the mathematical level, this joint spectral measure supplies the spectral measure of each one of the operators $A_1,..., A_n$; whereas, at the physical level, it provides the probabilities

$$P^{\mathbf{A}}_{\psi}(B) = \left\langle \psi \middle| E^{\mathbf{A}}(B)\psi \right\rangle = \mathrm{Tr}[\rho E^{\mathbf{A}}(B)] \ , \qquad \rho = |\psi\rangle\langle\psi| \ , \tag{2.2}$$

for all the *simultaneous* measurement outcomes of those observables on a system in a pure state represented by the normalized state vector ψ. For example, the spectral measures whose expectation values supply the probability measures in (1.16), as well as the probabilities themselves, can be expressed in the following respective forms,

$$(E^{\mathbf{Q}}(B)\hat{\psi})(\mathbf{x},t) = \chi_B(\mathbf{x})\hat{\psi}(\mathbf{x},t) \ , \qquad P^{\mathbf{Q}}_{\hat{\psi}_t}(B) = \left\langle \hat{\psi}_t \middle| E^{\mathbf{Q}}(B)\hat{\psi}_t \right\rangle, \tag{2.3a}$$

$$(E^{\mathbf{P}}(B)\tilde{\psi})(\mathbf{k},t) = \chi_B(\mathbf{k})\tilde{\psi}(\mathbf{k},t) \ , \qquad P^{\mathbf{P}}_{\tilde{\psi}_t}(B) = \left\langle \tilde{\psi}_t \middle| E^{\mathbf{P}}(B)\tilde{\psi}_t \right\rangle \ , \tag{2.3b}$$

where χ_B generically denotes the *characteristic function* of a subset B of some larger set (such as $\mathbf{R}^3$ in the case of the characteristic functions in (2.3)), which assumes the value 1 on that set, the value 0 outside that set.

A set of commuting self-adjoint operators with joint spectral measure $E^{\mathbf{A}}$ is called

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[9] The Borel sets in $\mathbf{R}^n$ are elements of the Boolean σ-algebra generated by all intervals in $\mathbf{R}^n$ – basically, by taking *countable* unions and intersections of intervals in $\mathbf{R}^n$. For further details on this and other topics related to (2.2)-(2.3), see (Prugovečki, 1981), Sec. 1.3 in Chapter II, Sec. 5.4 in Chapter III, and Sec. 1.3 in Chapter IV.

informationally complete (Prugovečki, 1977) with respect to a certain family of pure or mixed quantum states, generically represented by density operators ρ, if for any density operator ρ_1 in that family the equalities $\mathrm{Tr}[\rho_1 E^{\mathbf{A}}(B)] = \mathrm{Tr}[\rho_2 E^{\mathbf{A}}(B)]$ on all Borel sets B on which that spectral measure is defined implies that $\rho_2 = \rho_1$. Hence, the statement which we made in the concluding paragraphs of Sec. 3.1, to the effect that such standard complete sets of observables as those represented by position operators or by momentum operators, are *not* informationally complete, might appear, at first sight, to contradict orthodox quantum theory. We shall therefore demonstrate now that this is actually not the case.

According to Dirac's (1930, 1947) original definition of a complete set $\{A_1,...,A_n\}$ of observables, such a set is informationally complete with respect to the family of state vectors ψ which are *bona fide* eigenvectors corresponding to the simultaneous eigenvalues $\lambda_1,...,\lambda_n$ of the respective self-adjoint operators $A_1,...,A_n$: by that definition, if such a set $\{\lambda_1,...,\lambda_n\}$ is the outcome of an actual measurement, then the state vector ψ has been pinpointed modulo a physically irrelevant multiplicative constant. That does not help, however, with those state vectors which do *not* belong to the eigenspace corresponding to any simultaneous *bona fide* eigenvalues, i.e., to values $\lambda_1,...,\lambda_n$ which do not belong to the *point* spectra of the respective self-adjoint operators $A_1,...,A_n$.

In particular, the sets $\mathbf{Q} = \{Q^1,Q^2,Q^3\}$ and $\mathbf{P} = \{P_1,P_2,P_3\}$ of position and momentum operators have pure continuous spectra, so that they possess no *bona fide* eigenvalues whatsoever, and they are *not* informationally complete with respect to *any* family of states. For example, the two quantum states whose wave functions satisfy

$$\left|\hat{\psi}_1(\mathbf{x})\right|^2 \equiv \left|\hat{\psi}_2(\mathbf{x})\right|^2 = r^2(\mathbf{x}) \quad , \qquad \hat{\psi}_j(\mathbf{x}) = r(\mathbf{x})e^{i\theta_j(\mathbf{x})} \quad , \tag{2.4}$$

share the same probabilities for position measurement outcomes, but if

$$\theta_1(\mathbf{x}) \neq \theta_2(\mathbf{x}) + \text{const.} \ (\mathrm{mod}\, 2\pi) \tag{2.5a}$$

holds almost everywhere in $\mathbf{R}^3$, they obviously represent distinct states. Moreover, we can have in addition that

$$\theta_1(\mathbf{x}) = -\theta_2(-\mathbf{x}) + \text{const.} \ (\mathrm{mod}\, 2\pi) \quad , \tag{2.5b}$$

in which case it follows that

$$\begin{aligned} \tilde{\psi}_1(\mathbf{k}) &= (2\pi)^{-3/2} \int_{\mathbf{R}^3} \exp(-i\mathbf{k}\cdot\mathbf{x})\, \hat{\psi}_1(\mathbf{x}) d\mathbf{x} \\ &= (2\pi)^{-3/2} \int_{\mathbf{R}^3} \exp(i\mathbf{k}\cdot\mathbf{x})\, \hat{\psi}_2^*(\mathbf{x}) d\mathbf{x} = \tilde{\psi}_2^*(\mathbf{k}) \quad , \end{aligned} \tag{2.6}$$

so that the momentum probability distributions are also equal for such wave functions. Nevertheless, for choices of phases in the configuration-space representation which are not antisymmetric under space reflections, the two wave functions represent distinct states. Hence, even taken in conjunction, the complete sets $\mathbf{Q}$ and $\mathbf{P}$ cannot secure *informational* completeness for all pure states of an ensemble of single-particle systems.

Clearly, this conclusion remains valid for multi-particle systems, since by choosing in the configuration-space representation combinations of phases with the properties (2.5) for individual particles, we can deduce that we have an even greater lack of informational

completeness.

It might be thought that this conclusion can be somehow avoided by using δ-"functions" and plane waves, which in the above considered case of a single nonrelativistic quantum particle of zero spin would allow us to *formally* write

$$\langle \mathbf{x}' | \mathbf{x} \rangle = \delta^3(\mathbf{x} - \mathbf{x}') \ , \qquad \langle \mathbf{x}' | \mathbf{k} \rangle = (2\pi)^{-3/2} \exp(i\mathbf{k} \cdot \mathbf{x}') \ . \tag{2.7}$$

In fact, in some physics textbooks it is argued that "once we admit the generalized states $|\mathbf{x}\rangle$ to the Hilbert space we should also admit the scalar products $\langle \mathbf{x}|\mathbf{x}' \rangle$ to the space of allowed [wave] functions." (Kleinert, 1990, p. 23.)

However, neither the above δ-"functions", nor the above plane waves, are Lebesgue square-integrable functions. Hence, they certainly do not belong to the Hilbert space $L^2(\mathbf{R}^3)$ with the inner product (1.12). For that reason, von Neumann (1932) avoided the use of δ-"functions". Their mathematical nature was eventually clarified by L. Schwartz (1945), and the mathematically *correct* general treatment of the objects in (2.7) was subsequently supplied by the theory of rigged (Gel'fand *et al.*, 1964, 1968) and of equipped (Berezanskii, 1968, 1978) Hilbert spaces.

In these extensions of the Hilbert space framework the objects in (2.7) are described as elements of eigen*function* expansions, but *not* as eigen*vectors* of any operators in the original Hilbert space. Adaptations of these frameworks to quantum physics have been carried out by Antoine (1969, 1980), Prugovečki (1973), and others, by using various mathematical techniques. Regardless of which one of these mathematically *correct* approaches one adopts, it is clear that Dirac's "bra" and "ket" vectors, familiar to all physicists, are not related to a single Hilbert space $\mathcal{H}$, but to "triples" of topological vector spaces:

$$(\mathbf{x}|, (\mathbf{k}| \in \mathcal{H}_- \supset \mathcal{H} \supset \mathcal{H}_+ \ , \qquad (\mathbf{x}|, (\mathbf{k}| \notin \mathcal{H} = L^2(\mathbf{R}^3) \cong \mathcal{H}^* \ . \tag{2.8}$$

The use of round brackets in (2.8) is meant to underline the fact that $\mathcal{H}_-$ is, in general, a topological vector space which provides an *extension* of the original Hilbert space $\mathcal{H}$ of state vectors; whereas, $\mathcal{H}_+$ is dense subspace of $\mathcal{H}$ equipped with a topology that is finer than the norm topology of $\mathcal{H}$. The construction of the triple of spaces in (2.8) then makes $\mathcal{H}_-$ equal to the dual of $\mathcal{H}_+$, whereas $\mathcal{H}$ is identified with its own dual $\mathcal{H}^*$.

By modifying Dirac's "bra" and "ket" notation, we can write in such a formalism

$$(\mathbf{x}|\psi\rangle = \psi(\mathbf{x}) \ , \qquad \psi \in \mathcal{H}_+ \subset \mathcal{H} = L^2(\mathbf{R}^3) \ . \tag{2.9}$$

This modification of a very familiar notation is meant to underline a physically as well as mathematically very important point: the sesquilinear form on the left-hand side of the above equality is *not* an inner product, and the domain of the variable ψ appearing in it *cannot* be extended to the entire Hilbert space $\mathcal{H}$. This follows from the basic mathematical fact that the generic element of $\mathcal{H}$ in (2.9) is not a single function, but rather an equivalence class of almost everywhere (in the Lebesgue sense) equal functions, which are such that one can change the value of any one of these functions $\psi(\mathbf{x})$ at any given point $\mathbf{x}$ without leaving that equivalence class – namely, in physical terms, without changing the quantum state. Upon restricting oneself to a convenient dense subspace $\mathcal{H}_+$, one can choose a representative function $\psi(\mathbf{x})$ for which (2.9) holds true. However, that is *not* possible globally

on $\mathcal{H}$, so that when one writes

$$E^{\mathbf{Q}}(B) = \int_B |\mathbf{x}) \, d^3\mathbf{x} \, (\mathbf{x}| \ , \qquad E^{\mathbf{P}}(B) = \int_B |\mathbf{k}) \, d^3\mathbf{k} \, (\mathbf{k}| \ , \tag{2.10}$$

the above integrals can be applied only to vectors from the dense subspaces $\mathcal{H}_+$, which is chosen on mathematical grounds, rather than due to any physical significance. This means that physically one can speak of the probabilities in (2.3) for position and of momentum measurement outcomes within Borel regions B in configuration and momentum space, respectively, but *not* of probability densities at *single* points in configuration space or in momentum space for *arbitrary* wave functions.

These limitations on localizability and perfectly precise measurability of momenta are in complete concordance with Born's maxim that, in quantum theory: "Statements like 'A quantity x has a completely definite value' (expressed by a real number and represented by a point in the physical continuum) . . . have no physical meaning." (Born, 1956, p.167.) Moreover, since momentum operators do not commute with their position counterparts, in the case of position measurements it can be shown (Busch, 1985b) that such limitations also emerge from a general theorem of Wigner (1952), and of Araki and Yanase (1960), to the effect that in quantum theory,*"only quantities which commute with all additive conserved quantities are precisely measurable"* (Wigner, 1976, p. 298).

From the point of view of quantum state determination, it might be argued that the lack of informational completeness of position and momentum observables in quantum mechanics could be by-passed by invoking von Neumann's (1932) *postulate* that *all* self-adjoint operators in a Hilbert space of a quantum system represent observables, so that *ipso facto* the projector $|\psi\rangle\langle\psi|$ represents an observable for any normalized state vector ψ, and its "measurement" would determine the quantum state represented by ψ. However, in order to "measure" the "observable" $|\psi\rangle\langle\psi|$ one would first have to assign an "apparatus" to it, and that would require *prior* knowledge of that state, so that the argument would be circular. Hence, one would have to envisage that $|\psi\rangle\langle\psi|$ is an element of the spectral measure of *some* complete set of observables, and that the family of all such complete sets is rich enough for their spectral measures to contain all one-dimensional projectors, so that all self-adjoint operators in a Hilbert space of a quantum system could be considered to be at least functions of observables (Prugovečki, 1981, Chapter IV, Sec. 2). However, the discovery of various superselection rules (Wick *el al.*, 1952; Hegerfeldt *et al.*, 1968; Zurek, 1982; Omnès, 1990) has established that this postulate is definitely false for certain multi-particle systems in quantum field theory. Moreover, as pointed out by Wigner (1963, 1976, 1981), even in single-particle quantum theory we do *not know how* to measure even such simple "observables" as $Q^2P^2 + P^2Q^2$ or QP^2Q , and even if we knew, we *could not measure them even in principle* in the conventional sense of obtaining "sharp" (or, at least, arbitrarily precise) measurement outcomes (Busch, 1985). In Wigner's own words, "for some observables, in fact for the majority of them (such as xyp_z), nobody seriously believes that a measuring apparatus exists" (cf. Wheeler and Zurek, 1983, p. 338). Later on in the same article he also states: "On the other hand, most quantities which we believe to be able to measure, and surely all the important quantities such as position, momentum, fail to commute with all the conserved quantities [such as total angular momentum], so that

their measurement cannot be possible with a microscopic apparatus."

When Wigner made the above statement, he clearly had in mind[10] only the possibility of measurement of "sharp" values of position, of momentum, and of other well-known observables. In practice one cannot make perfectly precise measurements on observables with continuous spectra anyhow, so that "sharp" has to be interpreted as the possibility of *indefinitely* improving the precision of their measurement by perfecting the apparatus. As long as gravitational effects are ignored, the increase of measurement accuracy allowed by the earlier mentioned Wigner-Araki-Yanase theorem can be improved indefinitely *if* we *assume* that the "size" of the apparatus can be increased indefinitely. When, however, gravitational effects are taken into account, then, as discussed in Sec. 1.3, the Planck's length limits the accuracy of position measurements regardless of the "size" of the apparatus. The accuracy of momentum measurements is then also limited, since in the presence of gravity the time span between two consecutive position measurements in the time-of-flight – or any other measurements of velocity and momentum – cannot be increased indefinitely for the sake of improving momentum measurement accuracy at a given position measurement accuracy. Indeed, the strong equivalence principle has to be applied to such measurements. However, classically that principle is valid only locally, in sufficiently small neighborhoods of the worldline of the "apparatus", whose size is contingent on the desired measurement accuracy. Hence, as the aforementioned time span increases, the quantum "particle" representing the "system" is bound to cross the boundary of a thus prescribed neighborhood.

All this shows that the solution to the problem of informational completeness has to be sought through a re-evaluation of the entire role played by the conventional concept of complete sets of observables in quantum mechanics. Since complete sets of observables are completely characterized, both physically and mathematically, by the spectral measures associated with them, such a re-evaluation leads in a natural manner to an extrapolation of those projector-valued (PV) measures into positive operator-valued (POV) measures.

Indeed, ever since the 1950s some researchers into the foundations of quantum mechanics, and especially Ludwig (1953, 1958, 1983, 1985), have argued that the concept of PV-measure is too restrictive in the context of the quantum theory of measurement, and that in view of the fact that measurements of position, momentum and other observables can be carried out only with a finite precision, the projectors (2.1) used in computing their expectation values should be replaced by operators which generically are only positive[11]:

$$E \;:\; B \;\mapsto\; E(B) = E^*(B) \geq 0 \;, \qquad B \subset \mathbf{R}^n \;. \tag{2.11}$$

This entails the replacing of the PV-measures ordinarily used in the quantum theory with

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[10] Methods for measuring spatio-temporal distances with microscopic apparatuses were actually presented by Salecker and Wigner (1958), and the question of minimum mass required for a given degree of accuracy was posed and studied in that context. The limitations of "quantum clocks" of the type they proposed was later reconsidered by Peres (1980), and their use in defining an "internal clock time" in quantum field theory and quantum gravity was studied by Page and Wooters (1983). As we shall discuss in Sec. 3.7, the elements of quantum frames actually supply *natural* clocks, in de Broglie's (1923, 1924, 1979) sense of the word.

[11] A nonzero operator E is said to be *positive* if $\langle\psi|E\psi\rangle \geq 0$ for all vectors ψ in its domain within a given Hilbert space – in which case one writes $E \geq 0$. Ludwig (1983, 1985), Kraus (1983), and other researchers belonging to the Marburg school of quantum physics, call positive operators representing quantum mechanical measurement procedures *effects*, so that in their terminology a POV measure is referred to as an effect-valued measure.

POV measures, which, by their very definition, obey the σ-additivity law on all disjoint (and at most countably infinite) unions of Borel sets,

$$E(B_1 \cup B_2 \cup \cdots) = E(B_1) + E(B_2) + \cdots , \qquad B_i \cap B_j = \varnothing , \qquad i \neq j , \quad (2.12)$$

Hence, PV-measures indeed emerge as special cases of POV-measures.

The earlier given definition of informational completeness for commuting self-adjoint operators implicitly holds also for their joint spectral measures, so that it can be readily extended to POV measures: a POV measure E is called *informationally complete* with respect to an ensemble of identical systems if, for any two density operators ρ_1 and ρ_2 representing (pure or mixed) quantum states of that ensemble, the equalities $\mathrm{Tr}[\rho_1 E^{\mathbf{A}}(B)] = \mathrm{Tr}[\rho_2 E^{\mathbf{A}}(B)]$ can be true for all Borel sets B on which that POV measure is defined if and only if $\rho_1 = \rho_2$.

We note that, whereas the definition of informational completeness of commuting self-adjoint operators had been qualified with regard to the quantum states to which it was applicable, we have not imposed similar qualification on the informational completeness of generic POV measures. The reason is that even in those cases where some commuting operators represent complete sets of compatible observables we *never* have informational completeness for *all* states of the system; whereas, on the other hand, some POV measures can be associated with sets consisting of noncommuting self-adjoint operators (such as those for the "incompatible" observables of position and momentum), and in such cases of primary interest to us some of them *can* display universal informational completeness.

If one looks into the history of this somewhat controversial subject, it turns out that in the 1960s a variety of proofs were given (Urbanik, 1961; Varadarajan, 1962; Cohen, 1966; Gudder, 1968) which followed in the footsteps of von Neumann's (1932) proof of the "impossibility" of the simultaneous measurement of "incompatible" quantum observables. However, as is well-known to all researchers into the foundations of quantum mechanics, another widely cited proof of von Neumann's (1932), which allegedly established the "impossibility" of hidden variables, eventually turned to be circular. As a matter of fact, it was persuasively demonstrated by the formulation of Bell's (1964, 1966) inequalities that this entire issue required an experimental verdict – which was conclusively delivered by Aspect *et al.* (1981, 1982), and turned out to be *against* the existence of hidden variables.

On the other hand, it was independently realized by several researchers (She and Hefner, 1966; Prugovečki, 1966; Park and Margenau, 1968) that the problem of measurement of such "incompatible" quantum observables as position and momentum assumes entirely new aspects when the possibility is considered that their simultaneous measurement outcome might not be "sharp," i.e., that the measurement is not carried out with perfect precision – as is actually always the case in practice. In that case it can be shown (Prugovečki, 1967, 1976; Ali and Prugovečki, 1977) that not only is there no conflict with Heisenberg's uncertainty principle (so that the aforementioned "impossibility" proofs become irrelevant), but that actually an embodiment of Heisenberg's (1930) uncertainty principle for *single* measurements can be achieved thereby.

On the mathematical front, Davies and Lewis (1969), Davies (1976), Helstrom (1976), Holevo (1972), Prugovečki (1966, 1976), She and Hefner (1966), and many

others[12], were able to propose probability measures describing the outcome of such measurements, without giving rise to any inconsistencies with the orthodox theory of quantum measurement. Indeed, the latter deals exclusively with the highly idealized (and practically unrealizable) case of perfectly accurate measurement outcomes.

Of course, no uniqueness of probability measures for "unsharp" measurements can be expected to hold for any given system, since those outcomes depend as much on the employed "test bodies" or "apparatuses" (Gilmore, 1992) as on the system itself. For example, in the case of position measurements that are accurate to atomic order of magnitude, namely 10^{-8} cm, the role of such a test body could be played by a hydrogen atom. On the other hand, at the subatomic level, hadrons can take over the role of extended test bodies in position measurements accurate down to 10^{-13} cm order of magnitude. Beyond that, one needs relatively "pointlike" probes, which are provided by such leptons as the electrons.

One of the central theses in this monograph is that *all* elementary quantum objects are extended, and that the Planck length provides the lower bound on the linear dimensions of that extension, so that true pointlike localization loses its physical *raison d'être* at the most fundamental microlevel. As we saw in Sec. 1.3, this feature of the microworld was considered more than half-a-century ago by Born (1938, 1949), Dirac (1949), Heisenberg (1938, 1943), and other founders of quantum mechanics. In 1949 Dirac set forth the thesis that logically follows from such a physical feature of our universe in the clearest terms possible: "Present-day atomic theories involve the assumption of localizability, which is sufficient but it is *very likely to be too stringent. . . . A less drastic assumption may be adequate*, e.g., that there is a fundamental length λ . . . " (Dirac, 1949, p. 399 – emphasis added.)

It is to such a "less drastic assumption" that we turn our attention now.

3.3. Informationally Complete Nonrelativistic Quantum Frames

As we saw in the last part of Sec. 2.2, Einstein's (1905) definition of a spatio-temporal frame of reference had to be modified in order to take into account the fact that the concept of "rigid rod" is, strictly speaking, inconsistent with the tenets of special and general relativity. On the other hand, we saw in the preceding section that in his 1949 autobiographical notes Einstein pointed out that, at the microlevel, "measuring rods and clocks would have to be represented as solutions of the basic equations (objects consisting of moving atomic configurations), [and] not, as it were, as theoretically self-sufficient entities."

At the operational level, the physical definition of a frame of reference in terms of test particles, given at the end of Sec. 2.2, can be readily adapted to the quantum regime. Thus, in the nonrelativistic regime, a (global) *quantum* Galilei frame of reference can be envisaged as consisting of a system of "standard clocks" which measure "absolute time," and a spatial part, mathematically labeled by a triple $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ of vectors, and physically marked by means of non-interacting (neutral) *quantum* test particles moving by inertia. One of these particles then marks the origin O of that quantum frame, and six other identical test particles

[12] These independently discovered and re-discovered results led to a wealth of new ideas in previously unrelated branches of physics and mathematics. A short review article was therefore written by Busch *et al.* (1989), with the professed goal of making the physics community at large aware of the existence of all these results, and therefore prevent further wasteful duplication of effort. It provides an extensive bibliography on the subject. Some additional, more recent, references can be traced by using an article of Martens and de Muynck (1990).

A_a, $a=\pm1,\pm2,\pm3$, in its vicinity mark[13] its positive and negative axes. The operational verification of whether any such physical setup constitutes a quantum Galilei frame when taken in conjuction with the "standard clocks" in the immediate vicinity of all these test particles, and at relative *stochastic* rest to the spatial axes determined by them, would consist then of verifications of whether or not the test particles A_a, $a=\pm1,\pm2,\pm3$, are indeed at the chosen unit distance from O, and whether the axes joining O to each A_a, $a = \pm1,\pm2,\pm3$, are indeed at right angles to each other. This is very much as in the classical situation, with one key exception: when photons or massive test bodies are sent between the test particles constituting such a quantum frame, the compensations for the momenta transferred by the resulting recoils can be carried out only in accordance with Bohr's (1961) operational procedures, and therefore they can never be exact.

One of the key new features of such an arrangement, which is not present in its classical counterpart, is manifested in the idea of "stochastic rest." Indeed, if we envisage our quantum test particles as being strictly pointlike, and at absolute rest in relation to each other, then their "proper states" would have to be conventionally described, in the configuration representation associated with $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$, by the δ-"functions" in (2.7) centered at points with values $\mathbf{x}'\in\mathbf{R}^3$ that represent their relative stationary locations. However, not only is a δ-"function" unacceptable as a *bona fide* wave function, for all the reasons presented in Sec. 3.2, but such a conceptualization would be totally useless in setting up a spatio-temporal frame, which, with the passage of time, has to maintain, at least approximately, a prescribed geometric configuration. Moreover, as a degenerate case of Heisenberg's uncertainty principle, the state of instantaneous perfectly sharp localization at an instant t would entail a uniform probabilistic distribution of the 3-momenta $\mathbf{k}$, and therefore 3-velocities of such a pointlike object, leaving no possibility of predicting "where" it might be located in the most immediate future following that instant.

The way out of this epistemic *cul-de-sac* can be found in Born's earlier cited maxim that "statements like 'A quantity x has a completely definite value' (expressed by a real number and represented by a point in the physical continuum) . . . have no physical meaning." In that case, what *can* have physical meaning is the statement that a quantity X has a *stochastic value* (x,χ_x) (Prugovečki, 1976, 1984), consisting of *base value*[14] x and a *confidence function* χ_x. The confidence function is in general given in the form of a probability distribution assuming values at all the points $y\in\sigma(X)$ within the spectrum of the measurable quantity X, and provides the probabilities of observing statistical fluctuations from the measured base value x under the most ideal repeatability conditions for any class of experimental arrangements. We can also admit to this class of confidence functions the δ-"functions" $\delta(y-x)$, interpreted as a Dirac δ-measure δ_x, which assumes the value 1 on Borel subsets B that contain x, and the value zero otherwise. However, in the case where the quantity X possesses an *irreducible indeterminacy*, the measurement outcome of X can *not* be represented even *in principle* (i.e., not even in the *imagined* limit of an infinite se-

[13] The Bose-Einstein statistics do not interfere with this marking, since the "indistinguishability" of particles which it entails is in the wave function of such a system, and is related to exchanges of constituents which do not affect their geometric configurations. Of course, nonidentical test particles could also be used in this procedure.

[14] This base value could be the mean, the median, or some other value statistically related to the observed readings of measuring devices, and therefore in general it depends on the measurement procedures that are used and the design features of those devices.

quence of increasingly accurate measurement procedures that underlies all theories of measurement) by a stochastic value (x,δ_x).

In the case of position measurements performed in the nonrelativistic regime under the assumption that ℓ represents a fundamental length, which limits the precision of such measurements by imposing an *optimal* accuracy of the order of ℓ, we expect that this length would appear as a characteristic feature (such as a standard deviation) of the probability distributions provided by the confidence functions for the optimal stochastic values of such measurements. Of course, it would be rather trivial, and physically inconsequential to then merely take the probability densities in (1.16a) and "smear" them with some *ad hoc* confidence functions containing ℓ, in order to embed such a fundamental length into the conventional formulation of single-particle quantum mechanics. Instead, we shall resort to a *geometro-stochastic*[15] *method* of quantization (Prugovečki, 1978, 1984, 1992), in whose initial stages the same central significance is assigned to the classical phase space Γ as in the canonical method, but in which key aspects of the group-theoretical method outlined in Sec. 3.3 are also used. In this latter aspect it turns out that, at least in the nonrelativistic regime, this method shares some features with the method of geometric quantization (Śniatycki, 1980; Woodhouse, 1980; Hurt, 1983) developed by Kostant (1970) and Souriau (1970), since both these methods make use of L^2 spaces over classical phase space and of unitary representations.

The representations of the Galilei group which in the nonrelativistic regime underlie the geometro-stochastic method of quantization act in the case of systems of single particles without spin on the Hilbert space $L^2(\mathbf{R}^6)$, which consists of all the complex-valued functions $\psi(\mathbf{q},\mathbf{p})$ in the variables $\mathbf{q},\mathbf{p} \in \mathbf{R}^3$ that are square-integrable with respect to the Lebesgue measure on the phase space $\Gamma \cong \mathbf{R}^6$. This space carries the inner product

$$\langle \psi_1 | \psi_2 \rangle = \int_{\mathbf{R}^6} \psi_1^*(\mathbf{q},\mathbf{p}) \psi_2(\mathbf{q},\mathbf{p})\, d\mathbf{q}\, d\mathbf{p} \ , \tag{3.1}$$

and in it the free time evolution provided in conventional quantum mechanics by (1.14) assumes the form

$$\psi_t = \exp(-iH_0 t)\psi_0 \ , \qquad H_0 = (-1/2m)\Delta_{\mathbf{q}} \ . \tag{3.2}$$

The wave functions resulting from this time evolution provide for each fixed value of $\mathbf{p} \in \mathbf{R}^3$ solutions of the free Schrödinger equation in the $\mathbf{q}$-variables:

$$i\partial_t \psi(\mathbf{q},\mathbf{p},t) = (-1/2m)\Delta_{\mathbf{q}}\, \psi(\mathbf{q},\mathbf{p},t) \ , \qquad \psi(\mathbf{q},\mathbf{p},t) = \psi_t(\mathbf{q},\mathbf{p}) \ . \tag{3.3}$$

We can now look into the invariance of this equation under the following classical Galilei transformation in the extended nonrelativistic phase space $\mathbf{R}^7 = \Gamma \oplus \mathbf{R}^1$, which incorporates the absolute time dimension:

$$(b,\mathbf{a},\mathbf{v},R) : (\mathbf{q},\mathbf{p},t) \mapsto (\mathbf{q}' = \mathbf{a} + \mathbf{v}t + R\mathbf{q},\ \mathbf{p}' = m\mathbf{v} + R\mathbf{p},\ t' = t + b) \ . \tag{3.4}$$

[15] This method was originally called "stochastic," since it was based on stochastic values. However, it was then often confused with various stochastic mechanics formulations of quantum theory based on the *classical* theory of stochastic processes (Nelson, 1967, 1986; Guerra, 1981; Smolin, 1986; Namsrai, 1986), or with the "stochastic quantization" method in Euclidean quantum field theory (Parisi and Wu, 1981; Damgaard and Hüffel, 1987).

As in the case of (1.13) and (1.15), the above Schrödinger equation is left invariant by the following ray representation of the Galilei group,

$$U(b,\mathbf{a},\mathbf{v},R):\ \psi(\mathbf{q},\mathbf{p},t)\mapsto \psi'(\mathbf{q},\mathbf{p},t)=\exp\left[i\left(-\tfrac{1}{2}m\mathbf{v}^2(t-b)+m\mathbf{v}\cdot(\mathbf{q}-\mathbf{a})\right)\right]$$
$$\times\ \psi(R^{-1}[\mathbf{q}-\mathbf{a}-\mathbf{v}(t-b)],R^{-1}[\mathbf{p}-m\mathbf{v}],t-b)\ , \qquad (3.5)$$

It turns out (Prugovečki, 1978c,d) that this representation is unitary and satisfies the same group multiplication law as the one in (1.15):

$$U(b'',\mathbf{a}'',\mathbf{v}'',R'')=\exp\left\{-im\left[\tfrac{1}{2}\mathbf{v}'^2\,b+\mathbf{v}'\cdot R'\,\mathbf{a}\right]\right\}U(b',\mathbf{a}',\mathbf{v}',R')U(b,\mathbf{a},\mathbf{v},R)\ . \qquad (3.6)$$

However, as opposed to the one in (1.15), the representation in (3.5) is highly reducible. Its complete harmonic analysis (Ali and Prugovečki, 1986) reveals the existence of decompositions[16] into direct sums of irreducible representations for all integer-spin values, which can be interpreted as belonging to (extended) test particles constituting quantum frames. Indeed, each one of these irreducible subrepresentations is characterized by a unique *resolution generator* given by a vector ξ with the property that the projector $\mathbf{P}_\xi$ onto the subspace of such a subrepresentation displays the following essential feature[17]:

$$(\mathbf{P}_\xi\psi)(\mathbf{q},\mathbf{p})=\langle\xi_{\mathbf{q},\mathbf{p}}|\psi\rangle\ , \qquad \xi_{\mathbf{q},\mathbf{p}}:=U(0,\mathbf{q},\mathbf{p}/m,I)\xi\ , \qquad \psi\in L^2(\mathbf{R}^6)\ . \qquad (3.7)$$

Consequently, it is then easily verified that each such projector $\mathbf{P}_\xi$ can be expressed as an integral with an operator-valued integrand,

$$\mathbf{P}_\xi=\int_{\mathbf{R}^6}|\xi_{\mathbf{q},\mathbf{p}}\rangle d\mathbf{q}d\mathbf{p}\langle\xi_{\mathbf{q},\mathbf{p}}|\ , \qquad (3.8)$$

but with an ordinary measure of integration – which in the above case is the Lebesgue measure on $\mathbf{R}^6$ This mathematically characterizes such an integral as a Bochner integral (cf. Prugovečki, 1981, p. 480), since it is determined by the following ordinary integrals:

$$\langle\psi_1|\mathbf{P}_\xi\psi_2\rangle=\int_{\mathbf{R}^6}\langle\psi_1|\xi_{\mathbf{q},\mathbf{p}}\rangle d\mathbf{q}d\mathbf{p}\langle\xi_{\mathbf{q},\mathbf{p}}|\psi_2\rangle. \qquad (3.9)$$

Special note should be taken of the fact that, despite the formal similarity in appearance between (3.8) and the operator-valued integrals in (2.10), such a Bochner integral as the one in (3.8) is well-defined on the entire Hilbert space $L^2(\mathbf{R}^6)$. Indeed, the integral in (3.9) converges for arbitrary vectors $\psi_1,\psi_2\in L^2(\mathbf{R}^6)$, thus defining the outcome of the integra-

[16] Since the Galilei group is locally compact, but it is not compact, the well-known Peter-Weyl theorem (cf. Barut and Raczka, 1986, p. 172) does not apply to it, and such decompositions are not unique.

[17] It has been shown (Ali and Emch, 1986) that the choice of resolution generator for these POV measures is equivalent to the choice of polarization of phase space in the Kostant-Souriau geometric quantization method, and that it is related to operator ordering in canonical quantization (Ali and Doebner, 1990). A method of quantization proposed by Berezin (1975) bridges the technical gap between the Kostant-Souriau method and the nonrelativistic geometro-stochastic quantization method outlined in this chapter, since it makes extensive use of coherent states (cf. Odzijewicz, 1992). However, neither of these latter methodologies extend to the general relativistic regime, due to their emphasis on the existence of quantum counterparts of algebras of classical observables, which, on account of the difficulties with sharp localization in relativistic quantum theory that will be discussed in Sec. 3.6, confront such approaches with severe foundational problems already in the special relativistic regime.

tion in (3.8) unambiguously for *all* state vectors in $L^2(\mathbf{R}^6)$.

Equations (3.7) and (3.8) lead to the conclusion that each one of the closed subspaces $\mathbf{P}_\xi L^2(\mathbf{R}^6)$, which carries an irreducible subrepresentation of (3.5), is a reproducing kernel Hilbert space (Aronszajn, 1950), so that

$$(\mathbf{P}_\xi \psi)(\mathbf{q},\mathbf{p}) = \int_{\mathbf{R}^6} \left\langle \xi_{\mathbf{q},\mathbf{p}} \middle| \xi_{\mathbf{q}',\mathbf{p}'} \right\rangle \psi(\mathbf{q}',\mathbf{p}')\, d\mathbf{q}'\, d\mathbf{p}'. \tag{3.10}$$

If we restrict the integration in (3.10) to Borel sets B in $\mathbf{R}^6$, we obtain the POV-measure

$$E_\xi(B) = \int_B \left| \xi_{\mathbf{q},\mathbf{p}} \right\rangle d\mathbf{q}\, d\mathbf{p} \left\langle \xi_{\mathbf{q},\mathbf{p}} \right| \quad , \qquad B \subset \mathbf{R}^6 \ . \tag{3.11}$$

It is not obvious at first sight if the above POV-measure has any physical significance whatsoever. Its potential physical meaning will emerge very clearly, however, from the following *marginal values* of the expectation values of all wave functions ψ that belong to the irreducible subspace $\mathbf{P}_\xi L^2(\mathbf{R}^6)$:

$$\left\langle \psi \middle| E_\xi(\hat{B} \times \mathbf{R}^3) \psi \right\rangle = \int_{\mathbf{R}^3} d\mathbf{p} \int_{\hat{B}} d\mathbf{q} \left| \psi(\mathbf{q},\mathbf{p}) \right|^2 \ , \tag{3.12a}$$

$$\left\langle \psi \middle| E_\xi(\mathbf{R}^3 \times \tilde{B}) \psi \right\rangle = \int_{\mathbf{R}^3} d\mathbf{q} \int_{\tilde{B}} d\mathbf{p} \left| \psi(\mathbf{q},\mathbf{p}) \right|^2 \tag{3.12b}$$

To see this, one has to take advantage of the fact that to each resolution generator $\xi \in L^2(\mathbf{R}^6)$ can be assigned a (modulo a multiplicative constant) unique wave function

$$\tilde{\xi}^l_j(\mathbf{k}) = (2\pi)^{-3/2} e(|\mathbf{k}|)\, Y^l_j\big(\mathbf{k}/|\mathbf{k}|\big) \ , \qquad l = -j, -j+1, \ldots, +j \ , \tag{3.13}$$

from the momentum space representation of particles with integer-valued spin j, which is such that the map from that space onto the irreducible subspace $\mathbf{P}_\xi L^2(\mathbf{R}^6)$,

$$W_\xi \ : \ \ell^2(2j+1) \otimes L^2(\mathbf{R}^3) \ \rightarrow \ \mathbf{P}_\xi L^2(\mathbf{R}^6) \ , \tag{3.14a}$$

determined by the integral transform

$$(W_\xi \psi)(\mathbf{q},\mathbf{p}) = (2\pi)^{-3/2} \int_{\mathbf{R}^3} \exp(i\mathbf{q}\cdot\mathbf{k})\, \bar{e}(|\mathbf{p}-\mathbf{k}|)\, \overline{Y}^l_j\big((\mathbf{p}-\mathbf{k})/|\mathbf{p}-\mathbf{k}|\big)\, \tilde{\psi}_l(\mathbf{k})\, d\mathbf{k}, \tag{3.14b}$$

with the bar denoting complex conjugation, is a unitary operator for which

$$\mathbf{P}_\xi U(b,\mathbf{a},\mathbf{v},R) = W_\xi \tilde{U}^j(b,\mathbf{a},\mathbf{v},R) W_\xi^{-1} \ , \tag{3.15}$$

where the representation on the right-hand side of the above relation is the generalization of the one in (1.10) to the case of all spin values $j = 0,1,2,\ldots$ (Levy-Léblond, 1971):

$$\begin{aligned} &\tilde{U}^j(b,\mathbf{a},\mathbf{v},R) \ : \ \ \tilde{\psi}(\mathbf{k},t) \ \mapsto \ \tilde{\psi}'(\mathbf{k},t) \\ &= \exp\Big\{ i\Big[\tfrac{1}{2} m\mathbf{v}^2 (t-b) - \mathbf{k}\cdot(\mathbf{a}+\mathbf{v}(t-b)) \Big] \Big\} D^j(R) \tilde{\psi}(R^{-1}(\mathbf{k}-m\mathbf{v}), t-b) \ . \end{aligned} \tag{3.16}$$

The e-function in (3.14b) is, in general, a square-integrable function of $\mathbf{k} \in \mathbf{R}^3$; whereas, the Y-functions are the familiar spherical harmonics, and a summation in the repeated l-index is implicit in the above notation. It has to be emphasized that the existence of these e-functions is a *proved* mathematical fact (Prugovečki, 1978d; Ali and Prugovečki, 1986),

and not a feature "inserted by hand" into the formalism. Thus, the physical interpretation which emerges from (3.12) is an *intrinsic* aspect of the geometro-stochastic quantization method, and not an *ad hoc* device superimposed on the orthodox framework.

In the case of spin $j = 0$, this physical interpretation emerges from the easily provable result (Prugovečki, 1978d, 1984) that (1.11) and (3.14b) imply that the marginal values in (3.12) for position and momentum are related to the conventional probability densities in (1.16) for these physical observables as follows:

$$\left\langle \psi \middle| E_\xi(\hat{B} \times \mathbf{R}^3) \psi \right\rangle = \int_{\hat{B}} d\mathbf{q} \int_{\mathbf{R}^3} d\mathbf{x}\, \hat{\chi}^\xi_{\mathbf{q}}(\mathbf{x}) \left| \hat{\psi}(\mathbf{x}) \right|^2, \quad \hat{\chi}^\xi_{\mathbf{q}}(\mathbf{x}) = (2\pi)^3 \left| \hat{\xi}(\mathbf{x} - \mathbf{q}) \right|^2, \quad (3.17a)$$

$$\left\langle \psi \middle| E_\xi(\mathbf{R}^3 \times \tilde{B}) \psi \right\rangle = \int_{\tilde{B}} d\mathbf{p} \int_{\mathbf{R}^3} d\mathbf{k}\, \tilde{\chi}^\xi_{\mathbf{p}}(\mathbf{k}) \left| \tilde{\psi}(\mathbf{k}) \right|^2, \quad \tilde{\chi}^\xi_{\mathbf{p}}(\mathbf{k}) = (2\pi)^3 \left| \tilde{\xi}(\mathbf{k} - \mathbf{p}) \right|^2. \quad (3.17b)$$

These marginal values incorporate the confidence of the phase-space *stochastic* values

$$(\mathbf{q}, \hat{\chi}^\xi_{\mathbf{q}}) \times (\mathbf{p}, \tilde{\chi}^\xi_{\mathbf{p}}) \in \Gamma_\xi \cong \mathbf{R}^6 , \quad (3.18)$$

whose spreads obey the Heisenberg uncertainty principle. Indeed, according to a well-known theorem by Wigner (1932, 1979), no positive-definite distribution functions on the phase space $\Gamma \cong \mathbf{R}^6$ can be associated with either pure or mixed quantum states in such a manner that their marginal distributions (obtained by integrating them over $\mathbf{R}^3$ in $\mathbf{q}$ and $\mathbf{p}$, respectively) coincide with the conventional probability densities in (1.16) for the position and momentum measurements. Formally, such marginal distributions would be obtained by choosing the confidence functions in (3.17) equal to a δ-"function" at $\mathbf{q}$ and $\mathbf{p}$, so that the resulting probability measures could be interpreted as probability distributions for the simultaneous measurement of *sharp* stochastic position and of *sharp* stochastic momentum.

On the other hand, it is totally consistent (Prugovečki, 1967, 1976) with Heisenberg's (1930) interpretation of the uncertainty principle for simultaneous measurements of position and momentum on *single* quantum particles to interpret the expectation values

$$\left\langle \psi \middle| E_\xi(B) \psi \right\rangle = \int_B \left| \psi(\mathbf{q}, \mathbf{p}) \right|^2 d\mathbf{q} d\mathbf{p} , \qquad \psi \in \mathbf{P}_\xi L^2(\mathbf{R}^6) , \quad (3.19)$$

as providing the probabilities for the simultaneous measurement of the *stochastic phase space* values in (3.18). Such measurements can be then envisaged as being performed with quantum test bodies constituting a *quantum* frame (Prugovečki, 1978d, 1981d). In principle such a frame could be obtained by subjecting identical duplicates of a quantum test body to all the basic kinematical operations of spacetime translation and boost in order to obtain an array of kinematically correlated microdetectors, which can be then used for the spatio-temporal localization of quantum systems.

In mathematical terms, a *nonrelativistic quantum frame* $Q^{\boldsymbol{u}}$ associated with a classical Galilei frame $\boldsymbol{u}$ can be obtained by considering a resolution generator ξ to represent the *proper state vector* $\Phi^{\boldsymbol{u}}$ of such a test body marking the origin of that frame, and subjecting it to the set of transformations that describes those kinematical operations in the non-relativistic regime, i.e., in the case of system and test particles of the same mass m and spin-0, to the irreducible ray representation $U\mathbf{P}_\xi$ of the Galilei group, provided by (3.5) in the subspace $\mathbf{P}_\xi L^2(\mathbf{R}^6)$ of $L^2(\mathbf{R}^6)$. Generically, the outcome of this construction is a family

$$Q^{u} = \left\{ \Phi^{u}_{\zeta} = U(\zeta)\Phi^{u} \;\middle|\; \zeta = (t,\mathbf{q},\mathbf{v},I) \in \mathbf{R}^{7} \right\} , \tag{3.20}$$

of generalized coherent states that supply a continuous resolution of the identity in the Hilbert space which carries that irreducible representation. Hence, in the present case of system particles and test particles of the same mass m, the corresponding inertial frame at the (absolute Newtonian) time $t = 0$ is described by the coherent state vectors in $\mathbf{P}_{\xi}L^{2}(\mathbf{R}^{6})$:

$$\Phi^{u}_{(0,\mathbf{q},\mathbf{p}/m,I)} \cong \xi_{\mathbf{q},\mathbf{p}} = U(0,\mathbf{q},\mathbf{p}/m,I)\xi \cong U(0,\mathbf{q},\mathbf{v},I)\Phi^{u} . \tag{3.21}$$

It has been established (Ali and Prugovečki, 1977b) that a quantum frame, such as the one in (3.21), or more generally in (3.20), is informationally complete if and only if the wave function $\xi(\mathbf{q},\mathbf{p})$, representing the proper state vector Φ^{u} of the test particle marking its origin, is different from zero for (almost all with respect to the Lebesgue measure on $\mathbf{R}^{6}$) values $(\mathbf{q},\mathbf{p}) \in \Gamma \cong \mathbf{R}^{6}$. Thus, in that case the probabilities in (3.19) uniquely determine each single-particle quantum state. This condition is actually easy to fulfill, and the conclusion itself remains true for multi-particle states, as well as for non-zero spin values.

Furthermore, it has been established in another mathematically rigorous study (Ali and Prugovečki, 1977a) that the marginal POV measure in (3.12a) for the measurement of the position stochastic values in (3.18) is *informationally equivalent* to the PV measure in (2.3a) in the sense that, for any pair of state vectors $\psi_1, \psi_2 \in \mathbf{P}_{\xi}L^{2}(\mathbf{R}^{6})$, the validity of the equalities

$$\left\langle \psi_1 \middle| E_{\xi}(B \times \mathbf{R}^{3})\psi_1 \right\rangle = \left\langle \psi_2 \middle| E_{\xi}(B \times \mathbf{R}^{3})\psi_2 \right\rangle , \qquad B \subset \mathbf{R}^{3}, \tag{3.22}$$

for all Borel sets B in $\mathbf{R}^{3}$ implies the validity of the equalities

$$\left\langle \hat{\psi}_1 \middle| E^{\mathbf{Q}}(B)\hat{\psi}_1 \right\rangle = \left\langle \hat{\psi}_2 \middle| E^{\mathbf{Q}}(B)\hat{\psi}_2 \right\rangle , \qquad \hat{\psi}_1 = W_{\xi}^{-1}\psi_1, \hat{\psi}_2 = W_{\xi}^{-1}\psi_2 \in L^{2}(\mathbf{R}^{3}), \tag{3.23}$$

for the corresponding states in the configuration space representation, namely the informational equivalence of those states. Similarly, by the same type of proof, the marginal POV measure in (3.12b) for the measurement of the momentum stochastic values in (3.18) is informationally equivalent to the PV measure in (2.3b) for sharp momentum measurements. Thus, absolutely nothing is lost, and much is gained at the informational level by working with a stochastic phase-space representation provided by the irreducible Galilei group representation $U\mathbf{P}_{\xi}$ in the Hilbert space $\mathbf{P}_{\xi}L^{2}(\mathbf{R}^{6})$.

3.4. Sharp-Point Limits of Nonrelativistic Quantum Frames

The definition of a nonrelativistic quantum frame given in the preceding section displays two distinct facets, namely an operational one and a mathematical one.

Its operational aspect reflects the fact, explicitly demonstrated by Busch (1985b), that no arbitrarily accurate measurements of position are *in principle* feasible even at the nonrelativistic level, on account of a theorem by Wigner (1952), Araki and Yanase (1960) (cf. also Yanase, 1961) which demonstrated that, in Wigner's words, "no precise measurement of any quantity is possible unless [it] commutes with all conserved additive quantities." (Wigner, 1976, p. 313.) Since, in the ultimate analysis, any measurement involves evalua-

tions of position coincidences and relationships, this conclusion can be said to possess general epistemological validity. In turn, this leads to the possibility of *simultaneous* measurements of positions and momenta performed with those *finite* accuracies that are commensurate with the uncertainty principle. Practically implementable devices and schemes are by now known for such measurements (Yamamoto and Haus, 1986; Busch and Schroeck, 1989; Busch and Lahti, 1990; Busch *et al.*, 1989, 1991), so that their operational aspect has significance even at the present technological level of development.

The mathematical part of the definition of a nonrelativistic quantum frame, presented in the previous section, is founded on the concept of *proper* state vector $\Phi^{\boldsymbol{u}}$ of a quantum test body marking its origin. Each such a proper state vector is a *bona fide* element of a Hilbert space $\mathcal{H}$ containing all the state vectors of the system as well as of the quantum frame. However, it can be *imagined* that a sharp-point limit is taken in which that proper state vector transcends into that of an object that is *perfectly* localized at the origin of the classical Galilei frame $\boldsymbol{u}$ associated with $\Phi^{\boldsymbol{u}}$. The proper state vector of such a quantum point test particle would then obviously be described by a δ-"function" which, as we saw in Sec. 3.2, lies within an *extension* of the Hilbert space $\mathcal{H}$, but not in $\mathcal{H}$ itself.

The mathematical aspect of sharp localization envisaged by the orthodox interpretation of quantum mechanics reflects the fact that, in this interpretation, the part of the physical world which is under observation is divided into "system" and "apparatus," and that classical concepts and terminology are applied to the latter. Although in the nonrelativistic regime this leads to a *mathematically* consistent framework, as well as to an abundance of phenomenologically satisfactory models, it is seldom recognized that at a *foundational* level such a solution eventually proves to be unsatisfactory, due to a fundamental incompatibility between the language and the concepts of classical theories on one hand, and some of the observational features of quantum phenomena on the other. In fact, it can be argued that the genesis of the various paradoxes that plague conventional quantum theory (Tarrozzi and van der Merwe, 1988) does not lie in the nature of quantum phenomena themselves, but rather in the unwarranted *imposition* of classical language and conceptualizations, based on "yes-no" type of statements, in a realm which *de facto* does not admit such statements.

The introduction of the concept of quantum frame as an integral part of the formalism of quantum mechanics virtually removes (Prugovečki, 1985a) the root causes of these paradoxes. And yet, it has been confirmed by various studies (Prugovečki, 1976-78; Ali and Prugovečki, 1977) that the ensuing framework and its physical interpretation is consistent not only with the orthodox formalism and interpretation of quantum mechanics, but also with all the fundamental aspects of classical and quantum statistical mechanics (cf. Prugovečki, 1984, Chapter 3, for a review). This is in part due to the fact that the existence of the unitary transforms in (3.14), which mediate the translation of each single statement and result from the configuration and momentum representations, which constitute the basis of the orthodox formulation of quantum mechanics, into the stochastic phase space representation associated with a nonrelativistic quantum frame in (3.20). Hence, in the transition to these latter representations, absolutely no informational content available in the orthodox approach is ever lost, but merely systems of imprimitivity (Mackey, 1968; Wightman, 1962), that constitute the basis of the orthodox notion of quantum particle localizability with respect to classical Galilei frames, are replaced with systems of covariance, describing

localizability with respect to nonrelativistic quantum frames. Thus, for example, the systems of imprimitivity consisting of the PV measures in (2.10) and the respective transformation laws in (1.15) and (1.10), are replaced by the systems of covariance, consisting of the POV measures in (3.11) and the transformation laws to which (3.5) gives rise.

In general, for each change $\boldsymbol{u} \mapsto \boldsymbol{u}\cdot\mathcal{g}$ of some classical Galilei frame of reference $\boldsymbol{u}$, to which a Galilean transformation $\mathcal{g} = (b,\mathbf{a},\mathbf{v},R)$ gives rise, there is a corresponding change of a quantum frame – such as, for example,

$$\boldsymbol{u}\cdot\mathcal{g} \;\leftrightarrow\; Q^{\boldsymbol{u}\cdot\mathcal{g}} = \left\{\Phi_\zeta^{\boldsymbol{u}\cdot\mathcal{g}} = U(\mathcal{g}^{-1})\Phi_\zeta^{\boldsymbol{u}} \in L^2(\mathbf{R}^6) \,\middle|\, \zeta = (t,\mathbf{q},\mathbf{v},I)\right\} , \tag{4.1}$$

in the case of the quantum frame in (3.20). Such changes of quantum frame of reference give rise to the following transformation laws for probability amplitudes,

$$\Psi_{\boldsymbol{u}}(\zeta) := \left\langle \Phi_\zeta^{\boldsymbol{u}} \middle| \Psi \right\rangle \;\mapsto\; \Psi_{\boldsymbol{u}\cdot\mathcal{g}}(\zeta) = \big(U(\mathcal{g})\Psi_{\boldsymbol{u}}\big)(\zeta) \;, \qquad \Psi \in \mathcal{H} \;, \quad \|\Psi\| = 1, \tag{4.2}$$

and to corresponding changes of the stochastic phase space probability densities

$$\rho_{\boldsymbol{u}}(\Psi;\zeta) = \left\langle \Psi \middle| \Phi_\zeta^{\boldsymbol{u}} \right\rangle \left\langle \Phi_\zeta^{\boldsymbol{u}} \middle| \Psi \right\rangle = \left|\Psi_{\boldsymbol{u}}(\zeta)\right|^2 \;, \tag{4.3}$$

which supply the expectation values $\langle\Psi|E(B)\Psi\rangle$ for the POV measures $E(B)$ of the system of covariance for the chosen class of quantum frames. In this context, it is interesting to observe that these stochastic phase space probability densities are directly related to the geometric concept of Fubini-Study distance[18]

$$d(\bar\Psi_1,\bar\Psi_2) = 2\arccos\left|\left\langle \Psi_1 \middle| \Psi_2 \right\rangle\right| \;, \qquad \Psi_i \in \bar\Psi_i \subset \mathcal{H} \;, \quad \|\Psi_i\| = 1 \;, \tag{4.4}$$

between unit Hilbert rays of the general form

$$\bar\Psi = \left\{c\Psi/\|\Psi\| \,\middle|\, c \in \mathbf{C}^1,\ |c| = 1\right\} \subset \mathcal{H} \;, \qquad \Psi \neq \mathbf{0} \;, \tag{4.5}$$

which *uniquely* correspond to a pure quantum state described by state vectors in $\mathcal{H}$. Indeed, in view of (4.3) and (4.4) we have:

$$\rho_{\boldsymbol{u}}(\Psi;\zeta) = \cos^2\left(\tfrac{1}{2}d(\bar\Phi_\zeta^{\boldsymbol{u}},\bar\Psi)\right) . \tag{4.6}$$

In the case of the POV measures in (3.11), we obtain from these transformation laws that the stochastic phase space probability measures in (3.19) transform in such a manner that, for example, in the case of a nonrelativistic quantum particle of mass m and zero spin,

$$\left\langle \psi_t \middle| E_\xi(\hat B\times\tilde B)\psi_t \right\rangle = \left\langle \psi'_{t+b} \middle| E_\xi\Big((\mathbf{a}+\mathbf{v}t+R\cdot\hat B)\times(m\mathbf{v}+R\cdot\tilde B)\Big)\psi'_{t+b} \right\rangle \;. \tag{4.7}$$

Studies of measurement-theoretical procedures for the simultaneous measurement of position and momentum, as well as of other "incompatible" observables (Ali *et al.*, 1985-90; Busch *et al.*, 1985-91; Schroeck *et al.*, 1981-90), have led to a great variety of recent re-

[18] Cf. (Prugovečki, 1991c) or Secs. 3.7-3.9 of (Prugovečki, 1992) for further details on these geometric aspects, and Chapter 13, §2 in (Barut and Raczka, 1986), or Chapter IV, §2.1 in (Prugovečki, 1981), for a discussion of the significance of unit Hilbert rays (introduced in quantum theory by Dirac (1930)) in the formulation and interpretation of quantum theory in general.

sults which demonstrate the practical implementability as well as the theoretical usefulness of a generalized theory of quantum measurement based on such POV measures.

On the other hand, from the point of view of a notion of nonrelativistic spacetime structure that would incorporate a fundamental length ℓ, we would expect that some form of uniqueness of such POV measures would emerge from the concept of the *optimal* spacetime localizability that is feasible in the presence of such a fundamental length, and from the complementary optimally accurate simultaneous momentum determination. In view of the well-known properties of the ground states of nonrelativistic harmonic oscillators, it comes as no surprize that in the nonrelativistic regime such optimal spacetime localizability can be achieved (Ali and Prugovečki, 1977a, Theorem 4) *only* in relation to a quantum frame whose proper state vectors $\Phi^{\boldsymbol{u}}$ for the quantum test particle marking its origin is given by the phase space wave function

$$\xi^{(\ell)}(\mathbf{q},\mathbf{p}) = (2\pi)^{-3}\exp\left\{-\tfrac{1}{2}[(\mathbf{q}/2\ell)^2 + \ell^2\mathbf{p}^2 - i\mathbf{q}\cdot\mathbf{p}]\right\} , \tag{4.8}$$

whose momentum space representative, generically obtained by taking the inverse of the transformation in (3.14b), and equal to[19]

$$\tilde{\psi}(\mathbf{k}) = (W_\xi^{-1}\psi)(\mathbf{k}) = (2\pi)^{-3/2}\int_{\mathbf{R}^6}\exp(-i\mathbf{k}\cdot\mathbf{q})e(|\mathbf{k}-\mathbf{p}|)\,\psi(\mathbf{q},\mathbf{p})d\mathbf{q}d\mathbf{p} , \tag{4.9}$$

in the case of the spin value $j = 0$, is given by the wave function

$$\tilde{\xi}^{(\ell)}(\mathbf{k}) = (W_{\xi^{(\ell)}}^{-1}\xi^{(\ell)})(\mathbf{k}) = (\ell^2/2\pi^3)^{3/4}\exp(-\ell^2\mathbf{k}^2) . \tag{4.10a}$$

Upon explicitly computing its inverse Fourier transform by using standard methods for the integration of Gaussian integrals (cf., e.g., Schulman, 1981, p. 20), we obtain the configuration space representative of the resolution generator in (4.8):

$$\hat{\xi}^{(\ell)}(\mathbf{x}) = (8\pi^3\ell^2)^{-3/4}\exp(-\mathbf{x}^2/4\ell^2) . \tag{4.10b}$$

Consequently, the confidence functions in the following stochastic phase space values

$$(\mathbf{q},\hat{\chi}_{\mathbf{q}}^{(\ell)})\times(\mathbf{p},\tilde{\chi}_{\mathbf{p}}^{(\ell)}) \in \Gamma_{\xi^{(\ell)}} \cong \mathbf{R}^6, \tag{4.11}$$

which are obtained on the basis of (3.17), namely

$$\hat{\chi}_{\mathbf{q}}^{(\ell)}(\mathbf{x}) = (2\pi)^3\left|\hat{\xi}^{(\ell)}(\mathbf{x}-\mathbf{q})\right|^2 = (2\pi\ell^2)^{-3/2}\exp[-(\mathbf{x}-\mathbf{q})^2/2\ell^2] , \tag{4.12a}$$

$$\tilde{\chi}_{\mathbf{p}}^{(\ell)}(\mathbf{k}) = (2\pi)^3\left|\tilde{\xi}^{(\ell)}(\mathbf{k}-\mathbf{p})\right|^2 = (2\ell^2/\pi)^{3/2}\exp[-2\ell^2(\mathbf{k}-\mathbf{p})^2] , \tag{4.12b}$$

are seen to indeed correspond to the case of minimal uncertainty, for which the Heisenberg inequalities in (1.3) transcend into equalities.

In view of (4.12a), in the earlier mentioned sharp-point limit $\ell \to +0$ we get

$$\hat{\chi}_{\mathbf{q}}^{(\ell)}(\mathbf{x}) \xrightarrow[\ell\to+0]{} \delta^3(\mathbf{x}-\mathbf{q}) . \tag{4.13}$$

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[19] Cf. (Ali and Prugovečki, 1986), p. 25. For the sake of simplicity, in the present considerations we adopt the same values for the spin and mass of the system particles and of the test particles. In actuality any combination of these values is feasible, and basically all the cited results and conclusions remain unchanged (*ibid.*).

Hence, if we imagine an infinite sequence of quantum frames that are somehow tied to a given classical Galilei frame of reference, and consist of quantum test particles better and better localized in relation to that classical frame, then the marginal probability densities

$$\rho_u^{(\ell)}(\mathbf{q},t) = \int_{\mathbf{R}^3} |\psi(\mathbf{q},\mathbf{p},t)|^2 d\mathbf{p} = \int_{\mathbf{R}^3} \hat{\chi}_{\mathbf{q}}^{(\ell)}(\mathbf{x}) |\hat{\psi}(\mathbf{x},t)|^2 d\mathbf{x}, \qquad \psi \in \mathbf{P}_{\xi^{(\ell)}} L^2(\mathbf{R}^6), \quad (4.14)$$

for a given particle, measured in relation to the quantum frames in that sequence, converge to the probability densities for spatial localization in orthodox quantum mechanics,

$$\rho_u^{(\ell)}(\mathbf{q},t) \xrightarrow[\ell \to +0]{} \hat{\rho}_u(\mathbf{q},t) = |\hat{\psi}(\mathbf{q},t)|^2 , \quad (4.15)$$

at least for those quantum states that can be represented by (almost everywhere) continuous wave functions in the configuration representation.

This fact already provides a strong indication that the proposed interpretation of

$$\rho_u^{(\ell)}(\psi;\mathbf{q},\mathbf{p},t) = |\psi(\mathbf{q},\mathbf{p},t)|^2 \quad (4.16)$$

as a probability density for the simultaneous measurement of the stochastic phase space values in (4.11) is in agreement with orthodox theory. However, even more remarkable is the fact that the corresponding probability current,

$$\mathbf{j}_u^{(\ell)}(\mathbf{q},t) = \int_{\mathbf{R}^3} \mathbf{v} \rho_u^{(\ell)}(\psi;\mathbf{q},\mathbf{p},t) d\mathbf{p} = \int_{\mathbf{R}^3} \frac{\mathbf{p}}{m} |\psi(\mathbf{q},\mathbf{p},t)|^2 d\mathbf{p} , \quad (4.17)$$

obtained from the interpretation of $|\psi(\mathbf{q},\mathbf{p},t)|^2$ in (4.16) as a probability density in phase space (and therefore of $\mathbf{v} = \mathbf{p}/m$ as a stochastic 3-velocity) in the same manner as its counterpart in classical statistical mechanics (Balescu, 1975), satisfies the continuity equation

$$\partial_t \rho_u^{(\ell)}(\mathbf{q},t) + \nabla \cdot \mathbf{j}_u^{(\ell)}(\mathbf{q},t) = 0 . \quad (4.18)$$

Moreover, although no close relationship could have been expected on *a priori* grounds between this current and the well-known probability current

$$\mathbf{j}_u(\mathbf{x},t) = (2im)^{-1} \hat{\psi}^*(\mathbf{x},t) \vec{\nabla} \hat{\psi}(\mathbf{x},t) , \qquad \hat{\psi}_1 \vec{\nabla} \hat{\psi}_2 := \hat{\psi}_1 \nabla \hat{\psi}_2 - \hat{\psi}_2 \nabla \hat{\psi}_1 , \quad (4.19)$$

in orthodox quantum mechanics (cf., e.g., Galindo and Pascual, 1991), it turns out that in the sharp-point limit the former merges into the latter:

$$\mathbf{j}_u^{(\ell)}(\mathbf{q},t) \xrightarrow[\ell \to +0]{} \mathbf{j}_u(\mathbf{q},t) . \quad (4.20)$$

All these results were first established and proved (Prugovečki, 1978a) for the above treated case of optimal stochastic phase space localization, but were later extended (Ali *et al.*, 1981) to all resolution generators possessing real configuration space wave function representatives. Their validity can hardly be expected to be merely a mathematical accident, especially when amplified by the outcomes of extensive studies – reviewed in (Prugovečki, 1984, Chapter 3) and in (Ali, 1985) – of the close connections between corresponding classical and quantum statistical concepts and results, which emerge from a conceptually and mathematically unified nonrelativistic framework for the formulation of classical *and* quantum statistical mechanics.

Perhaps the most remarkable result of these studies is provided by a unified treatment of classical and quantum Boltzmann equations (Prugovečki, 1978b), whereby a quantum version of the well-known classical Boltzmann equation (cf. Balescu, 1975) is obtained, which incorporates a quantum scattering differential cross-section derived in a manner analogous to that used in the classical context, and is based on a supermatrix $\langle \mathbf{k}';\ell|\boldsymbol{T}^{\ell}|\mathbf{k};\ell\rangle$:

$$\sigma^{\ell}_{\mathbf{k}}(\theta,\varphi) = (2\ell^2/\pi)^{3/2}\int_0^{\infty} dk\, k^2 \int_{\mathbf{R}^3} d\mathbf{k}' \exp[-2\ell^2(\mathbf{k}-\mathbf{k}')^2](m/k')\left\langle \mathbf{k}';\ell\middle|\boldsymbol{T}^{\ell}\middle|\mathbf{k};\ell\right\rangle. \quad (4.21)$$

This supermatrix is derivable from a superoperator $\boldsymbol{T}^{\ell} = -i\boldsymbol{H}_{\mathrm{I}}^{\ell}\boldsymbol{\Omega}_{+}^{\ell}$ that acts in a master Liouville space containing density operators for quantum as well as classical ensembles. In the classical regime, the formula obtained by replacing the quantum superoperator $\boldsymbol{T}^{\ell}$ in (4.21) with its classical superoperator counterpart $\boldsymbol{T}^{\mathrm{cl}}$, produces in the formal momentum sharp-point limit $\ell \to +\infty$ the δ-"function" $\delta^3(\mathbf{k}'-\mathbf{k})$.

At the interpretational level, such a limit can be associated in the quantum regime with sharp-momentum measurements, corresponding to those *gedanken* limiting procedures whereby position measurements are carried out with a finite precision corresponding to a fixed value for ℓ, but the macroscopic spatial separations in the time-of-flight or other standard types of momentum measurements is conceived to become infinite (Prugovečki, 1978a,b). Under analogous circumstances of increasingly accurate measurements of classical momenta, the resulting classical supermatrix-based scattering differential cross-section turns out to be equal to the standard classical scattering differential cross-section derived within the conventional framework (cf., e.g., Balescu, 1975, pp 387-390):

$$\sigma^{\mathrm{cl}}_{\mathbf{k}}(\theta,\varphi) = (b\,db\,d\varphi)/d\Omega = m\int_0^{\infty} dk k\left\langle \mathbf{k}';\mathrm{cl}\middle|\boldsymbol{T}^{\mathrm{cl}}\middle|\mathbf{k};\mathrm{cl}\right\rangle, \qquad \boldsymbol{T}^{\mathrm{cl}} = -i\boldsymbol{H}_{\mathrm{I}}^{\mathrm{cl}}\boldsymbol{\Omega}_{+}^{\mathrm{cl}} \quad (4.22)$$

In the quantum regime, the formula in (4.21) produces numerical values which are in agreement with those derived from the well-known differential scattering cross-section formula based on a T-matrix (Messiah, 1962; Taylor, 1972) under physically reasonable assumptions (such as those described on p. 177 of Taylor, 1972) on the orders of magnitude of basic parameters in typical scattering experiments. However, in general these two formulae do not coincide even in the limit $\ell \to +\infty$, which formally yields a formula similar to the integral in (4.22) – and was subsequently derived (Turner and Snider, 1980) within the conventional framework for quantum statistical mechanics. Moreover, there is a fundamental *physical* distinction between the methods required in the physically and mathematically *rigorous* derivations of these respective formulae: *any* rigorous derivation of the conventional scattering cross-section formula requires (Newton, 1979) an averaging procedure associated with a planar target or a beam cross-section (cf., e.g., Messiah, 1962, Chapter X, §§5-6; Prugovečki, 1981, pp. 430-435; Taylor, 1972, Sec. 10-d); whereas, just as in the classical case of (4.22), no such averaging is required in the derivation of (4.21) (cf. Prugovečki, 1978b, pp. 237-246), so that (4.21) depicts the differential scattering cross-section for ensembles of isolated pairs of scattering particles.

This illustrates the possibility of using the idea of fundamental length even in the nonrelativistic regime, to uncover facts about physical systems that might remain hidden when one is relying exclusively on the orthodox framework for nonrelativistic quantum theory.

3.5. Path Integration and Nonrelativistic Quantum Frames

Although useful, the idea of fundamental length ℓ is not essential to the solution of the problem of quantum particle localizability[20] in the nonrelativistic regime, where the Newtonian conceptualization of spatio-temporal localization relies exclusively on Euclidean geometries, which are deeply rooted in the Newtonian version of absolute space and time. However, in this section we shall demonstrate that, even in the nonrelativistic regime, this idea can resolve the mathematical and physical problems which one encounters in the path-integration method for the formulation of nonrelativistic quantum evolution (Feynman, 1948). The solution of these problems is quite essential to the formulation of the concept of quantum-geometric propagation in the next chapter, since that formulation relies on the concept of parallel transport along broken paths consisting of arcs of geodesics, and requires the use of path-integration methods that are free of inconsistencies.

The physics community at large was first made aware of the path integration method by a paper which Feynman published in 1948. That seminal paper soon received widespread attention, and was later reprinted on pp. 321-341 of the well-known collection of key papers on quantum electrodynamics edited by Schwinger (1958). In it, Feynman presented the path integration method as a "space-time approach to non-relativistic quantum mechanics" that was meant to provide an *alternative* to the, by that time, very well-established formulation of quantum mechanical time-evolution, based on the Schrödinger and the Heisenberg pictures of quantum mechanics.

As recounted by Feynman in his Nobel prize acceptance speech (Feynman, 1966), in which he extensively discussed the origins of his ideas on path integration, the principal source of those ideas lay in a 1933 paper by Dirac, originally published in a somewhat obscure journal, but later reprinted on pp. 312-320 of (Schwinger, 1958). Feynman's key epistemic observation was that, if $(q_{t+dt}|q_t)$ denotes the transition amplitude resulting from the *quantum* propagation of a nonrelativistic particle during the infinitesimal Newtonian time span from t to $t + dt$, then by the statement (made in Dirac's 1933 paper[21]) that "the result $(q_{t+dt}|q_t)$ corresponds to $\exp[iLdt/h]$, Dirac meant that they were proportional." (Feynman, 1966, p. 703.)

In his 1948 paper on path integration, Feynman denoted that proportionality constant by $1/A$, and computed its value to be $(m/2\pi it)^{3/2}$ for a quantum particle of mass m moving in three spatial dimensions. This computation was based on a formal derivation of the Schrödinger equation from his path integral formula. However, this result was obtained at the price of assigning well-defined values to certain "integrals [which] are oscillatory and not defined" (Feynman, 1948, p. 376), since they actually *do not exist* either as Riemann improper integrals, or as Lebesgue integrals. In fact, as later established by M. Kac (1959), Feynman's (1948) entire derivation assumes a mathematically legitimate meaning only for

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[20] The quantum locality problems emerging from the well-known EPR paradox (Einstein *et al.*, 1935) are related to the propagation of state-reduction effects that are, strictly speaking, at odds only with our classical and *relativistic* conceptualization of spacetime structure; of course, as such, they are profoundly affected by the introduction of a quantum notion of spacetime, which incorporates a fundamental length ℓ (Prugovečki, 1985a).

[21] Cf. p. 316 of Schwinger, 1958, where L is defined as the *classical* Lagrangian of a particle. As discussed in (Gleick, 1992), pp. 128-132, as well as (Schweber, 1986b), p. 462, this, at first sight, innocuous observation played a key epistemic role during the entire formative period in which Feynman gradually developed his ideas on the path-integration method of formulating quantum evolution.

potentials which allow analytic continuations of quantum propagators to pure imaginary time, in which case, however, it produces a solution of the heat equation (or, equivalently at the mathematical level, of the diffusion equation for Brownian motion), rather than of the Schrödinger equation. This necessitates, *after* a solution of the heat equation is actually obtained, yet another analytic continuation back to real time in order to obtain a legitimate solution of the Schrödinger equation for a quantum process.

Albeit rigorous, and the predecessor of all the still popular Euclidean-metric methods in the relativistic regime (Rivers, 1987), this kind of analytic continuation to imaginary time is not totally satisfactory from the mathematical point of view, since it is rigorously applicable only to a limited class of potentials, which does not include many of the potentials that occur in practice. It is even less satisfactory from the physical point of view, since it forfeits Feynman's original aspirations – described in considerable detail in (Schweber, 1986b) – for a "visualization of [quantum] space-time processes" in *real* time.

To illustrate and then analyze these points, let us consider the nonrelativistic Feynman path integral method based on the following propagators of pointlike quantum particles without spin, formally expressed as follows (Feynman and Hibbs, 1965):

$$K(\mathbf{x}'',t'';\mathbf{x}',t') := \langle \mathbf{x}(t'') | \mathbf{x}(t') \rangle = \left\langle \mathbf{x}'' \middle| e^{-iH_0(t''-t')} \middle| \mathbf{x}' \right\rangle \ , \tag{5.1a}$$

$$|\mathbf{x}',t'\rangle = \hat{U}(t',\mathbf{x}',\mathbf{0},I)|\mathbf{0},0\rangle \ , \qquad |\mathbf{x}'',t''\rangle = \hat{U}(t'',\mathbf{x}'',\mathbf{0},I)|\mathbf{0},0\rangle \tag{5.1b}$$

These propagators are conventionally interpreted as transition amplitudes between an initial state vector $|\mathbf{x}',t'\rangle$ and a final state vector $|\mathbf{x}'',t''\rangle$, obtained by translating the state vector $|\mathbf{0},0\rangle$, situated at the origin of a Galilei frame in a Newtonian spacetime, to the spacetime locations $(\mathbf{x}',t')$ and $(\mathbf{x}'',t'')$. Hence, they are supposed to describe all the probability densities for a particle to reach the spatial location $\mathbf{x}(t'')$ at given instant t'' if it has started from the location $\mathbf{x}(t')$ at time t'.

This interpretation is, however, merely heuristic at the mathematical as well as at the physical level: mathematically, because the time evolution operator in (5.1a) is applied to such objects as $|\mathbf{x}(t')\rangle$ which, as we saw in Sec. 3.2, do not belong to the Hilbert space $L^2(\mathbf{R}^3)$; physically, because the square of the absolute value of the propagator in (5.1a) does not yield a *bona fide* probability density, since it is seen from the well-known explicit expression for the free Feynman propagator in (5.1a) (cf. Feynman and Hibbs, 1965; Schulman, 1981), namely from

$$K(\mathbf{x}'',t'';\mathbf{x}',t') = (m/2\pi it)^{3/2}\exp[i(m/2t)(\mathbf{x}''-\mathbf{x}')^2] \ , \qquad t = t''-t' \ , \tag{5.2}$$

that this absolute value is a constant, and therefore not square-integrable in $\mathbf{x}'' \in \mathbf{R}^3$.

The first difficulty can be removed by the use of the rigged or equipped Hilbert spaces mentioned in Sec. 3.2, but the second one remains, and can be dealt with only by regarding the propagator of a point quantum particle as a renormalized limit (involving an infinite renormalization) of propagators for state vectors in $L^2(\mathbf{R}^3)$.

However, at the heuristic level[22], a straightforward application of Trotter's product

[22] Trotter's product formula constitutes part of a mathematically rigorous theorem in Hilbert space (cf., e.g., Prugovečki, 1981, p. 585), but its use in deriving (5.3) (cf. Schulman, 1981, Chapter 1) is merely formal.

formula to (5.1a) yields Feynman's (1948) well-known action integral

$$K(\mathbf{x}'',t'';\mathbf{x}',t') = \int \exp[iS(\mathbf{x}(t'');\mathbf{x}(t'))]\ \mathcal{D}\mathbf{x}(t)\ . \tag{5.3a}$$

This formula is based on the classical action over a straight path γ from $\mathbf{x}(t')$ to $\mathbf{x}(t'')$,

$$S(\mathbf{x}(t'');\mathbf{x}(t')) = \int_\gamma L[\dot{\mathbf{x}}(t),\mathbf{x}(t)]dt\ ,\quad \gamma = \left\{\mathbf{x}(t)\middle|\ t' \le t \le t''\right\}\ , \tag{5.3b}$$

which in the case considered at present contains a free classical Lagrangian L that is only velocity-dependent. Thus, according to Feynman's visualizable, simple and very elegant physical interpretation, the path integral in (5.3a) results from the "summing" of the action in (5.3b) over all possible broken polygonal paths connecting the initial point $\mathbf{x}' = \mathbf{x}(t')$ at time t' to the final point $\mathbf{x}'' = \mathbf{x}(t'')$ at time $t'' > t'$, and obtained by slicing the Newtonian spacetime segment between t' and t'' with hyperplanes $\sigma(t_n)$ corresponding to time values $t' = t_0 < t_1 < \cdots < t_N = t''$, then connecting all pairs of points $\mathbf{x}(t_{n-1})$ and $\mathbf{x}(t_n)$ on two neighboring Newtonian spacetime slices $\sigma(t_{n-1})$ and $\sigma(t_n)$ by straight lines, and in the end taking the limit $\varepsilon = t_n - t_{n-1} = (t'' - t')/N \to +0$.

To give the needed mathematical legitimacy to this physical interpretation, Feynman suggested replacing in the above procedure "$\hbar$ by $\hbar(1-i\delta)$, for example, where δ is a small positive number, and then let[ting] $\delta \to 0$" (Feynman, 1948, p. 376). A later paper by Gel'fand and Yaglom (1960) actually tried to provide a mathematically rigorous formulation of Feynman path integration by following this very suggestion, but it was soon afterwards shown by Cameron (1960, 1962) to have failed in that task. Of course, during the three decades since that time there have been numerous attempts at a rigorous formulation of Feynman path integrals (cf. Albeverio and Høegh-Krohn, 1976; DeWitt-Morette *et al.*, 1979; etc). However, as a rule they all either had to invoke physically unacceptable restrictions, or to make the transition to the "Euclidean spacetime regime," i.e., to resort to analytic continuations to imaginary time, which invalidates the main physical purpose of the path integration method that Feynman had in mind, namely that of a visualizable description of *actual* quantum processes, taking place in *real* physical time.

Feynman's other suggestion as to how to impart mathematical legitimacy to his formulation of path integration in nonrelativistic quantum mechanics consisted of the observation that a "convergence factor is automatically provided" (Feynman, 1948, p. 376) if the path integration procedure is applied to an initial state vector ψ which belongs to a Hilbert space (and is therefore represented by a bona fide wave function), rather than to such a singular mathematical entity as the ket-vector $|\mathbf{x}\rangle$, which in the configuration representation is (formally) represented by a δ-"function" at $\mathbf{x} \in \mathbf{R}^3$. And indeed, it can be immediately deduced from the rigorous version of Trotter's product formula (mentioned in Footnote 22) that for any two self-adjoint operators H_0 and H_I with a common domain of definition one has

$$\left\langle \psi' \middle| e^{-i(H_0+H_I)(t''-t')}\psi \right\rangle = \lim_{\varepsilon\to+0}\left\langle \psi' \middle| \left(e^{-iH_I\varepsilon}e^{-iH_0\varepsilon}\right)^N \psi \right\rangle\ ,\quad \varepsilon = (t''-t')/N\ . \tag{5.4}$$

However, whereas (5.4) is totally rigorous, the claim (Schulman, 1981) that a Feynman propagator actually results from it cannot be justified at a mathematically rigorous level.

That such is the case can be seen when one tries to rigorously derive the path integral in (5.2) by this method even in the absence of the interaction term H_I. Indeed, if one begins the derivation by first multiplying both sides of the formal equality

$$K(\mathbf{x}'',t'';\mathbf{x}',t') = \lim_{\varepsilon\to+0}\int\langle\mathbf{x}(t_N)|\mathbf{x}(t_{N-1})\rangle\prod_{n=N-1}^{1}\langle\mathbf{x}(t_n)|\mathbf{x}(t_{n-1})\rangle\, d\mathbf{x}(t_n)\ , \qquad (5.5)$$

with an initial wave function $\psi(\mathbf{x}',t')$, and integrating over $\mathbf{x}'$, then for sufficiently well-behaved wave functions the *iterative* integrals obtained by performing the $\mathbf{x}(t_n)$-integrations in the *prescribed* time-sequence do indeed exist. However, the integrations cannot be performed in an arbitrary order since Fubini's theorem (cf., e.g., Prugovečki, 1981, p. 96) cannot be applied, and therefore the existence of a *functional over paths* cannot be deduced, since one is dealing with only an iterative series of integrations. The basic mathematical reason for this fact is very simple: a Lebesgue integral, such as the one in (5.5), exists if and only if its integrand is integrable in the absolute sense (*ibid.*, Chapter II, Sec. 3.4). However, the absolute value of the Feynman propagator in (5.1a) is seen from (5.2) to be of constant value, so that when such propagators are inserted into such integrals as those in Eq. (2-31) of Feynman and Hibbs (1965), those integrals do not converge in the Lebesgue sense; moreover, even if improper Riemann integrals are used, the infinite-interval limits defining such integrals merely produce oscillatory terms. Hence, the "path integration" resulting from such a procedure cannot be actually carried out by integrating *independently* over all the vertices of broken paths (which would legitimatize the claim that the polygonal paths with those vertices are indeed the actual variables of integration), but only by integrating *sequentially* in $\mathbf{x}(t_1), \mathbf{x}(t_2), \ldots$. However, this last procedure merely recovers Dirac's self-reproducing-kernel formula for time evolution in the Schrödinger picture – cf. Eq. (11) on p. 316 of (Schwinger, 1958).

This physically and mathematically undesirable situation changes radically in the case of path integrals formulated by means of propagators of quantum particles optimally localized in the stochastic phase space based on the fundamental length ℓ. In that situation, the stochastic phase space counterpart of the free Feynman propagator in (5.1a) is:

$$K^{(\ell)}(\mathbf{q}'',\mathbf{p}'',t'';\mathbf{q}',\mathbf{p}',t') = \left\langle \xi^{(\ell)}_{\mathbf{q}'',\mathbf{p}''}\middle| e^{-iH_0(t''-t')}\xi^{(\ell)}_{\mathbf{q}',\mathbf{p}'}\right\rangle\ , \qquad (5.6a)$$

$$\xi^{(\ell)}_{\mathbf{q}',\mathbf{p}'} = U(0,\mathbf{q}',\mathbf{p}'/m,I)\xi^{(\ell)}\ , \qquad \xi^{(\ell)}_{\mathbf{q}'',\mathbf{p}''} = U(0,\mathbf{q}'',\mathbf{p}''/m,I)\xi^{(\ell)}. \qquad (5.6b)$$

Indeed, this propagator is obtained by exactly the same kinematical procedures from the wave function in (4.8), describing the proper state vector of a quantum particle at the origin of a nonrelativistic quantum frame, as the propagator (5.1a) of a pointlike quantum particle is obtained from a δ-"wave function" at the origin of a classical Galilei frame. Moreover, by working in the momentum representation, we obtain from (1.10), (4.10a) and (5.6b)

$$K^{(\ell)}(\mathbf{q}'',\mathbf{p}'',t'';\mathbf{q}',\mathbf{p}',t') = \int_{\mathbf{R}^3}\tilde{\xi}^{(\ell)*}_{\mathbf{q}'',\mathbf{p}'',t''}(\mathbf{k})\tilde{\xi}^{(\ell)}_{\mathbf{q}',\mathbf{p}',t'}(\mathbf{k})d\mathbf{k}\ , \qquad (5.7a)$$

$$\tilde{\xi}^{(\ell)}_{\mathbf{q},\mathbf{p},t}(\mathbf{k}) = (\ell^2/2\pi^3)^{3/4}\exp[-\ell^2(\mathbf{k}-\mathbf{p})^2 - i\mathbf{k}\cdot\mathbf{q} + i\mathbf{k}^2(t/2m)]\ , \qquad (5.7b)$$

so that, by using the standard formulae on Gaussian integrals (cf.. e.g., Schulman, 1981, p. 20), we can explicitly compute the integral in (5.7a), with the following result:

$$K^{(\ell)}(\mathbf{q}'',\mathbf{p}'',t'';\mathbf{q}',\mathbf{p}',t') = (2\pi)^{-3}(\ell^2/\ell_{m,t}^2)^{3/2}\exp\left\{-[(\mathbf{q}''-\mathbf{q}')^2/8\ell_{m,t}^2]\right\}$$

$$\times\exp\left\{-(\ell^2/2\ell_{m,t}^2)\left[\ell^2(\mathbf{p}''-\mathbf{p}')^2 - i(\mathbf{q}''-\mathbf{q}')\cdot(\mathbf{p}''+\mathbf{p}') + (it/2m)(\mathbf{p}''^2+\mathbf{p}'^2)\right]\right\}, \quad (5.8a)$$

$$\ell_{m,t}^2 := \ell^2 + (i/4m)t\ , \qquad t = t''-t' \in \mathbf{R}^1 \quad (5.8b)$$

Consequently, we see that in the sharp-point limit $\ell \to +0$ the propagator in (5.6) indeed converges to the Feynman propagator in (5.2),

$$(\pi/2\ell^2)^{3/2}K^{(\ell)}(\mathbf{q}'',\mathbf{p}'',t'';\mathbf{q}',\mathbf{p}',t') \xrightarrow[\ell\to+0]{} K(\mathbf{q}'',t'';\mathbf{q}',t')\ , \quad (5.9)$$

but only after a renormalization is performed, which becomes infinite in that limit. In fact, for $t' = t''$, that limit has to be interpreted in the sense of distributions, since it equals a δ-"function," which is not an ordinary but only a generalized function (Gel'fand and Shilov, 1964). On the other hand, the fact that the renormalization constant in (5.9) becomes infinite in the sharp-point limit emerges as one of the root causes of the earlier discussed difficulties with Feynman propagators and path integrals.

We shall now indicate how to derive[23] all the key results of Feynman's (1948) path-integral approach to nonrelativistic quantum mechanics by using the stochastic phase space propagator in (5.7) instead of the one in (5.1). In this process we shall demonstrate how all the aforementioned difficulties disappear.

First of all, we had the interpretational difficulty originating from the fact that the Feynman propagator in (5.1a) is neither determined by a *bona fide* inner product, nor does it in turn determine a *bona fide* probability density, since the square of its absolute value is constant. This fundamental difficulty is removed once the fundamental length ℓ is introduced, since the propagator in (5.6a) is determined by a *bona fide* inner product, and it in turn determines a *bona fide* probability density, on account of the fact that, in accordance with (3.9), (3.13) and (4.9), we have

$$(2\pi)^3\int_{\mathbf{R}^6}\left|K^{(\ell)}(\mathbf{q}'',\mathbf{p}'',t'';\mathbf{q}',\mathbf{p}',t')\right|^2 d\mathbf{q}''d\mathbf{p}'' \ = \ 1 \quad (5.10)$$

Moreover, according to the physical interpretation discussed in the previous section, for a particle optimally localized at time t' at the stochastic location $\mathbf{q}'$, and having there the stochastic 3-momentum $\mathbf{p}'$, this density provides the probabilities of its reaching at the time t'' various stochastic locations $\mathbf{q}''$, and displaying there various stochastic 3-momenta $\mathbf{p}''$.

Second, from the generic expression (3.8), and the fact that the operator it represents is an orthogonal projection operator, and therefore equal to its own adjoint and to its own square, we obtain

$$K^{(\ell)}(\mathbf{q}'',\mathbf{p}'',t'';\mathbf{q}',\mathbf{p}',t') = K^{(\ell)*}(\mathbf{q}',\mathbf{p}',t';\mathbf{q}'',\mathbf{p}'',t'')$$

$$= \int_{\mathbf{R}^6} K^{(\ell)}(\mathbf{q}'',\mathbf{p}'',t'';\mathbf{q},\mathbf{p},t)K^{(\ell)}(\mathbf{q},\mathbf{p},t;\mathbf{q}',\mathbf{p}',t')\,d\mathbf{q}d\mathbf{p}\ . \quad (5.11)$$

[23] All the subsequent results have been actually derived (Prugovečki, 1981a) for the case of generic resolution generators – cf. Secs. 1.8-1.11 of (Prugovečki, 1984) for a review and more detailed discussion. However, the sharp-point limit in (5.9) requires a very specific type of renormalization factor, and the requirement of its existence restricts that choice.

By iterating the last expression at the time values $t' = t_0 < t_1 < \cdots < t_N = t''$, we arrive, upon setting $\mathbf{q}' = \mathbf{q}_0$, $\mathbf{p}' = \mathbf{p}_0$, $\mathbf{q}'' = \mathbf{q}_N$ and $\mathbf{p}'' = \mathbf{p}_N$, at the following counterpart of (5.5),

$$K^{(\ell)}(\mathbf{q}'',\mathbf{p}'',t'';\mathbf{q}',\mathbf{p}',t') = \lim_{\varepsilon\to+0}\int K^{(\ell)}(\mathbf{q}_N,\mathbf{p}_N,t_N;\mathbf{q}_{N-1},\mathbf{p}_{N-1},t_{N-1}) \times \prod_{n=N-1}^{1} K^{(\ell)}(\mathbf{q}_n,\mathbf{p}_n,t_n;\mathbf{q}_{n-1},\mathbf{p}_{n-1},t_{n-1})\, d\mathbf{q}_n d\mathbf{p}_n \ . \quad (5.12)$$

In this expression all the integrals exist in a *bona fide* Lebesgue sense, and the limit in it exists pointwise as $\varepsilon = \max(t_n - t_{n-1}) \to +0$. This is easily seen to be the case, since the presence of the Gaussian factors in *each* one of the phase space propagators in (5.12) secures (by the earlier mentioned Fubini theorem) the *simultaneous* integrability in *all* the $6(N-1)$ variables of integration that occur on the right-hand side of (5.12). In turn, this leads to a phase-space path integral, which is mathematically closely related to coherent state representations of path integration (cf., e.g., Schulman, 1981, Chapter 27), but which, as opposed to the latter, has a direct physical significance that emerges from the optimal type of nonrelativistic localizability of quantum systems in the presence of a fundamental length ℓ.

This physical feature manifests itself very clearly in the generic case of those time-dependent interactions $H_I(t)$ of a quantum particle with external sources which are given by potentials $V(\mathbf{q},t)$. The total Hamiltonian of such a system can be expressed directly within the Hilbert space $L^2(\mathbf{R}^6)$,

$$H(t) = H_0 + H_I(t) \ , \qquad H_0 = \mathbf{P}^2/2m \ , \qquad H_I(t) = V(\mathbf{Q},t) \ , \quad (5.13)$$

by using the position and momentum operators that can be associated with the phase-space representation $U(b,\mathbf{a},\mathbf{v},R)$ in (3.5) in accordance with (1.17):

$$Q^j = -\frac{i}{m}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[U\big(t,0,\mathbf{v}_{(j)}(\varepsilon),I\big)-\mathbf{1}\Big] \ , \qquad v^i_{(j)}(\varepsilon) = \varepsilon\,\delta^i_j \ , \quad (5.14a)$$

$$P_j = i\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[U\big(t,\mathbf{a}_{(j)}(\varepsilon),0,I\big)-\mathbf{1}\Big] \ , \qquad a^i_{(j)}(\varepsilon) = \varepsilon\,\delta^i_j \ . \quad (5.14b)$$

Indeed, when these self-adjoint operators are restricted to any of the irreducible subspaces $\mathbf{P}_\xi L^2(\mathbf{R}^6)$ they become, on account of (3.14), unitarily equivalent to their configuration space counterparts in (1.18). It is also easily seen that they can be expressed as follows:

$$P_j = -i\,\partial/\partial q^j \ , \qquad Q^j = q^j + i\,\partial/\partial p^j \ , \qquad j = 1,2,3 \quad (5.15)$$

For analytic potentials $V(\mathbf{q},t)$ the action of $H_I(t)$ in $\mathbf{P}_{\xi(t)}L^2(\mathbf{R}^6)$ can be approximated by

$$V(\mathbf{Q}_{\rm st},t) \ , \qquad (Q^j_{\rm st}\psi)(\mathbf{q},\mathbf{p},t) = q^j\psi(\mathbf{q},\mathbf{p},t) \ , \qquad j = 1,2,3 \ . \quad (5.16)$$

It is then easily computed from (3.14) that the action of such a phase-space interaction Hamiltonian is related, from the point of view of the configuration representation, to that of a potential that is "smeared out" in accordance with (4.12a) and (4.14):

$$(W^{-1}_{\xi^{(\ell)}} V(\mathbf{Q}_{\rm st},t) W_{\xi^{(\ell)}} \hat{\psi})(\mathbf{x},t) = V^{(\ell)}(\mathbf{x},t)\hat{\psi}(\mathbf{x},t) \ , \tag{5.17a}$$

$$V^{(\ell)}(\mathbf{x},t) = \int_{\mathbf{R}^3} \hat{\chi}^{(\ell)}_{\mathbf{q}}(\mathbf{x}) V(\mathbf{x},t) d\mathbf{x} \ . \tag{5.17b}$$

In turn, in the sharp-point limit $\ell \to +0$, such a "smeared out" potential becomes equal to a potential acting on a point particle.

However, in that same limit the free phase-space propagators in (5.8) require the infinite renormalization in (5.9) in order to merge into their Feynman counterparts in (5.1a). Consequently, the action integrals over paths in phase space for the propagator

$$K(\mathbf{q}'',\mathbf{p}'',t'';\mathbf{q}',\mathbf{p}',t') = \left\langle \xi_{\mathbf{q}'',\mathbf{p}''} \middle| \, T e^{-i\int_{t'}^{t''} H(t)dt} \xi_{\mathbf{q}',\mathbf{p}'} \right\rangle \ , \tag{5.18}$$

also require, in the limit $\ell \to +0$, infinite renormalization in order to formally merge into the action integral in (5.3) for the corresponding Feynman path integral representing the configuration space propagator

$$K(\mathbf{x}'',t'';\mathbf{x}',t') = \left\langle \mathbf{x}'' \middle| \, T e^{-i\int_{t'}^{t''} \hat{H}(t)dt} \middle| \mathbf{x}' \right\rangle \ , \qquad \hat{H}(t) = W^{-1}_{\xi^{(\ell)}} H(t) W_{\xi^{(\ell)}} \mathbf{P}_{\xi^{(\ell)}} \ . \tag{5.19}$$

On the other hand, the stochastic phase space path integrals exist in a mathematically rigorous sense, since the application of the Trotter product formula to (5.18) yields

$$K(\mathbf{q}'',\mathbf{p}'',t'';\mathbf{q}',\mathbf{p}',t') = \lim_{\varepsilon \to +0} \left\langle \xi_{\mathbf{q}'',\mathbf{p}''} \middle| \prod_{n=N}^{1} e^{-iH_I(t_n)(t_n - t_{n-1})} e^{-iH_0(t_n - t_{n-1})} \xi_{\mathbf{q}',\mathbf{p}'} \right\rangle \ , \tag{5.20}$$

in a mathematically legitimate manner. Hence, upon using (3.8) in between the elements of each consecutive pair of operators in (5.20), we obtain the expression

$$K(\mathbf{q}'',\mathbf{p}'',t'';\mathbf{q}',\mathbf{p}',t') = \lim_{\varepsilon \to +0} \int d\mathbf{q}'_N d\mathbf{p}'_N \prod_{n=N-1}^{1} d\mathbf{q}'_n d\mathbf{p}'_n d\mathbf{q}_n d\mathbf{p}_n$$

$$\times \prod_{n=N}^{1} \left\langle \xi_{\mathbf{q}_n,\mathbf{p}_n} \middle| e^{-iH_I(t_n)(t_n - t_{n-1})} \xi_{\mathbf{q}'_n,\mathbf{p}'_n} \right\rangle K^{(\ell)}(\mathbf{q}'_n,\mathbf{p}'_n,t_n;\mathbf{q}_{n-1},\mathbf{p}_{n-1},t_{n-1}) \ , \tag{5.21}$$

from which a path integral formula for an action integral in stochastic phase space can be derived (Prugovečki, 1981a; 1984),

$$K(\mathbf{q}'',\mathbf{p}'',t'';\mathbf{q}',\mathbf{p}',t') = \int \exp[iS^{(\ell)}(\mathbf{q}(t''),\mathbf{p}(t'');\mathbf{q}(t'),\mathbf{p}(t'))] \, \mathcal{D}^{(\ell)}[\mathbf{q}(t),\mathbf{p}(t)] \ , \tag{5.22}$$

which is at the formal level analogous to the one for (5.19).

It has to be emphasized that the definition of the above stochastic phase space path integrals is based on the Riemannian rather than on the Lebesgue notion of integration. Indeed, for a quantum process in *real* time there is *no* measure over paths analogous to a Wiener measure (Kac, 1959), such is the case for the Feynman-Kac integrals (cf., e.g., Glimm and Jaffee, 1987) that can be *formally* obtained by replacing in (5.3) the real variable t with the purely imaginary variable $t' = it$. Of course, this does not make these stochastic phase space integrals mathematically any less well-defined, since the Riemannian

method of integration is mathematically as rigorous as the Lebesgue method, and merely more restricted in its scope than the latter in the case of integration over finite-dimensional manifolds. Rather, this necessary use of Riemannian integration merely underlies the fundamental fact that in the quantum regime there are no probability measures over paths, since in quantum theory it is the probability amplitudes for various paths, rather than probability densities for such paths, that are being added in quantum processes in accordance with the superposition principle. This feature distinguishes all *quantum* stochastic processes (Prugovečki, 1981a; 1984) from all *classical* stochastic processes – which constitute the principal subject of interest in the conventional theory of stochastic processes. It also is the main reason why the innumerable attempts at basing quantum mechanical path integration in the *real* time variable t on the conventional theory of stochastic processes, which requires the existence of *bona fide* measures over its paths, have basically ended up in failure.

3.6. Poincaré Covariance and Relativistic Quantum Localizability

The acknowledged founder of relativistic quantum mechanics is P.A.M. Dirac (1928). However, it was also Dirac who eventually became its severest critic, and even declared: "One must seek a new relativistic quantum mechanics and one's prime concern must be to base it on sound mathematics." (Dirac, 1978b, p. 6.)

As explained in Sec. 1.4, the reasons for Dirac's dissatisfaction with the ultimate status of the field of quantum physics which he himself had founded sprang primarily from the problems with divergences in relativistic quantum field theory. The origins of these divergences can be traced all the way to unresolved problems with the conventional notions of the quantum relativistic localizability of particles and fields. We shall now describe and analyze some of these problems, concentrating mainly on the simple case of a quantum particle of rest mass m and zero spin, where the concomitant mathematical technicalities are least liable to obscure the essential physical issues.

As is well-known, in the relativistic regime the role of the Galilei group is taken over by the (full) Poincaré group. This group is also known under the name of inhomogeneous Lorentz group, so that if the homogeneous Lorentz group is viewed as a group of matrices and is denoted by $\mathrm{SO}(3,1)$, then the full Poincaré group is denoted by $\mathrm{ISO}(3,1)$. The elements of $\mathrm{ISO}(3,1)$ incorporate, in addition to Lorentz transformations Λ, also spacetime translations, which can be labeled by 4-tuples $a \in \mathbf{R}^4$. Hence, these elements, which we shall call Poincaré transformations, act on 4-tuples x of Minkowski coordinates as follows:

$$(a,\Lambda) : x \mapsto x' = a + \Lambda x , \qquad (a,\Lambda) \in \mathrm{ISO}(3,1) , \qquad x \in \mathbf{R}^4 . \tag{6.1}$$

By performing on the same 4-tuple x two such transformations in succession, it is immediately seen that the group multiplication law for the Poincaré group is given by

$$(a',\Lambda')(a,\Lambda) = (a' + \Lambda' a, \Lambda'\Lambda) , \qquad a',a \in \mathbf{R}^4 , \qquad \Lambda',\Lambda \in \mathrm{SO}(3,1) . \tag{6.2}$$

As a consequence, the Poincaré group emerges as a ten-dimensional Lie group, whose maximal connected subgroup is the *restricted* Poincaré group $\mathrm{ISO}_0(3,1)$, which corresponds to the restricted Lorentz group $\mathrm{SO}_0(3,1)$ that we have already encountered in Sec.

2.6. Consequently, $\mathrm{ISO}_0(3,1)$ is the largest subgroup of $\mathrm{ISO}(3,1)$ which does not incorporate any space and/or time reflections. We shall work in this and the next two chapters exclusively with $\mathrm{ISO}_0(3,1)$, rather than with $\mathrm{ISO}(3,1)$, and therefore in the sequel we shall refer to $\mathrm{ISO}_0(3,1)$ simply as the Poincaré group.

The Poincaré group $\mathrm{ISO}_0(3,1)$ is a locally compact and connected Lie group to which the method of induced representation has been applied successfully, so that by now all its irreducible unitary representations are known and have been classified.[24] There are many analogies between these representations and those of the Galilei group, which is not surprising in view of the fact that the Galilei group can be obtained from the Poincaré group by a group contraction[25] in the limit $c \to \infty$ of infinite speed of light *in vacuo*. There is also, however, one key difference between these representations: whereas the Galilei group has *bona fide* ray representations, and those representations give rise to the (nonrelativistic) canonical commutation relations, that is not the case with the Poincaré group, since all its unitary ray representations are equivalent to its ordinary vector representations. From the mathematical point of view, this is the first indication that the conventional notion of spacetime localization, transferred from the classical to the quantum special relativistic regime without any basic modifications, might run into difficulties in the quantum context.

For the case of a quantum particle of rest mass m and of zero spin, the unitary representation of the Poincaré group $\mathrm{ISO}_0(3,1)$ that is obtained by the inducing method is equivalent (cf. Thaller, 1992, Sec. 3.2) to the following one,

$$\tilde{U}(a,\Lambda) \;:\; \tilde{\varphi}(k) \mapsto \tilde{\varphi}'(k) = \exp(ia\cdot k)\,\tilde{\varphi}(\Lambda^{-1}k)\;, \quad (a,\Lambda) \in \mathrm{ISO}_0(3,1)\;. \tag{6.3}$$

This representation acts in the Hilbert space (cf. Bogolubov *et al.*, 1975, Sec. 6.2)

$$L^2(\boldsymbol{V}_m^+, d\Omega_m)\;, \qquad \boldsymbol{V}_m^+ = \left\{ k \,\middle|\, k\cdot k := \eta_{\mu\nu}k^\mu k^\nu = m^2,\; k^0 > 0 \right\} \subset \mathbf{R}^4, \tag{6.4}$$

consisting of functions which are square-integrable, over the forward mass hyperboloid in (6.4) for the rest mass m, with respect to the (unique modulo a multiplicative constant) invariant measure

$$d\Omega_m(k) = \delta(k^2 - m^2)d^4k \cong d\Omega_m(\mathbf{k}) = \tfrac{1}{2}(\mathbf{k}^2 + m^2)^{-1/2}d\mathbf{k}\;, \quad k = (k^0, \mathbf{k}) \in \boldsymbol{V}_m^+\;. \tag{6.5}$$

The Hilbert space in (6.4) carries the inner product

$$\left\langle \tilde{\varphi}_1 \middle| \tilde{\varphi}_2 \right\rangle = \tfrac{1}{2}\int_{\mathbf{R}^3} \tilde{\varphi}_1^*(k)\tilde{\varphi}_2(k)(\mathbf{k}^2 + m^2)^{-1/2}\,d\mathbf{k} = \int_{\boldsymbol{V}_m^+} \tilde{\varphi}_1^*(k)\tilde{\varphi}_2(k)\,d\Omega_m(k)\;, \tag{6.6}$$

and provides the momentum representation of conventional relativistic quantum mechanics for spinless particles of rest mass m (cf., e.g., Schweber, 1961, Chapter 3; Bjorken and Drell, 1964, Chapter 9). This is due to the fact that in this Hilbert space the PV measure

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[24] Cf. (Barut and Raczka, 1986), Chapter 17. For half-integer spin, the covering group ISL(2,C) of ISO(3,1) has to be used in the inducing procedure. However, all these representations were first derived by Wigner (1939) well before Mackey's (1951-53) general method was formulated – cf. (Ohnuki, 1976) for Wigner's approach, and Sec. 3.2 of (Thaller, 1992) for a detailed derivation of the connection between Mackey's and Wigner's approaches.

[25] Cf. Inönü and Wigner (1952, 1953). A general discussion of Lie algebra contractions can be found in (Barut and Raczka, 1986), Chapter 17, §8; whereas, a review of the basic results on contractions pertaining to the Poincaré group and the "little groups" determining its representations can be found in Chapter VIII of (Kim and Noz, 1986).

$$\tilde{E}(B):\ \tilde{\varphi}(k)\ \mapsto\ \chi_B(k)\tilde{\varphi}(k)\ ,\qquad \tilde{\varphi}\in L^2(\boldsymbol{V}_m^+,d\Omega_m)\ ,\quad B\subset \boldsymbol{V}_m^+\ , \tag{6.7}$$

supplies the probabilities of sharp 4-momentum measurement outcomes with $k\in B$,

$$\left\langle\tilde{\varphi}\middle|\tilde{E}(B)\tilde{\varphi}\right\rangle = \int_B |\tilde{\varphi}(k)|^2\, d\Omega_m(k)\ ,\qquad B\subset \boldsymbol{V}_m^+\ , \tag{6.8}$$

with respect to a given global and classical Lorentz frame of reference. In fact, the combination of (6.7) and (6.8) represents the relativistic counterpart of (2.3b). Indeed, it can be easily seen that if in a relativistic state the preponderant 3-velocities $\mathbf{v}$ of the relative motion with respect to that Lorentz frame display a spread of magnitudes which are small in relation to the speed of light (so that the magnitudes of their 3-momenta $\mathbf{k}$ are small in relation to m) then, after an appropriate renormalization, the probabilities in (6.8) can be approximated by those in (2.3b). Moreover, on account of (6.3), the above expectation values display the correct Poincaré covariance properties under changes of such Lorentz frames:

$$\left\langle\tilde{\varphi}'\middle|\tilde{E}(\Lambda B)\tilde{\varphi}'\right\rangle = \left\langle\tilde{\varphi}\middle|\tilde{E}(B)\tilde{\varphi}\right\rangle\ ,\qquad (a,\Lambda)\ :\ k\mapsto k'=\Lambda k\ ,\quad k\in \boldsymbol{V}_m^+ \tag{6.9}$$

The transition to the conventional configuration representation can be effected by following the formal analogy with (1.11), and therefore setting

$$\hat{\varphi}(x) = (2\pi)^{-3/2}\int_{k^0>0}\exp(-ik\cdot x)\tilde{\varphi}(k)\,d\Omega_m(k)\ ,\qquad k\cdot x := k^\mu\eta_{\mu\nu}x^\nu\ . \tag{6.10}$$

If the above transformation is assumed to be unitary, then the resulting functions constitute a Hilbert space with an inner product which (on a suitable dense set) has the form

$$\left\langle\hat{\varphi}_1\middle|\hat{\varphi}_2\right\rangle = i\int_{\mathbf{R}^3}\hat{\varphi}_1^*(x)\overleftrightarrow{\partial}_0\hat{\varphi}_2(x)\,d\mathbf{x}\ ,\qquad \hat{\varphi}_1^*\overleftrightarrow{\partial}_0\hat{\varphi}_2 := \hat{\varphi}_1^*\partial\hat{\varphi}_2/\partial x^0 - \hat{\varphi}_2\,\partial\hat{\varphi}_1^*/\partial x^0\ , \tag{6.11}$$

and is left invariant by the time-evolution given by

$$\hat{\varphi}_{x^0} = \exp(-i\hat{P}_0x^0)\hat{\varphi}_0\ ,\qquad \hat{P}_0 = \left(\hat{\mathbf{P}}^2+m^2\right)^{1/2}, \tag{6.12}$$

in which the 3-momentum operators in $\mathbf{P}$ formally coincide with those in (1.18). Hence, the wave functions obeying (6.12) satisfy the Klein-Gordon equation,

$$\left(\partial_\mu\partial^\mu+m^2\right)\hat{\varphi}(x)=0\ ,\qquad \hat{\varphi}(x^0,\mathbf{x}) := \hat{\varphi}_{x^0}(\mathbf{x})\ , \tag{6.13}$$

and the inner product in (6.11) can be written in the manifestly covariant form

$$\left\langle\hat{\varphi}_1\middle|\hat{\varphi}_2\right\rangle = i\int\hat{\varphi}_1^*(x)\overleftrightarrow{\partial}_\mu\hat{\varphi}_2(x)\,d\sigma^\mu(x)\ ,\qquad d\sigma^\mu(x)=n^\mu(x)d\sigma(x)\ , \tag{6.14}$$

in which n^μ, $\mu=0,1,2,3$, are the components of the future-pointing unit vectors normal to the chosen maximal spacelike hypersurface σ in Minkowski space over which the integration in (6.14) is carried out.

It follows very easily from (6.3), (6.10) and the Poincaré invariance of the measure in (6.10) that, as was the case with the transition from (1.10) to (1.15) in the nonrelativistic regime, the transformations in (6.3) give rise to representation

$$\hat{U}(a,\Lambda)\ :\ \hat{\varphi}(x)\ \mapsto\ \hat{\varphi}'(x)=\hat{\varphi}(\Lambda^{-1}(x-a))\ ,\qquad (a,\Lambda)\in \mathrm{ISO}_0(3,1)\ . \tag{6.15}$$

However, at this point the formal analogies between the nonrelativistic and the relativistic situations stop completely, since the counterparts of the limits in (1.17a) no longer provide position operators. In fact, the three operators of multiplication by $x^j, j = 1,2,3$, are not even symmetric (i.e., Hermitian) with respect to the inner product in (6.14).

Of course, as can be seen from a review article of Kálnay (1971), there has been a great variety of attempts to cope with this problem, of which the best-known is the one by Newton and Wigner (1949). However, a theorem by Hegerfeldt (1974), which was eventually extended to multi-particle systems and to various types of modifications of the conventional concept of quantum localization (cf. Hegerfeldt and Ruijsenaars, 1980; Hegerfeldt, 1989), has *proved* the incompatibility of Einstein causality with any form of quantum localization in relation to classical Lorentz frames. Since these results are valid for any spin, the only conclusion that can be drawn is that "every attempt to provide a precise definition of a position coordinate [for a quantum point particle] stands in direct contradiction with special relativity" (Wigner, 1976, p. 313).

Another route that has been attempted in order to deal with the localization problem in special relativistic regime was to dispense with any notion of relativistic position observable, and to rely on covariant and conserved relativistic currents, in an attempt to interpret their timelike component $j^0(x)$ as a probability density. The prototype of all such currents is the Klein-Gordon current,

$$j^\mu(x) = (i/2m)\,\hat{\varphi}^*(x)\,\overleftrightarrow{\partial}^\mu\,\hat{\varphi}(x)\ , \tag{6.16}$$

which is supposed to provide the probability density for observing a relativistic spin zero particle at some point x in Minkowski space.

Historically, it was realized from the outset that the timelike component $j^0(x)$ of this current is indefinite on the space of all solutions of the Klein-Gordon equation. However, after that equation was integrated into quantum field theory, it became clear that only those solutions of the Klein-Gordon equation that correspond to positive energies are acceptable wave functions for single quantum point particles; whereas, those corresponding to negative energies have to be assigned to antiparticles. This has led in some textbooks on this subject to the *incorrect* claim that, for such positive-energy wave functions, the timelike component $j^0(x)$ of the Klein-Gordon current is positive-definite in Minkowski space, and that therefore "a consistent theory [of localization in Minkowski space] can be developed for a free [relativistic spin-zero] particle" (Schweber, 1961, p. 56). This turned out to be totally false since, as eventually *rigorously* proved by Gerlach *et al* (1967, 1969), the opposite is actually true: for *any* positive-energy solution of the Klein-Gordon equation there are points in Minkowski space where $j^0(x) < 0$, and other points where $j^0(x) > 0$, so that $j^0(x)$ is *never* positive-definite. Hence, the Klein-Gordon current is never a probability current, and the problem of sharp localization of massive spin-zero quantum particles in relation to classical Lorentz frames is seen to be insoluble.

This observation actually remains true for arbitrary spin values, as well as for quantum particles of zero mass. Nevertheless, it is sometimes claimed that no problems with the conventional notion of particle localizability occur in the case of relativistic quantum particles of spin-1/2, since their wave functions satisfy the Dirac equation instead of the Klein-Gordon equation. However, in view of the existence of Foldy-Wouthuysen

(1950) transformations, which transform a Dirac bispinor wave function into one whose four components obey the Klein-Gordon equation, it was observed already in the 1970s that, with regard to the problem of localization, "the Klein-Gordon equation is neither better nor worse than the Dirac equation" (Wightman, 1972, p. 98).

When faced with this fact, some physicists claim that the localization problem of relativistic quantum point particles of spin-1/2 is actually solved in the conventional sense by the Dirac current, since the timelike component of that current is positive-definite for all solutions of the Dirac equation. However, it was rigorously proved (Thaller, 1984, 1992) that *all* positive-energy solutions of the Dirac equation are different from zero at *all* points in Minkowski space. This means that *none* of these solutions can describe a state of a spin-1/2 quantum *point* particle that can be localized in the conventional sense, i.e., that can be localized within some bounded region of space or of spacetime delineated with respect to some classical inertial frame. Indeed, the wave function of a state localized in such a bounded region would obviously have to vanish outside that region. Hence, the conventional special relativistic concept of localizability of quantum point particles is inconsistent also for the spin-1/2 case. The same remains true for all other spin values.

A standard type of attempt at by-passing all these fundamental inconsistencies within the conventional quantum theoretical framework resorts to the rationalization that, on account of the phenomenon of pair-creation that accompanies the high-energy collision processes required by high-precision position measurements, the solution to the quantum relativistic localization problem is unrelated to that of single-body relativistic quantum systems, and should be sought only in the context of quantum *field* localization. This attempted escape route ignores, however, the fact that *all* fields, i.e., classical or quantum, *require test bodies* for their detection and localization, so that such suggestions merely lead to a vicious circle.

This last fact, and some of its consequences, were pointed out already in Sec. 1.4, and will be further discussed in Chapter 5. However, since some of the quantum field theorists who subscribe to this rationalization incorrectly attribute it to Pauli, it is worth quoting from a 1951 letter of W. Pauli to M. Fierz, which underlines this very point: "In my [i.e., Pauli's] opinion the snag in quantized field theory nevertheless remains *quite the same* as in non-quantized field theory: it should be the case that a field would not be mathematically or logically *conceivable* without the experimental bodies required for its measurement. . . . [However,] the true relation of complementarity between the possibility of perceiving the *same* objects either as fields or as experimental bodies . . . is not expressed in the formalism used today." (Cf. Laurikainen, 1988, p. 222.)

3.7. Poincaré Covariance and Quantum Lorentz Frames

The operational definition of a Galilei quantum frame of reference was given in Sec. 3.3 in terms of freely propagating quantum test particles, geometrically arrayed by means of spacetime translations and velocity boosts. That definition can be immediately adapted to the special relativistic regime in the same manner in which Einstein (1905) adapted to that regime the classical definition of a Galilei frame, namely by assuming that the speed of light *in vacuo* is frame independent, and by using then that postulate for the synchronization of

clocks. However, in the quantum relativistic regime, one major conceptual simplification emerges: in principle, the test particles constituting that frame supply the *natural* standards for gauging spacetime distances, so that, at least at the conceptual level, there is no need to introduce "rigid rods" or "standard clocks," whose exact physical nature actually remains unspecified. Indeed, according to a fundamental epistemic idea of de Broglie (1923, 1924), any elementary quantum object of rest mass m represents a *natural* clock with period $T = 2\pi/m$ in Planck natural units. Hence, a truly fundamental choice, independent of all conventions, exists at the microlevel for the gauging of all spatio-temporal relationships.

In fact, the role of macroscopic "unit rods," such as the standard meter in Paris, has already been supplanted in experimental praxis by microscopic standards that have emerged from the realization that "the best physical definition of length and time is provided by a particular light or radio source which acts as standard for both length and time" (Arzeliés, 1966, p. 21). In a micro-setting, the measurement of spatial separation between two nearby "material points" by means of electromagnetic radiation can be accomplished by taking advantage of interference phenomena, which enable the counting of the number of wave-lengths needed to cover the distance between those two "points." In principle, the same procedure can be carried out with beams of massive test particles, such as electrons, or – in the case that electromagnetic phenomena cannot be ignored in a given measurement situation – with any of the stable neutral elementary particles discovered thus far (Kelly *et al.*, 1980). Indeed, it should be recalled that the replacement of a source of electromagnetic radiation, used in an optical microscope for the measurement of spatial separations, with a source of massive elementary particles is already *fait accompli* in an electron microscope.

All this shows that de Broglie's fundamental idea implies that a *natural* standard for the gauging of *all* spatio-temporal relationships *is inscribed in every single bit of matter* in existence. Although apparently this became forgotten in more recent times, it was this very idea that occupied a central place in his 1923-24 publications, that hailed the birth of quantum mechanics in 1925, and provided Schrödinger with the decisive impetus towards his formulation of wave mechanics (Moore, 1989). In the 1970s, de Broglie recalled its epistemic origins as follows: "When, in 1922-1923, I had my first ideas about wave mechanics, I was guided by the vision of constructing a true physical synthesis, resting upon precise concepts, of the coexistence of waves and particles. I never questioned then the nature of the physical reality of waves and particles. . . . I also noticed that if the particle was regarded as containing the rest energy $M_0c^2 = h\nu_0$, it was natural to compare it with a small clock of proper frequency ν_0, such that, when the clock moves with the speed βc, its internal frequency differs from that of the wave and equals $\nu_0(1-\beta^2)^{1/2}$." (de Broglie, 1979, pp. 6-7.)

In describing his later pursuits of this idea, de Broglie went on to say: "The account given in my *Thèse* had the disadvantage of applying only to the particular case of a plane monochromatic wave, which is never rigorously realized in nature, by virtue of the inevitable existence of spectral width." (*Ibid.*, p. 7.)

It is this latter fact that largely motivated his proposal of "pilot waves," that postulated the existence of a basic dichotomy in nature, whereby the motion of a quantum "particle" is guided by a "wave," so that "even in the general case of a wave which is not plane monochromatic, the internal vibration of the particle remains constantly in phase with the

wave carrying it." (*Ibid.*, p. 9.) This led him to the conclusion that "the rest mass M_0, which appears in the expression for the [particle energy] W and for [its 3-momentum] $\mathbf{p}$ is not in general equal to the rest mass m_0 usually attributed to the particle." (*Ibid.*, pp. 9-10.)

The presence in the quantum geometry framework of the fundamental length ℓ altogether dispenses with monochromatic waves, since it implicitly introduces a corresponding fundamental "spectral width" within the framework. Hence, it modifies de Broglie's original idea of an absolutely sharp natural time, "which is never rigorously realized in nature," into that of a *mean* natural (proper) time. The replacement of global with local frames in the next chapter will insure that such a proper time is kept *locally*, and in accordance with the fundamental epistemic features of Einstein's *general* relativistic conceptualization of an ultimately valid description of nature. Moreover, the introduction in subsequent chapters of a quantum-geometric mode of propagation within bundles of quantum frames or superframes will remove the need for "pilot waves," and therefore also remove the wave-particle dichotomy that gives rise to the distinction between the two rest masses M_0 and m_0 in de Broglie's (1924, 1990) framework. In fact, at the deeper epistemological level, the replacement of classical geometries, upon which de Broglie based his foundational ideas, with quantum geometries obviates the need for all such extraneous assumptions about the nature of quantum reality in the same manner in which, in classical physics, the replacement of Euclidean geometry with the geometry of Minkowski space obviated the need for considering Lorentz contractions of electrons as real classical phenomena.

The construction of mathematical representatives of special-relativistic, and therefore *global* quantum Lorentz frames, represents the first step in the formulation of truly quantum geometries for relativistic quantum mechanics over a curved base spacetime manifold, which will be provided in the next chapter. Such a construction can be carried out by adapting the first stages of the geometro-stochastic method of quantization, described in Sec. 3.3, from the nonrelativistic to the special relativistic regime. This can be achieved in a straightforward manner, namely, by replacing all the nonrelativistic concepts of Sec. 3.3 with their special relativistic counterparts (Prugovečki, 1978c,d; 1984).

First of all, it has to be noted that, in view of the insuperable intertwining of time and space in relativistic theory, the special relativistic phase space of a classical particle is given by the seven-dimensional manifold[26]

$$M^4 \times \boldsymbol{V}_m^+ \subset \mathbf{R}^8 \ , \qquad M^4 := (\mathbf{R}^4, \boldsymbol{\eta}) \ , \tag{7.1}$$

so that the counterpart of the (six-dimensional) nonrelativistic phase space is given by a *reference hypersurface*

$$\Sigma_m = \sigma \times \boldsymbol{V}_m^+ \subset M^4 \times \boldsymbol{V}_m^+ \ , \tag{7.2}$$

in which σ is a spacelike hyperplane within the Minkowski space M^4, and corresponds to some constant value of the time variable q^0, such as $q^0 = 0$, with respect to a given classical Lorentz frame. Consequently, the relativistic counterpart of the Hilbert space $L^2(\mathbf{R}^6)$ in

[26] In CGR a relativistic phase space is defined (Ehlers, 1971) as a submanifold of the cotangent bundle $T^*\mathbf{M}$, obtained by restricting the domain of the covectors in each cotangent space to the forward mass hyperboloid in (6.4) for the rest mass of each considered particle. In the special-relativistic context each such phase space can be therefore identified with the direct topological product of Minkowski space and the forward mass hyperboloid.

Sec. 3.3 is the Hilbert space $L^2(\Sigma_m)$, which consists of all the complex-valued functions $\varphi(q, p)$ in the variables $q = (0,\mathbf{q})$ and $p = (p^0,\mathbf{p})$, with $\mathbf{q},\mathbf{p} \in \mathbf{R}^3$ and $p^0 = (\mathbf{p}^2+m^2)^{1/2}$, and carries the inner product

$$\langle \varphi_1 | \varphi_2 \rangle = \int_{q^0=0} \varphi_1^*(q,p)\,\varphi_2(q,p)\,d\mathbf{q}\,d\mathbf{p} \ . \tag{7.3}$$

By analogy with (3.2), and in agreement with (6.12), we then introduce in this Hilbert space the free time evolution

$$\varphi_{q^0} = \exp(-iP_0 q^0)\varphi_0 \ , \qquad P_0 = \left(\mathbf{P}^2 + m^2\right)^{1/2} , \qquad \mathbf{P}^2 = -\Delta_{\mathbf{q}} \ . \tag{7.4}$$

This leads, as in the case of (6.13), to the the following Klein-Gordon equations:

$$\left(\partial_\mu \partial^\mu + m^2\right)\varphi(q,p) = 0 \ , \qquad \partial_\mu := \partial/\partial q^\mu \ , \qquad p \in \boldsymbol{V}_m^+ \tag{7.5}$$

Within the relativistic phase space in (7.1) all the Poincaré transformations are represented by mappings of the following form:

$$(a,\Lambda) \ : \ (q,p) \ \mapsto \ (q',p') = (a + \Lambda q \ , \Lambda p) \ , \qquad q \in \mathbf{R}^4 \ , \ p \in \boldsymbol{V}_m^+ \ . \tag{7.6}$$

Hence, proceeding by analogy with (3.5), and in agreement with (6.15) and (7.6), we introduce in the Hilbert space $L^2(\Sigma_m)$ the following representation of the Poincaré group:

$$U(a,\Lambda) \ : \ \varphi(q,p) \ \mapsto \ \varphi'(q,p) = \varphi(\Lambda^{-1}(q-a), \Lambda^{-1}p) \ . \tag{7.7}$$

It is then easily verified that this representation satisfies the same group multiplication law as the representations of the Poincaré group considered in the preceding section,

$$U(a',\Lambda')U(a,\Lambda) = U(a'+\Lambda' a, \Lambda'\Lambda) \ , \tag{7.8}$$

but that, as opposed to those representations, it is highly reducible.

The spectral decomposition of the representation in (7.7) can be carried out (Ali and Prugovečki, 1986) in basically the same manner as that of its nonrelativistic counterpart (3.5), despite the fact that (7.7) is an ordinary vector representation; whereas, (3.5) is a ray representation which is not equivalent to a vector one. Indeed, subsequent results by Ali *et al.* (1989-90) and by De Bièvre (1989b) have established the existence of an underlying general theory of coherent states for certain types of locally compact groups.[27] This was expected, on account of the striking analogies between the Galilei and the Poincaré cases, provided by the generalization in (Prugovečki, 1978c) of an already generalized notion of coherent states, formulated and developed by Perelomov (1972, 1986). The need for such a further generalization became very apparent after the phase space representation in (7.7) was first introduced in (Prugovečki, 1978d).

This need emerged from the *physical* requirement of a truly *parallel* treatment of the phase space quantum localizability problem under Galilei and Poincaré types of covariance.

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[27] The technical details of such constructions, as well as the prospects for further applications and generalization, are discussed from a general group-theoretical point of view by Ali and Antoine (1989), and by Antoine (1990). Particular derivations of this kind of generalized coherent states for the de Sitter group can be found in (Ali *et al.*, 1990) for two spacetime dimensions, and in (Drechsler and Prugovečki, 1991; Drechsler, 1992) for four spacetime dimensions. For a review of earlier results, see Secs. 2.3-2.7 in (Prugovečki, 1984).

Indeed, comparison with Perelomov's (1972, 1986) prescription reveals that the states $\xi_{\mathbf{q},\mathbf{p}}$ in (3.7) are generalized coherent states in his sense only for the *isochronous* Galilei subgroup, corresponding to t = const. slices of Newtonian spacetime, but not for the entire Galilei group, which includes time translations. In fact, if Perelomov's method of construction of coherent states is applied to the entire Galilei group, it results in unphysical wave functions, namely, in wave functions which are square integrable in the time as well as in the space variables. Hence, such wave functions can describe only quantum objects which materialize out of nowhere, and which, with the passage of time, rescind back into nowhere, even when left all by themselves. This, of course, would violate the most basic postulates of energy and momentum conservation, total probability conservation, etc., which are supposed to hold in all formulations of quantum theory.

Due to the lack of absolute simultaneity in special relativity, the Poincaré group possesses no isochronous subgroups. Hence, the special relativistic counterparts of (3.7),

$$(\mathbf{P}_\eta\varphi)(q,p) = \langle \eta_{q,p}|\varphi\rangle \ , \qquad \eta_{q,p} := U(q,\Lambda_{p/m})\eta \ , \qquad \varphi \in L^2(\Sigma_m) \ , \tag{7.9}$$

incorporate Lorentz boosts Λ_v to the 4-velocity $v = p/m$ that are the relativistic counterparts of the Galilei boosts in (3.7), but give rise to continuous resolutions of the projectors

$$\mathbf{P}_\eta \ = \ \int_{q^0=0} |\eta_{q,p}\rangle \, d\mathbf{q}\, d\mathbf{p} \langle \eta_{q,p}| \tag{7.10}$$

onto irreducible subspaces of (7.7) that contain generalized coherent states for the entire Poincaré group which, of necessity, are *not* coherent states in the sense of Perelomov. Mathematically, this is due to the fact that the action of time translations upon each resolution generator η, uniquely characterizing[28] each irreducible subrepresentation of (7.7),

$$\mathbf{P}_\eta U = W_\eta \tilde{U} W_\eta^{-1} \ , \qquad W_\eta : \ \ell^2(2j+1) \otimes L^2(\mathbf{V}_m^+, d\Omega_m) \to \mathbf{P}_\eta L^2(\Sigma_m) \ , \tag{7.11a}$$

$$(\mathbf{P}_\eta\varphi)(q,p) = \left(m/4\pi^3\right)^{1/2} \int_{\mathbf{V}_m^+} \exp(-iq\cdot k) e(p\cdot k/m) \overline{Y}_j^l(R_{k,\Lambda_{p/m}}\hat{\mathbf{k}}) \tilde{\varphi}_l(k) d\Omega_m(k) \ , \tag{7.11b}$$

for rest mass m and integer-valued spin j, is related neither to a character of the subgroup of time translations, nor to the isotropy group for the quantum state represented by that resolution generator. Physically, this fact is in keeping with the requirement that the projectors in (7.10) should be related to manifestly covariant POV measures

$$E_\eta(B) = \int_B |\eta_{q,p}\rangle \, d\Sigma(q,p) \langle \eta_{q,p}| \ , \qquad \tilde{\eta}^l(k) = \left(m/4\pi^3\right)^{1/2} e(k^0) Y_j^l(\mathbf{k}/|\mathbf{k}|) \ , \tag{7.12}$$

in which $d\Sigma$ is the unique (modulo a positive multiplicative constant) covariant measure on the hypersurface Σ_m within the relativistic phase space (7.1),

$$d\Sigma(q,p) = 2p_\mu \, \delta(p^2 - m^2) d\sigma^\mu(q) d^4p \ = 2p_\mu \, d\sigma^\mu(q) d\Omega_m(p) \ , \tag{7.13}$$

and B is any Borel set within Σ_m, so that

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[28] The R-operator acting in the case of non-zero spin j upon the normalized 3-momentum $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ in (7.11b) represents a Wigner 3-rotation – cf. Eq. (1.10b) in Sec. 7.1. As in the nonrelativistic case, the present relativistic formalism can accommodate any combination of spins and rest masses for the test particles and for the system particles (Prugovečki, 1980, 1984; Ali and Prugovečki, 1986; Brooke and Schroeck, 1989).

$$\left\langle \varphi \middle| E_\eta(B) \varphi \right\rangle = \int_B \left|\varphi(q,p)\right|^2 d\Sigma(q,p) \ , \qquad \varphi \in \mathbf{P}_\eta L^2(\Sigma_m) \ , \tag{7.14}$$

can be interpreted as the Poincaré covariant counterparts of the probabilities in (3.19).

All these features emerge naturally from the mathematical fact that the inner product in (7.3) can be written in the following manifestly covariant form,

$$\left\langle \varphi_1 \middle| \varphi_2 \right\rangle = \int \varphi_1^*(q,p)\varphi_2(q,p)\, d\Sigma(q,p) \ , \tag{7.15}$$

once the mathematically arbitrary constant in $d\Sigma$ is adjusted so that, in any global Lorentz frame of reference in which σ is a hyperplane corresponding to constant q^0, the element of measure $d\Sigma(q,p)$ in (7.13) equals $d\mathbf{q}d\mathbf{p}$ in the Minkowski coordinates with respect to that frame – and therefore (7.15) assumes the form (7.3). Indeed, it turns out (cf. Prugovečki, 1984, pp. 85-86) that each reference hypersurface Σ_m can be identified with the homogeneous space (cf., e.g., Barut and Raczka, 1986) $\mathrm{ISO}_0(3,1)/(\mathcal{T}\times \mathrm{SO}(3))$ of left cosets with respect to the direct product $\mathcal{T}\times \mathrm{SO}(3)$ of the time-translation subgroup $\mathcal{T}$ with the spatial rotation subgroup SO(3) of the Poincaré group $\mathrm{ISO}_0(3,1)$. In that case $d\Sigma$ gives rise to a left-invariant measure on that space. On the other hand, the same Σ_m can be also identified with the homogeneous space $(\mathcal{T}\times \mathrm{SO}(3))\backslash \mathrm{ISO}_0(3,1)$ of right cosets, and in that case it is $d\Sigma/p^0$ that gives rise to a right-invariant measure on that space.

The existence of the following conserved and Poincaré covariant probability current[29]

$$j_\eta^\mu(q) = 2\int_{V_m^+} v^\mu \left|\varphi(q,p)\right|^2 d\Omega_m(p) = \int_{\mathbf{R}^3} \frac{p^\mu}{mp^0} \left|\varphi(q,p)\right|^2 d\mathbf{p} \ , \tag{7.16a}$$

$$\partial_\mu j_\eta^\mu(q) = 0 \ , \qquad \partial_\mu := \partial/\partial q^\mu \ , \tag{7.16b}$$

$$j_\eta^\mu(q) \ \mapsto \ j_\eta^\mu(q') = \Lambda^\mu{}_\nu \, j_\eta^\nu(q) \ , \qquad q' = a + \Lambda q \ , \tag{7.16c}$$

actually ensures that (7.15) remains true for reference hypersurfaces Σ_m of the from (7.2) for which σ is an arbitrary maximal spacelike hypersurface within the Minkowski space M^4. Hence, the action of the projector in (7.9) can be expressed in the form

$$(\mathbf{P}_\eta \varphi)(q,p) = \int \left\langle \eta_{q,p} \middle| \eta_{q',p'} \right\rangle \varphi(q',p')\, d\Sigma(q',p') \ , \tag{7.17}$$

where the integration can be carried out over *any* such reference hypersurfaces Σ_m. This relation also establishes that, just as in the nonrelativistic situation considered in Sec. 3.3, each one of the irreducible subspaces $\mathbf{P}_\eta L^2(\Sigma_m)$ is a reproducing kernel Hilbert space.

In complete analogy with the nonrelativistic case considered in Sec. 3.3, a *special relativistic quantum frame* $\mathcal{Q}^{\boldsymbol{u}}$ can be now associated with a classical Lorentz frame $\boldsymbol{u}$ by considering a resolution generator η to represent the proper state vector $\boldsymbol{\Phi}^{\boldsymbol{u}}$ of such a test body marking the origin of that frame, and subjecting it to the set of transformations that

[29] The existence and properties of this probability current, as well as of the "charge current" in (7.20a), were established in (Prugovečki, 1978c) for resolution generators η with real momentum space representatives that corresponded to $j = 0$ in (7.11), but it was later shown (Ali and Prugovečki, 1986; Ali *et al.*, 1981, 1988) that all the results derived in that paper (and partly described in this section) actually remain true in the generic case.

describes the basic kinematical operations of spacetime translation and boost – whose elements are mathematically provided for the test particles considered in this section by the irreducible representation $U\mathbf{P}_\eta$ in (7.11), acting in the subspace $\mathbf{P}_\eta L^2(\Sigma_m)$ of $L^2(\Sigma_m)$. Generically, the outcome of such a construction is a family

$$Q^{\boldsymbol{u}} = \left\{ \Phi^{\boldsymbol{u}}_\zeta = U(q, \Lambda_{p/m})\Phi^{\boldsymbol{u}} \;\middle|\; \zeta = (q,p) \in M^4 \times \mathbf{V}_m^+ \right\} , \tag{7.18a}$$

of generalized coherent states supplying the mathematical representatives of such a quantum frame. For example, in the present case of system of particles of the same rest mass m as the test particles, these generalized coherent states are provided by the state vectors

$$\Phi^{\boldsymbol{u}}_\zeta \cong \eta_{q,p} = U(q, \Lambda_{p/m})\eta \in \mathbf{P}_\eta L^2(\Sigma_m) . \tag{7.18b}$$

In view of the form (6.14) of the inner product in the conventional configuration representation of relativistic quantum mechanics for a spin-zero particle, it is significant to observe that in each of the irreducible subspaces $\mathbf{P}_\eta L^2(\Sigma_m)$ the inner product in (7.15) can be written in the alternative form (cf. Prugovečki, 1984, Sec. 2.6),

$$\langle \varphi_1 | \varphi_2 \rangle = im^{-2} \hat{Z}_m(\eta) \int \varphi_1^*(q,p) \overleftrightarrow{\partial}_\mu \varphi_2(q,p)\, d\sigma^\mu(q) d\Omega_m(p) , \tag{7.19a}$$

$$\hat{Z}_m^{-1}(\eta) = (2\pi)^3 m^{-2} \int_{V_m^+} \left| e(k^0) \right|^2 d\Omega_m(k) , \qquad \tilde{\eta}(k) = (m/4\pi^3)^{1/2} e(k^0) . \tag{7.19b}$$

It has to be stressed, however, that as opposed to (7.15), the form (7.19a) is *not* valid on the entire Hilbert space $L^2(\Sigma_m)$, since the value of the normalization constant in (7.19a) depends critically on the choice of the resolution generator η with momentum space representative $\tilde{\eta}$ in (7.19b). On the other hand, on each irreducible subspace $\mathbf{P}_\eta L^2(\Sigma_m)$, the following conserved and Poincaré covariant "charge current" is associated with (7.19a):

$$J^\mu_\eta(q) = im^{-2} \hat{Z}_m(\eta) \int_{V_m^+} \varphi^*(q,p) \overleftrightarrow{\partial}_\mu \varphi(q,p)\, d\Omega_m(p) , \quad \varphi^* \overleftrightarrow{\partial}_\mu \varphi := \varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^* , \tag{7.20a}$$

$$\partial_\mu J^\mu_\eta(q) = 0 , \qquad J^\mu_\eta(q) \;\mapsto\; J^\mu_\eta(q') = \Lambda^\mu{}_\nu J^\nu_\eta(q) . \tag{7.20b}$$

However, as in the case of the Klein-Gordon charge current in (6.16), and as opposed to the probability current in (7.16), the timelike component of this "charge current" is not positive-definite, so that it cannot be assigned any probabilistic interpretation.

Clearly, it is difficult to believe that the existence of the probability current in (7.16), whose spacelike components are totally analogous to their nonrelativistic counterparts in (4.17), and timelike component to the probability density in (4.14), is a matter of mere mathematical coincidence. And indeed, when (4.14) and (4.17) are formulated for the case of generic nonrelativistic resolution generators ξ which can be identified with the relativistic counterparts η, then it can be shown that the probability currents in (7.16) merge in the limit $c \to \infty$ into their nonrelativistic counterparts. On the other hand, although in the next section we will show that there exists a relativistic counterpart of the ℓ-based fundamental quantum frames of Sec. 3.4, the corresponding probability current in (7.16a) does *not* converge in the sharp-point limit $\ell \to 0$ of those special relativistic frames to the Klein-Gordon current in (6.16), but rather it diverges. Furthermore, despite its formal similarity

to the Klein-Gordon current in (6.16), in the same sharp-point limit $\ell \to 0$ the "charge current" in (7.20a) does not merge into the one in (6.16) either, except in the asymptotic limit $m \to \infty$ of infinitely massive test particles – i.e., in the limit where the physical behavior of a quantum test particle assumes distinctly classical features.[30]

All this demonstrates that, whereas in the nonrelativistic quantum mechanics the introduction of a fundamental length ℓ could not be deemed as absolutely necessary on the basis of foundational considerations, that is no longer the case already in the special relativistic quantum regime: in the presence of such a fundamental length, it is a *proven* fact that *there is* a consistent theory of special relativistic quantum localization, based on conserved and Poincaré covariant *probability* currents for arbitrary rest masses and spins; whereas, in the conventional framework such an all-engulfing theory *does not exist*, and the sharp-point limit of the aforementioned consistent theory of special relativistic quantum localization does not exist either.

3.8. Fundamental Special-Relativistic Quantum Lorentz Frames

The conventional concept of special-relativistic quantum particle localizability rests on two major premises: 1) the acceptability of the classical concept of a frame of reference in the micro-domain, which implicitly gives rise to the retention of classical Minkowski geometry in that domain; 2) the conceptualization of a quantum particle as a pointlike entity immersed within a classical spacetime continuum, whose structure is described by such a geometry.

One of the earliest measurement-theoretical arguments indicating that these conventional concepts might be inadequate in the quantum special relativistic regime was given by Heisenberg (1938b) as part of the motivation for his introduction of a fundamental length in quantum physics. That argument was more recently reconsidered in the context of the thesis that "the concept of rest energy of [a quantum] particle, and thus the concept of the particle becomes dubious" (von Borzeszkowski and Treder, 1988, p. 102) in that regime. As presented in this last context, the argument consists of the observation that if the *non*relativistic uncertainty relations in (1.3) are retained for a relativistic quantum particle of rest mass m, which moves (approximately) along, say, the direction of the x-axis of an inertial frame, and if E denotes its energy and p_x its 3-momentum with respect to that axis, then in the ultra-relativistic limit, where its velocity approaches c, we would have:

$$\Delta x \, \Delta E/c \sim \Delta x \, \Delta p_x \gtrsim \hbar \, . \tag{8.1a}$$

Hence, if one assumes that the uncertainty for position measurements of that particle is basically determined by its Compton wavelength, one obtains the following approximate inequality (cf. von Borzeszkowski and Treder, 1988, Eq. (6.85)),

$$\Delta E \gtrsim \hbar c/\Delta x \sim mc^2 \, , \qquad \Delta x \sim \hbar/mc \, , \tag{8.1b}$$

for the uncertainty in energy of a quantum "particle" of rest energy mc^2. Hence, according to this last inequality, the measurement-theoretical uncertainty in the energy of that particle would appear to be greater than its rest energy.

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[30] Cf. (Prugovečki, 1984), Secs. 2.5-2.8 for a review of the derivation of all the aforementioned results, and for a more detailed discussion of their significance.

In view of the fact that the formulation of the conserved and Poincaré covariant *bona fide* probability current in (7.16) required the introduction of the relativistic phase space into the quantum framework, it is noteworthy that the above argument relies on the hidden premise that the spatial micro-localization of a quantum point particle with respect to the micro-constituents of an inertial frame can be achieved in a 4-velocity independent manner. However, the *assumption* that the classically-founded conceptualization of a velocity-independent special relativistic metric can be retained at the microlevel has been brought into question already by Max Born (1938, 1949) – the generally acknowledged progenitor of the orthodox interpretation of quantum mechanics (Born, 1926). Indeed, in a 1949 paper, in which he reviewed a theory of reciprocity between spacetime and 4-momentum relationships that represented one of the earliest attempts at introducing in a consistent manner a fundamental length in quantum physics, Born remarked the following: "Ordinary relativity is based on the invariance of the 4-dimensional distance, or its square $R = x_k x^k$. Can one really define the distance of two particles in sub-atomic dimensions independently of their velocity? This seems to me not evident at all." (Born, 1949, p. 208.)

Born became involved with research into classical relativity right after its inception of relativity theory (cf. Born, 1978), and had conducted an extensive correspondence with Einstein over many years (Born, 1971). Hence, the conventional velocity-independent definition $q^2 = \eta_{\mu\nu} q^\mu q^\nu$ of the special relativistic "square-distance" of two quantum particles would have been at least as "obvious" to him as to anybody else, were it merely a matter of his conceptualizing "sub-atomic particles" as "material points" immersed within a classical relativistic spacetime continuum. However, by 1938 Born had clearly gone well beyond such a naive conceptualization. He was prompted to do that by the observation that the "square-magnitude" $p^2 = \eta_{\mu\nu} p^\mu p^\nu$ of the 4-momentum p is totally analogous to the "square-distance" $q^2 = \eta_{\mu\nu} q^\mu q^\nu$, but, whereas the entire body of research into elementary particles indicated that the former displays in Nature discrete values, the latter was simply *assumed* to display a classically conceptualized continuum of values even at the microlevel.

Upon noting the symmetry displayed in the nonrelativistic regime by the Hamilton equations, canonical commutation relations, angular momentum operators, as well as many other well-established physical relationships and quantities, under the "reciprocity transformation" $q^\mu \mapsto p^\mu$, $p^\mu \mapsto -q^\mu$, Born (1938) conjectured that such a reciprocity symmetry might extend also to the relativistic realm. This led him to the idea that not only do the rest masses of elementary particles assume discrete values, but also do their "sizes," and that those "sizes" are governed by the eigenvalues of a "quantum metric operator"

$$D^2 = Q^\mu Q_\mu + P^\mu P_\mu \quad , \qquad Q^\mu = \eta^{\mu\nu} Q_\nu \quad , \quad P_\mu = \eta_{\mu\nu} P^\nu \quad , \tag{8.2}$$

which embodies the following *relativistic* canonical commutation relations:

$$\left[Q_\mu, P_\nu\right] = -i\eta_{\mu\nu} \quad , \qquad \left[Q_\mu, Q_\nu\right] = \left[P_\mu, P_\nu\right] = 0 \quad . \tag{8.3}$$

Together with a number of collaborators, Born pursued research into reciprocity theory for well over a decade. But although this research led to some partial numerical successes, it could not cope with such foundational problems as those of causality or of gauge and translational invariance, since it did not manage to develop quantum geometries into which the notion of quantum metric operator could be incorporated at a fundamental

level, so that those foundational problems could be avoided.[31]

On the other hand, Born's (1938) basic idea of quantum metric operator did not altogether die out, since it was later incorporated by Yukawa (1950, 1953) into his nonlocal quantum field theory. This eventually led to the relativistic harmonic oscillator model of Katayama and Yukawa (1968), and of Takabayashi (1967), in which the operator D^2 in (8.2) is interpreted as a Hamiltonian. In turn, subsequent researchers (Fujimura *et al.*, 1970, 1971; Feynman *et al.*, 1971; Kim and Noz, 1986) reformulated this model into a quark-antiquark model for mesons, by interpreting the Q-operators in it in terms of multiplications with the components of a 4-vector "variable x [which] measures the space-time separation between the quarks" (Kim and Noz, 1986, p. 109) that are tied together by a harmonic oscillator potential $x^2/2$.

In this latter interpretation, the eigenvalue problem associated with (8.2) and (8.3) is formulated by imposing boundary conditions at infinity which make the wave function for the relative motion of the pointlike quark and antiquark square-integrable over all of $\mathbf{R}^4$. However, this kind of boundary conditions render the spectrum of the resulting internal Hamiltonian unbounded from below, so that *ad hoc* subsidiary conditions have to be then imposed to remove an infinity of unwanted "time-like oscillations" (Kim and Noz, p. 112). Moreover, from the point of view of quantum dynamics, a wave function that is square integrable in space *as well as* in time is of questionable physical significance even in the nonrelativistic regime, since it suggests that the entity it describes disappears *spontaneously* from existence the further we look into the distant past, or into the distant future.

On the other hand, the relativistic canonical commutation relations in (8.3) can be also realized (Prugovečki, 1981b; Brooke and Prugovečki, 1985) by means of operators which act on the coordinate fluctuation amplitudes that describe the probabilities of measurement outcomes with respect to the *quantum* Lorentz frames in (7.18a). In that case no square-integrability conditions have to be imposed in timelike directions. This results in a spectrum for the quantum metric operator in (8.2) which is bounded from below, and whose ground eigenvalue is nondegenerate (cf. Prugovečki, 1984, Sec. 4.5). The corresponding *quantum metric ground state* is given in the momentum representation by the wave function[32]

$$\tilde{\eta}^{(\ell,m)}(k) = (m/4\pi^3)^{1/2} e^{(\ell)}(k^0) = (m^3 \tilde{Z}_{\ell,m})^{-1/2} \exp(-\ell k^0) \ , \tag{8.4a}$$

$$\tilde{Z}_{\ell,m} = (2\pi)^4 \mathrm{K}_2(2\ell m)/2\ell m^2 \ , \qquad k = (k^0, \mathbf{k}) \in V_m^+ \ , \tag{8.4b}$$

where K_2 denotes a modified Bessel function. This quantum metric ground state gives rise to the following *fundamental* quantum Lorentz frame,

$$Q^{u}_{\ell,m} = \left\{ U(q, \Lambda_{p/m}) \Phi^{u}_{\ell,m} \,\middle|\, (q,p) \in M^4 \times V_m^+ \right\} , \quad \Phi^{u}_{\ell,m} \equiv \eta^{(\ell,m)} \in \mathbf{P}_{\eta^{(\ell,m)}} L^2(\Sigma_m) \ , \tag{8.5}$$

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[31] According to H.S. Green, who was one of Born's collaborators in the extensive research programme into reciprocity theory, it was Pauli who drew Born's attention to these problems, which to Born eventually "appeared insuperable" (Green, 1982 – private communication). Perhaps for that reason there is hardly any mention of this entire programme in Born's (1978) autobiography, which was published posthumously by his children.

[32] Cf. Prugovečki, 1984, Sec. 2.9. The choice of ℓ, rather than of $2\ell^2 m$, as a factor of k^0 in the exponential function in (8.4a) is dictated by the *unified* treatment of all the massive *and* massless quantum-geometric fields in Chapters 5-8. The factor m^3 multiplying the normalization constant introduced in (8.4b) results from the use in this chapter of 4-momentum variable p, instead of the 4-velocity variable v adopted in those later chapters.

constructed from *geometro-stochastic excitons* in their ground state. The hierarchy of remaining geometro-stochastic exciton states is reminiscent of the hierarchy of excited states in early versions of bosonic strings formulated within the "old covariant approach" (Green *et al.*, 1987) to string theory,[33] and gives rise to a "tower" of quantum frames that can be constructed from those states (Brooke and Prugovečki, 1984).

Although such reciprocity-based considerations might eventually turn out to have a deeper physical significance, in this monograph we shall limit ourselves to the fundamental special-relativistic quantum frames in (8.5), and to their counterparts for spin-1/2 values (cf. Sec. 6.1). Indeed, even if one disregards reciprocity theory, such choices of quantum frames commend themselves as fundamental by virtue of the fact that, upon suitable renormalization, the special-relativistic stochastic phase space propagators built from them for various rest masses and spins converge, in the sharp-point limit $\ell \to 0$, to corresponding Feynman propagators. For example, in the spin-zero case, we can introduce in accordance with (7.18b) the following counterpart of the nonrelativistic propagator in (5.6):

$$K^{(\ell,m)}(q'',p'';q',p') = \left\langle \eta^{(\ell,m)}_{q'',p''} \middle| \eta^{(\ell,m)}_{q',p'} \right\rangle . \tag{8.6a}$$

According to (7.11b) and (8.4b) we then have

$$K^{(\ell,m)}(q'',p'';q',p') = m^3 \tilde{Z}^{-1}_{\ell,m} \int_{V_m^+} \exp\{[i(q'-q'') - (\ell/m)(p'+p'')] \cdot k\} \, d\Omega_m(k) . \tag{8.6b}$$

This can be compared to the corresponding Feynman propagator, which is usually written in the configuration representation in the form (cf. Itzykson and Zuber, 1980, p. 124)

$$K_F(x''-x') = -\frac{1}{(2\pi)^4} \lim_{\varepsilon \to +0} \int d^4k \frac{\exp[-ik\cdot(x''-x')]}{k^2 - m^2 + i\varepsilon} , \tag{8.7a}$$

but can be decomposed into a future-oriented and a past-oriented component, so that, e.g., for the former we have (cf. Itzykson and Zuber, 1980, p. 34; Schweber, 1960, p. 444):

$$K_F(x''-x') = i(2\pi)^{-3} \int_{V_m^+} \exp[ik\cdot(x'-x'')] d\Omega_m(k) , \qquad x''^0 > x'^0 . \tag{8.7b}$$

We therefore immediately see that

$$i(2\pi m)^{-3} \tilde{Z}_{\ell,m} K^{(\ell,m)}(q'',p'';q',p') \xrightarrow[\ell \to +0]{} K_F(q''-q') , \qquad q''^0 > q'^0 . \tag{8.8}$$

On the other hand, all the mathematical and physical difficulties encountered by the conventional approach to path integration in the nonrelativistic regime, that were briefly reviewed in Sec. 3.5, are exacerbated in the special relativistic regime by the fact that, as discussed in Sec. 3.7, there is no consistent conventional notion of relativistic quantum particle localization. In contradistinction, all the mathematically and physically *consistent*

[33] When the same approach is applied to massless excitons (Prugovečki, 1988a; 1989b), a closer analogy with string states results (cf. Prugovečki, 1992, Sec. 12.5). However, just as was the case in the early stages in the development of string theory, originally such considerations were used in the treatment of hadrons as geometro-stochastic excitons. That led to a mass formula (Prugovečki, 1981b), which produced Regge trajectories that were actually found to be in good agreement with the available experimental data (Brooke and Guz, 1984).

features of the stochastic phase space approach, reviewed in Sec. 3.5 in the nonrelativistic context, are retained in the present special-relativistic regime (Prugovečki, 1981a; 1984). In particular, for a free particle which is known to have had at the base spacetime location q' the stochastic 4-momentum p', the propagator in (8.6) provides all the probabilities of its reaching the various stochastic locations q'' along a spacelike hypersurface σ constituting part of a reference hypersurface Σ_m in (7.2), and of displaying there various stochastic 4-momenta p''. This is due to the fact that the following counterpart of (5.10) remains true:

$$(4\pi^3/m)\int \left|K^{(\ell,m)}(q'',p'';q',p')\right|^2 d\Sigma(q'',p'') \;=\; 1\,. \tag{8.9}$$

Moreover, due to the self-adjointness and the idempotency of the orthogonal projection operator in (7.17), the propagator in (8.6) also has the following two properties:

$$\begin{aligned} K^{(\ell,m)}(q'',p'';q',p') \;&=\; K^{(\ell,m)*}(q',p';q'',p'') \\ &= \int K^{(\ell,m)}(q'',p'';q,p)K^{(\ell,m)}(q,p;q',p')\,d\Sigma(q,p)\,. \end{aligned} \tag{8.10}$$

From the above reproducibility property a counterpart of (5.12) can be derived, and a consistent path-integral formulation of special-relativistic quantum propagation in the presence of any external 4-potentials can be obtained.[34]

In addition to harmonizing with Born's (1938, 1949) reciprocity ideas about the symmetric role played in Nature by the position and momentum variables, thus providing the means as to "how to introduce the fundamental length to physics in some natural way" (Dirac, 1963, p. 50), the adoption of the fundamental quantum frames in (8.5) renders the probability amplitudes with respect to those frames to be analytic functions in the variables $z = q - i\ell p$. As we shall show in the next chapter, this secures the informational completeness of those frames in the special as well as the general relativistic regimes. Moreover, it also mediates in a most natural way the replacement in quantum theory of real with complex variables – which is an idea that was strongly advocated by Dirac on many occasions.

Indeed, as pointed out in a recent scientific biography of Dirac, dealing with his achievements and his views on quantum physics, Dirac found the theory of functions of a complex variable "to be of 'exceptional beauty' and hence likely to lead to deep physical insight." (Kragh, 1990, pp. 282.) As is well-known, Dirac was guided in many of his research endeavors, including the one that led him to his famous equation (Dirac, 1928), by the idea of "mathematical beauty." In the present instance, this methodological guide led him to the suggestion of replacing in quantum mechanics the fundamental concept of "observable" with something more general, that could accommodate the introduction of complex variables in quantum theory. Kragh describes Dirac's basic reasoning as follows: "In quantum mechanics the state of a system is usually represented by a function of real variables, the domains of which are the eigenvalues of certain observables. In 1937, Dirac suggested that the condition of realness be dropped and the variables be considered as complex quantities so that the representatives of dynamical variables could be worked out

[34] Cf. Sec. 2.10 in (Prugovečki, 1984) for a review. It is significant that this theory extends to the spin-1/2 case (Prugovečki, 1980), and that it leads to a consistent theory of single-particle bound states in external potentials which displays Lamb-shift features (Ali and Prugovečki, 1981).

with the powerful mathematical machinery belonging to the theory of complex functions. If dynamical variables are treated as complex quantities, they can no longer be associated with physical observables. Dirac admitted this loss of physical understanding but did not regard the increased level of abstraction as a disadvantage." (*Ibid.*, pp. 282-283.)

The adoption of the quantum frames in (8.5) as fundamental makes the introduction of complex variables into relativistic quantum theory totally natural. Moreover, such an adoption not only removes the possibility of any "loss of physical understanding," but it actually enhances that physical understanding, since it replaces the concept of "quantum observable" with the even more fundamental concept of "quantum frame of reference."

Of course, the systematic use of complex variables has been central to the entire S-matrix programme (Cushing, 1990) in the 1950s and the 1960s. However, the present approach injects such variables into a special-relativistic quantum framework which can be extrapolated in a most natural manner to the general relativistic regime. The presence of the complex variables $z = q - i\ell p$ in the quantum metric fluctuation amplitudes represented by the *local* quantum-geometric propagators as $\Delta^{(\pm)}$ and $S^{(\pm)}$ in the next chapter, will ensure that in the special-relativistic regime those local propagators equal, upon renormalization, the analytic extensions (in the sense of distributions) of their conventional Feynman counterparts. In view of the status of contemporary experimental high energy technology, which is still far from being able to probe spatio-temporal distances and energies of "Planckian" orders of magnitude, this feature is bound to secure numerical agreement at the formal perturbative level, and within the spatio-temporal orders of magnitude experimentally reached thus far, between conventional quantum field theoretical models and their quantum-geometric counterparts. On the other hand, the presence of $\ell > 0$ ensures the non-singularity of quantum-geometric propagators, thus avoiding the principal cause of divergencies that plague the conventional framework for quantum field theory.

Thus, in addition to adhering to Dirac's recommendation that "the research worker, in his efforts to express the fundamental laws of Nature in mathematical form, should strive mainly for mathematical beauty" (Dirac, 1939, p. 124), the quantum-geometric approach also abides by the twin criterion of mathematical soundness. It was this twin criterion that motivated Dirac to urge researchers to "seek a new relativistic quantum mechanics," and in so doing, to make sure "to base it on sound mathematics." (Dirac, 1978b, p. 6.)

It will be in the direction designated by these twin aesthetic and ethical methodological principles advocated by Dirac that we shall seek in the subsequent chapters the unification of the fibre-theoretical framework for general relativity, presented in Chapter 2, with the stochastic phase space quantum framework presented in this chapter.

Chapter 4

Quantum Geometry over a Classical Base Spacetime

In the semi-classical regime employed by the conventional quantum field theory in curved spacetime, all the matter and nongravitational radiation fields are quantized, whereas the gravitational field itself is left unquantized. This suggests that the first step towards the consistent unification of general relativity and quantum theory should consist of replacing, within the fibre-theoretical formulation of classical spacetime presented in Chapter 2, the concept of local classical frames with that of local *quantum* frames, constructed in accordance with the general procedures presented in Chapters 2 and 3. The outcome of such a construction is a principal quantum frame bundle over a Lorentzian spacetime manifold. General relativistic quantum physics can be then formulated in various quantum bundles associated with such a principal quantum frame bundle in the same vein in which general relativistic classical physics can be formulated in various tensor bundles associated with a principal classical frame bundle. In particular, the parallel transport within quantum bundles over Lorentzian base manifolds will be governed by the Levi-Civita connection for those base manifolds. This kind of parallel transport can be then used in the formulation of a quantum-geometric mode of propagation, which takes place along broken paths constructed out of arcs of the geodesics for that connection.

4.1. Quantum Frame Bundles and Associated Bundles

We saw in Sec. 1.4 that the retention of conventional quantum field theoretical methods in the semi-classical general relativisitic regime, where the Minkowski geometry is replaced by the classical Lorentzian geometry of a curved spacetime manifold, but all the matter and nongravitational radiation fields are quantized, leads to *ex nihilo* particle creation. This has spurred many statements to the effect that the concept of quantum particle, which underlies orthodox nonrelativistic quantum mechanics, is "nebulous," and actually ill-defined in the presence of a gravitational field that gives rise to the curvature of that spacetime manifold. In this context, the following is often asserted: "Part of the nebulousness of the particle concept is its *global* nature." (Birrell and Davies, 1982, p. 49.)

The introduction of the concept of quantum frame in the preceding chapter enables the adoption of a local point of view towards the formulation of quantum theory over any Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ which is space and time oriented in the sense described in Sec. 2.6. In such a manifold we shall concentrate our attention on the *Poincaré frame bundle* $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, which corresponds to the Lorentz frame bundle $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ consisting of orthonormal frames $\{\boldsymbol{e}_i \mid i = 0,1,2,3\}$ whose $\boldsymbol{e}_0$-vectors point in the future direction, and whose spatial triples $\{\boldsymbol{e}_a \mid a = 1,2,3\}$ are right-handed. The elements of the Poincaré frame

bundle $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ are the affine frames $\{(\boldsymbol{a},\boldsymbol{e}_i)|\ i = 0,1,2,3\}$ (cf. Choquet-Bruhat *et al.*, 1987, p. 10), obtained by translating each local Lorentz frame $\{\boldsymbol{e}_i \mid i = 0,1,2,3\}$ by various amounts described by the 4-vectors $\boldsymbol{a}$ within the tangent space within which that frame lies. Consequently, the structure group of the principal bundle $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ is obviously equal to the Poincaré group $\mathrm{ISO}_0(3,1)$, which we have already encountered in Sec. 3.6.

In CGR the extension of the Lorentz frame bundle $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ into the Poincaré frame bundle $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ can be always carried out (Trautman, 1980, 1982), but it does not provide any particular advantages (cf. Sec. 4.3). However, as soon as quantum propagation is considered, such an extension becomes mandatory, on account of the fact that in the special relativistic regime the 4-momenta operators for *free* quantum propagation are the generators of spacetime translations of global Lorentz frames. Hence, if the strong equivalence principle is valid at the microscopic level, then this feature has to be retained for the *free-fall* propagation of local quantum Lorentz frames in the presence of a gravitational field.

In view of the stipulations of that principle, the operational definitions of classical or quantum *local* Poincaré frames are obtained from the corresponding operational definitions of classical or quantum *global* Lorentz frames, provided in the preceding chapter in the special relativistic context, by replacing the free motion in that regime with the free-fall motion taking place in the presence of a gravitational field. Naturally, in the presence of such a field the spatio-temporal separation of the constituents of those frames have to be commensurate with the strength of that field, i.e., it has to be restricted to separations over which the curvature effects are negligible, in accordance with the various features of the strong equivalence principle singled out in Sec. 2.6. Hence, strictly speaking, the origins of operationally feasible classical *local* Poincaré frames $\{(\boldsymbol{a},\boldsymbol{e}_i)\}$ can be located in relation to the origins of the corresponding classical *local* Lorentz frames $\{\boldsymbol{e}_i\}$ only at those "infinitesimal" values of $\boldsymbol{a}$ which are commensurate with the strength of the gravitational field around the location of that origin. Naturally, the same holds true of the quantum local Poincaré frames that are "soldered" to their classical counterparts.

The mathematical realization of such an imagined soldering of quantum to classical local frames can be carried out by building the *standard quantum frame bundle*

$$Q\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}}) = P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}}) \times_{\mathbf{G}} \left\{ U(a,\Lambda) Q_{\ell}^{\boldsymbol{u}_0} \middle|\ (a,\Lambda) \in \mathbf{G} \right\}\ , \quad \mathbf{G} = \mathrm{ISO}_0(3,1)\ , \tag{1.1}$$

from the fundamental quantum frame (cf. (3.8.5)) in the Hilbert space $L^2(\Sigma)$ over a fixed reference hypersurface $\Sigma = \Sigma_1$ corresponding to $m = 1$ and to σ with $q^0 = 0$ in (3.7.2):

$$Q_{\ell}^{\boldsymbol{u}_0} = \left\{ U(q,\Lambda_v) \Phi_{\ell,1}^{\boldsymbol{u}_0} \middle|\ q \in \mathbf{R}^4,\ v \in \boldsymbol{V}^+ \equiv \boldsymbol{V}_1^+ \right\} \subset \mathbf{P}_{\eta(\ell)} L^2(\Sigma)\ . \tag{1.2a}$$

This set of fundamental quantum frames is associated to all the global Lorentz frames in $(\mathbf{R}^4,\boldsymbol{\eta})$ obtained from the standard frame $\boldsymbol{u}_0$ in $\mathbf{R}^4$ (cf. (5.12) in Chapter 5), so that

$$\boldsymbol{u}_0 \cdot \mathrm{ISO}_0(3,1) = \left\{ \boldsymbol{u} = \boldsymbol{u}_0 \cdot (a,\Lambda) \middle|\ (a,\Lambda) \in \mathrm{ISO}_0(3,1) \right\}\ . \tag{1.2b}$$

Technically, the construction in (1.1) can be implemented by means of a mathematical procedure normally employed in building associated bundles, and usually described as the taking of the $\mathbf{G}$-product of a principal bundle – namely $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ in the present situation –

with a typical fibre $\mathbf{F}$, which in the present context is given by the set of quantum frames in (1.1). In this case, as well as other cases where the $\mathbf{G}$-product will be used in later chapters, this somewhat abstract mathematical procedure can be heuristically imagined as a kind of "soldering" of one type of objects, such as quantum frames in the present instance, to another type of objects that already belong to a principal bundle – such as the corresponding local frames belonging to $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$. For the sake of mathematical precision, we shall now provide all the technical details of the general construction of $\mathbf{G}$-products.

For any given topological space $\mathbf{F}$, let $(\mathbf{P}, \Pi, \mathbf{M}, \mathbf{G})$ be a principal bundle whose structure group $\mathbf{G}$ acts on $\mathbf{F}$ from the left[1] as a group of transformations – such as is the case with the diffeomorphisms $U(g): \mathbf{F} \to \mathbf{F}$, $g \in \mathbf{G}$, that act from the left, $U(g): \Psi \mapsto g \cdot \Psi$, on the typical fibre $\mathbf{F}$ in the definition of Sec. 2.4 for a fibre bundle $(\mathbf{E}, \pi, \mathbf{M}, \mathbf{F})$ associated to a principal bundle $(\mathbf{P}, \Pi, \mathbf{M}, \mathbf{G})$. The $\mathbf{G}$-*product* $\mathbf{P} \times_{\mathbf{G}} \mathbf{F}$ of $\mathbf{P}$ and $\mathbf{F}$ is then defined as follows: the elements g of the structure group $\mathbf{G}$ are taken to act from the right on the topological product $\mathbf{P} \times \mathbf{F}$, by mapping each pair $(\boldsymbol{u}, \Psi) \in \mathbf{P} \times \mathbf{F}$ into the new pair $(\boldsymbol{u}, \Psi) \cdot g := (\boldsymbol{u} \cdot g, g^{-1} \cdot \Psi) \in \mathbf{P} \times \mathbf{F}$; the set $\mathbf{P} \times_{\mathbf{G}} \mathbf{F}$ is then defined as the family of equivalence classes of elements in $\mathbf{P} \times \mathbf{F}$ – whereby any two elements $(\boldsymbol{u}', \Psi')$, $(\boldsymbol{u}'', \Psi'') \in \mathbf{P} \times \mathbf{F}$ are deemed equivalent if there is some $g \in \mathbf{G}$ for which $(\boldsymbol{u}'', \Psi'') = (\boldsymbol{u}', \Psi') \cdot g$.

The map which takes each $(\boldsymbol{u}, \Psi) \in \mathbf{P} \times \mathbf{F}$ into $x = \Pi(\boldsymbol{u}) \in \mathbf{M}$ induces a projection map $\pi: \mathbf{P} \times_{\mathbf{G}} \mathbf{F} \to \mathbf{M}$, which is continuous if $\mathbf{P} \times_{\mathbf{G}} \mathbf{F}$ is endowed with the natural topology induced in it by the topological product of $\mathbf{P}$ and $\mathbf{F}$. Furthermore, if $\mathbf{F}$ is a manifold, and if $\mathbf{G}$ acts on it from the left as a Lie group of transformations, then a differentiable structure (specified by an atlas of charts) can be introduced in $\mathbf{P} \times_{\mathbf{G}} \mathbf{F}$, by first noting that each $x \in \mathbf{M}$ has a neighborhood $\mathcal{N}$ for which the topological space $\Pi^{-1}(\mathcal{N})$ is homeomorphic to $\mathcal{N} \times \mathbf{G}$, and then imposing the requirement that the set $\pi^{-1}(\mathcal{N})$ should be an open submanifold of $\mathbf{P} \times_{\mathbf{G}} \mathbf{F}$ that is diffeomorphic to the manifold $\mathcal{N} \times \mathbf{F}$. It then follows that $(\mathbf{P} \times_{\mathbf{G}} \mathbf{F}, \pi, \mathbf{M}, \mathbf{F})$ is a fibre bundle associated with the originally chosen principal bundle $(\mathbf{P}, \Pi, \mathbf{M}, \mathbf{G})$ in the sense defined in Sec. 2.4, and that the projection map $\mathbf{P} \times \mathbf{F} \to \mathbf{P} \times_{\mathbf{G}} \mathbf{F}$, defined by the assignment $(\boldsymbol{u}, \Psi) \mapsto \Psi = \{(\boldsymbol{u}, \Psi) \cdot g \mid g \in \mathbf{G}\} \in \mathbf{P} \times_{\mathbf{G}} \mathbf{F}$ of pairs to the equivalence classes containing those pairs, gives rise to the diffeomorphisms

$$\sigma_x^{\boldsymbol{u}}: \Psi \mapsto \Psi \in \mathbf{F}, \qquad \boldsymbol{u} \in \Pi^{-1}(x), \qquad \Psi \in \mathbf{F}_x = \pi^{-1}(x), \tag{1.3}$$

between the fibres $\pi^{-1}(x)$ of that associated bundle and its standard fibre $\mathbf{F}$.

In the case of the $\mathbf{G}$-product in (1.1), as well as in the construction of other quantum bundles, we shall refer to the maps in (1.3) as *soldering maps*, since they can be conceived as giving rise to the "soldering" of quantum frames, or of other types of quantum objects, to the frames in a classical frame bundle. In fact, the term "soldered bundle" was often used by Cartan in his pioneering work on fibre bundles, but it eventually became displaced by the less picturesque term of associated bundle. However, in the case of a tangent bundle $T\mathbf{M}$ associated to the principal bundle $GL\mathbf{M}$, the soldering map in (1.3) assumes the form

$$\sigma_x^{\boldsymbol{u}}: \boldsymbol{X} \mapsto (\theta^0(\boldsymbol{X}), \theta^1(\boldsymbol{X}), \theta^2(\boldsymbol{X}), \theta^3(\boldsymbol{X})) \in \mathbf{R}^4, \qquad \boldsymbol{X} \in T_x\mathbf{M}, \quad x = \Pi\boldsymbol{u}, \tag{1.4}$$

[1] Cf. Footnote 14 in Chapter 2. Further mathematical details on G-bundle products can be found in (Kobayashi and Nomizu, 1963), p. 54. A presentation aimed primarily at physicists can be found in Sec. 3.3 of (Isham, 1989).

which gives rise, on the spaces tangent to the linear frame bundle $GL\mathbf{M}$, to the following $\mathbf{R}^4$-valued canonical form (cf. Kobayashi and Nomizu, 1963, p. 118),

$$\boldsymbol{\theta} : \mathbf{X} \mapsto \boldsymbol{\sigma}_x^{\mu}(\Pi_* \mathbf{X}) \in \mathbf{R}^4 , \qquad \mathbf{X} \in T_{\boldsymbol{u}} GL\mathbf{M} , \quad \boldsymbol{u} = \{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\} \in GL\mathbf{M} , \tag{1.5}$$

for which the term "soldering form" is still being used as a synonym.[2]

Clearly, the standard quantum frame bundle $Q\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ specified by the $\mathbf{G}$-product in (1.1) is itself a principal bundle, and it is obviously equivalent to the Poincaré frame bundle $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$. Consequently, it can be used in the definition of the $\mathbf{G}$-product

$$\mathbf{E} = Q\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}}) \times_{\mathbf{G}} \mathbf{F} , \qquad \mathbf{G} = \mathrm{ISO}_0(3,1) , \qquad \mathbf{F} = \mathbf{P}_{\eta^{(\ell)}} L^2(\Sigma) , \tag{1.6}$$

constituting a *Klein-Gordon quantum bundle*, within which the propagation of states of single quantum particles of spin zero and rest mass $m = 1$ can take place. The soldering maps in (1.3) are then expressed by the mappings

$$\boldsymbol{\sigma}_x^{\boldsymbol{u}} : \boldsymbol{\Psi} \mapsto \boldsymbol{\Psi} \in \mathbf{F} , \qquad \boldsymbol{u} = (\boldsymbol{a}, \boldsymbol{e}_i) \in P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}}) , \qquad \boldsymbol{\Psi} \in \mathbf{F}_x \subset \mathbf{E} , \tag{1.7}$$

which, according to (3.7.9), (3.7.18) and (1.2), assign the following *coordinate wave function* to the local state vectors $\boldsymbol{\Psi} \in \mathbf{F}_x$,

$$\Psi_x^{\boldsymbol{u}}(\zeta) = \left\langle U(q, \Lambda_v) \Phi_\ell^{\boldsymbol{u}} \middle| \boldsymbol{\sigma}_x^{\boldsymbol{u}}(\boldsymbol{\Psi}) \right\rangle , \qquad \zeta := q - i\ell v , \quad (q, v) \in \mathbf{R}^4 \times \boldsymbol{V}^+ \subset \mathbf{R}^8 . \tag{1.8}$$

In these coordinate wave functions the 4-tuple ζ of the complex variables, whose real part is represented by the components of q and the imaginary part by the components of the 4-velocity variable v, appears naturally in view of the fact that, by (3.7.11) and (3.8.4), we can express the inner product in (1.8) as follows:

$$\Psi_x^{\boldsymbol{u}}(\zeta) = \tilde{Z}_\ell^{-1/2} \int_{\boldsymbol{V}^+} \exp(-i\zeta \cdot u)\, \tilde{\Psi}_x^{\boldsymbol{u}}(u)\, d\Omega(u) , \quad \tilde{\Psi}_x^{\boldsymbol{u}} = W_{\eta^{(\ell)}}^{-1} \Psi_x^{\boldsymbol{u}} \in L^2(\boldsymbol{V}^+, d\Omega), \tag{1.9a}$$

$$d\Omega(u) = \delta(u^2 - 1)\, d^4u , \qquad \boldsymbol{V}^+ = \left\{ u \,\middle|\, u^2 := \eta_{\mu\nu} u^\mu u^\nu = 1 , \; u^0 > 0 \right\} , \tag{1.9b}$$

$$\tilde{Z}_\ell = 4\pi^3 \int_{\mathbf{R}^3} \exp(-2\ell\sqrt{1 + \mathbf{v}^2})\, d\mathbf{v} = 8\pi^4 \mathrm{K}_2(2\ell)/\ell . \tag{1.9c}$$

For each choice of section $\boldsymbol{s}$ of the Poincaré frame bundle $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, the soldering mappings in (1.7) provide the following local trivialization maps (cf. (2.4.1)):

$$\boldsymbol{\phi}^{\boldsymbol{s}} : \boldsymbol{\Psi} \mapsto (x, \boldsymbol{\sigma}_x^{\boldsymbol{u}} \boldsymbol{\Psi}) \in \mathbf{M}^{\boldsymbol{s}} \times \mathbf{F} , \qquad \boldsymbol{\Psi} \in \mathbf{F}_x \subset \boldsymbol{\pi}^{-1}(\mathbf{M}^{\boldsymbol{s}}) , \qquad \boldsymbol{u} = \boldsymbol{s}(x) , \tag{1.10}$$

On account of (3.7.7), with the transition from one local Poincaré frame $\boldsymbol{u}$ to another such frame $\boldsymbol{u}'$ in the same fibre, the wave function amplitudes in (1.8) transform as follows:

$$U(b, \Lambda) : \Psi_x^{\boldsymbol{u}}(\zeta) \mapsto \Psi_x^{\boldsymbol{u}'}(\zeta) = \Psi_x^{\boldsymbol{u}}(\Lambda^{-1}(q - b) - i\ell\Lambda^{-1}v) , \qquad \boldsymbol{u}' = \boldsymbol{u} \cdot (b, \Lambda) . \tag{1.11}$$

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[2] Cf. (Choquet-Bruhat *et al.*, 1987), p. 376. The concept of "soldered bundle" has been strongly advocated in physical applications by Drechsler (1975-1992), who uses, however, the term "soldering form" to describe the canonical form $\boldsymbol{\theta}$ plus the covariant derivative of the affine vector field $\boldsymbol{a}$ that appears in any section of the general affine frame bundle $GA\mathbf{M}$, corresponding to the general linear frame bundle $GL\mathbf{M}$ (cf. Sec. 4.3).

This implies that with a change of the Poincaré gauge, corresponding in general to any transition from a local section $\boldsymbol{s}$ of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ to another local section $\boldsymbol{s}'$, and implemented, in accordance with (2.5.20), by the Poincaré transformations

$$(b(x),\Lambda(x)) \;:\; (\boldsymbol{a}(x),\boldsymbol{e}_i(x)) \;\mapsto\; (\boldsymbol{a}'(x),\boldsymbol{e}'_i(x)) \;, \qquad x\in \mathbf{M}^{\boldsymbol{s}}\cap\mathbf{M}^{\boldsymbol{s}'} \;, \tag{1.12a}$$

we shall have the following changes of these wave function amplitudes,

$$\boldsymbol{\sigma}^{\boldsymbol{u}'}_x\circ(\boldsymbol{\sigma}^{\boldsymbol{u}}_x)^{-1} \;:\; \Psi^{\boldsymbol{u}}_x \;\mapsto\; \Psi^{\boldsymbol{u}'}_x = U(b(x),\Lambda(x))\Psi^{\boldsymbol{u}}_x \;, \qquad x\in \mathbf{M}^{\boldsymbol{s}}\cap\mathbf{M}^{\boldsymbol{s}'} \;, \tag{1.12b}$$

for the local state vectors Ψ in the corresponding Klein-Gordon fibres $\mathbf{F}_x$. Hence, if the base manifold $\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ coincides with the Minkowski space $(\mathbf{R}^4,\boldsymbol{\eta})$, then a global Lorentz frame $\mathcal{L}=\{\boldsymbol{e}_i(O)|\; i=0,1,2,3\}$, with its origin at $O\in\mathbf{M}\cong\mathbf{R}^4$, can be identified as in (2.3.9) with a global Poincaré moving frame given by the following cross-section of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$:

$$\boldsymbol{s}_0(\mathcal{L}) = \left\{(\boldsymbol{a}(x),\boldsymbol{e}_i(x))\middle|\; \boldsymbol{a}(x) = -x^i\boldsymbol{e}_i(x)\in T_x\mathbf{M},\;\; x=x^i\boldsymbol{e}_i(O)\in\mathbf{R}^4\right\}\subset P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}}). \tag{1.13}$$

Correspondingly, a local state vector Ψ can be then identified with a special-relativistic state vector which is represented in relation to $\mathcal{L}$ by a wave function in the Hilbert space $\mathbf{F}$ in (1.6), whose values equal those of the wave function amplitudes of (1.8) in the Poincaré gauge given by the global section in (1.13). In view of the Poincaré gauge invariance displayed in (1.11)-(1.12), this identification is invariant under changes of global Lorentz frames $\mathcal{L}$, since all such changes induce corresponding changes of the global Poincaré gauges of the type (1.13). Hence, the special relativistic framework of Sec. 3.8 is recovered as a special case of the present Poincaré gauge invariant framework.

4.2. The Internal Hilbert Structure of Quantum Bundles

In the transition from the special relativistic to the general relativistic regime, the Minkowski space $(\mathbf{R}^4,\boldsymbol{\eta})$ assumes the role of typical fibre of the tangent bundle $T\mathbf{M}$ associated with the Lorentz frame bundle $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$. As a consequence, the fibres $T_x\mathbf{M}$ of the bundle $T\mathbf{M}$ inherit the metric structure of $(\mathbf{R}^4,\boldsymbol{\eta})$. This mediates, with the help of the strong equivalence principle, the transference of key concepts and results from the special relativistic to the general relativistic regime. In particular, this inherited metric structure enables the definition in the tangent spaces $T_x\mathbf{M}$ of timelike, spacelike and null vectors, of 4-velocity hyperboloids, etc., that are essential to the physical interpretation of all CGR models.

In a totally analogous manner, after the transition is made from the quantum mechanics on special-relativistic stochastic phase spaces, described in Secs. 3.7 and 3.8, to the present general relativistic semi-classical quantum geometries, the Hilbert space $\mathbf{F}$ in (1.6), which played the role of state-vector space in the special relativistic context, assumes the role of typical fibre of the quantum bundle $\mathbf{E}\equiv(\mathbf{E},\boldsymbol{\pi},\mathbf{M},\mathbf{F})$. Consequently, the fibres $\mathbf{F}_x$ of the quantum bundle $\mathbf{E}$ will inherit the metric structure of $\mathbf{F}$, which is basically that of a Hilbert space. The more detailed pinpointing of that structure, which emerges from the fact that $\mathbf{F}$ is in addition a reproducing kernel Hilbert space of a very specific type, will enable us to transfer physical ideas and results, formulated in the preceding chapter in the special

relativistic context, to the present general relativistic and semi-classical regime.

In this regime, the local state vectors Ψ in each one of the fibres $\mathbf{F}_x$ can be regarded, in accordance with (1.8), as functions of the four complex components of the variable

$$\zeta = a + (q^k - iv^k)e_k \in T_x\mathbf{M}^{\mathbf{C}}, \quad (a + q^i e_i, v^i e_i) \in T_x\mathbf{M} \times V_x^+ \subset T_x\mathbf{M} \times T_x\mathbf{M} , \quad (2.1)$$

which assumes values within the complexified tangent space $T_x\mathbf{M}^{\mathbf{C}}$, obtained by extending the real linear space $T_x\mathbf{M}$ into the complex linear space $T_x\mathbf{M}^{\mathbf{C}}$ in the same manner in which $\mathbf{R}^4$ can be extended into $\mathbf{C}^4$ (cf. Nakahara, 1990, p. 274). These fibres are Hilbert spaces which carry the inner products

$$\langle \Psi_1 | \Psi_2 \rangle = \int \Psi_1^*(\zeta)\Psi_2(\zeta)\, d\Sigma(\zeta) , \qquad \Psi_1, \Psi_2 \in \mathbf{F}_x , \quad (2.2)$$

in which, in accordance with the general conclusions reached in Sec. 3.7, the integration can be performed with respect to the measure

$$d\Sigma(\zeta) = 2v_k\, \delta(v^2 - 1)\, d\sigma^k(q) d^4v = 2v_k\, d\sigma^k(q) d\Omega(v) , \quad (2.3)$$

along any of the reference hypersurfaces $\Sigma = \sigma \times V^+$ in $T_x\mathbf{M}^{\mathbf{C}}$ that correspond to some maximal spacelike hypersurface σ in $T_x\mathbf{M}$. In view of (3.7.19), this inner product can be written in the alternative form

$$\langle \Psi_1 | \Psi_2 \rangle = i\hat{Z}_\ell \int \Psi_1^*(\zeta) \overleftrightarrow{\partial}_k \Psi_2(\zeta)\, d\sigma^k(q) d\Omega(v) , \qquad \partial_k := \partial/\partial q^k . \quad (2.4)$$

The value of the normalization constant in it, which ensures the equality of the right-hand sides of the relations in (2.2) and (2.4), can be deduced from (3.7.19b), (3.8.4) and (1.9c), upon making the appropriate adjustments to the use of 4-velocity variables $v \in V^+$. That value can be computed to be explicitly given by (cf. Prugovečki, 1984, Sec. 2.6)

$$\hat{Z}_\ell = (2\pi)^{-3}\, \tilde{Z}_\ell \Big(\int_{V^+} \exp(-2\ell v^0) d\Omega(v) \Big)^{-1} = \mathrm{K}_2(2\ell)/\mathrm{K}_1(2\ell) , \quad (2.5)$$

in terms of the modified Bessel functions K_1 and K_2. We note, however, that the normalization constants in (1.9c) and in (2.5) diverge in the sharp-point limit $\ell \to +0$:

$$\tilde{Z}_\ell = (4\pi^4/\ell^3) + O(\ell^{-1}) \xrightarrow[\ell \to +0]{} +\infty , \quad (2.6a)$$

$$\hat{Z}_\ell = \ell^{-1} + O(1) \xrightarrow[\ell \to +0]{} +\infty . \quad (2.6b)$$

Each standard quantum frame in $Q\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ is soldered, in accordance with (1.1) and (1.2), to a Poincaré frame $\boldsymbol{u}(x) \in P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, and can be expressed in the form

$$Q_\ell^{\boldsymbol{u}(x)} = \left\{ \Phi_\zeta^{\boldsymbol{u}(x)} \in \mathbf{F}_x \;\middle|\; \zeta = a + (q^k - iv^k)e_k \in T_x\mathbf{M}^{\mathbf{C}},\ q \in \mathbf{R}^4,\ v \in V^+ \right\} . \quad (2.7)$$

The elements of such a frame are generalized coherent states which, according to (3.7.11) and (1.7)-(1.9), can be mathematically depicted as follows:

$$\Phi_\zeta^{\boldsymbol{u}(x)}(\zeta') = \langle \Phi_{\zeta'}^{\boldsymbol{u}(x)} | \Phi_\zeta^{\boldsymbol{u}(x)} \rangle = \tilde{Z}_\ell^{-1} \int_{V^+} \exp[i(\bar{\zeta} - \zeta') \cdot u]\, d\Omega(u) , \quad (2.8a)$$

$$\zeta = a + \zeta^k e_k\,,\ \zeta' = a + \zeta'^k e_k \in T_x\mathbf{M}^{\mathbf{C}}, \quad \overline{\zeta}^k = q^k + i\ell v^k,\ \zeta'^k = q'^k - i\ell v'^k\ . \tag{2.8b}$$

The integration in (2.8a) can be performed using basic properties of Bessel functions (cf. Prugovečki, 1984, Sec. 2.9), with the result that all the quantum frame elements can be explicitly displayed as Poincaré gauge invariant functions of complex 4-vector variables:

$$\Phi_\zeta^{u(x)}(\zeta') = 2\pi\tilde{Z}_\ell^{-1}\mathrm{K}_1\Big(\sqrt{-\zeta^\lozenge\cdot\zeta^\lozenge}\Big)\Big/\sqrt{-\zeta^\lozenge\cdot\zeta^\lozenge}\ , \tag{2.9a}$$

$$\zeta^\lozenge\cdot\zeta^\lozenge = \eta_{ik}\zeta^{\lozenge i}\zeta^{\lozenge k}\,, \qquad \zeta^{\lozenge k} = (q^k - q'^k) + i\ell(v^k + v'^k)\ . \tag{2.9b}$$

In view of (3.7.10), each quantum frame in (2.8) supplies a continuous resolution of the identity in the fibre $\mathbf{F}_x$ to which its elements belong:

$$\int\Big|\Phi_\zeta^{u(x)}\Big\rangle\, d\Sigma(\zeta)\,\Big\langle\Phi_\zeta^{u(x)}\Big| = \mathbf{1}_x\ . \tag{2.10}$$

Consequently, any vector Ψ that belongs to $\mathbf{F}_x$ can be expanded as follows:

$$\Psi = \Psi^\zeta\Phi_\zeta^{u(x)} := \int d\Sigma(\zeta)\Big\langle\Phi_\zeta^{u(x)}\Big|\Psi\Big\rangle\Phi_\zeta^{u(x)}\,, \qquad \Psi^\zeta := \Big\langle\Phi_\zeta^{u(x)}\Big|\Psi\Big\rangle\,. \tag{2.11}$$

where the values assumed by the above inner product coincide with those assumed by the coordinate wave function in (1.8). Hence, in complete analogy with the role played by the local frames $u(x) \in L\mathbf{M}_0(g^{\mathrm{L}})$ in CGR, the coordinate wave functions of a local quantum state vector Ψ can be always obtained by expanding that vector in a local quantum frame given by (2.7)-(2.9). On account of (2.9), the outcomes of these expansions are coordinate wave functions that are analytic in the complex-valued gauge variable ζ. Thus, it is only if one takes the formal sharp-point limit $\ell \to +0$ that the infinities which plague conventional quantum field theory will appear when the present framework is extended (cf. Chapters 5 and 6) into one whose quantum bundles can accommodate quantum fields.

The standard quantum frames in (2.7) have been formulated in terms of the generalized coherent states in (2.9) which correspond to spin zero and rest mass $m = 1$ in Planck units. However, the extension to arbitrary values for rest mass and spin is quite straightforward, and will be briefly outlined next.

The quantum frames for a generic rest mass value m and zero spin differ from the standard ones in (2.7)-(2.9) only by a change of normalization constant, which corresponds to a change to gauge variables that incorporate that mass value:

$$\Phi_{m;\zeta}^{u(x)}(\zeta') = (m^2\tilde{Z}_\ell/\tilde{Z}_{\ell,m})\,\Phi_{m\zeta}^{u(x)}(m\zeta')\,, \qquad \tilde{Z}_{\ell,m} = 8\pi^4\,\mathrm{K}_2(2\ell m)/\ell m^2\ . \tag{2.12}$$

The value of the renormalization constant in (2.12) follows from (3.7.11) and (3.8.4), after the 4-momentum variable $p = mv$ used in Sec. 3.7 is replaced with the 4-velocity variable v, but it can be also easily verified by using the following reproducibility properties

$$\int\Delta_x^{(+)}(\zeta';\zeta)\,\Delta_x^{(+)}(\zeta;\zeta'')\,d\Sigma(\zeta) = -i\Delta_x^{(+)}(\zeta';\zeta'')\ , \tag{2.13a}$$

$$\Delta_x^{(+)}(\zeta';\zeta) := -i\,\Phi_{m;\zeta}^{u(x)}(\zeta')\ , \quad \zeta = q - i\ell v\ , \quad \zeta' = q' - i\ell v'\ , \tag{2.13b}$$

that follow directly from the counterpart of (2.10) for mass m, as in the case of (3.7.17)

and (3.8.10). The notation in (2.13) is chosen by analogy with the familiar notation for 2-point "functions" (namely distributions) for commutators of creation and annihilation field operators in conventional quantum field theory (cf., e.g., Schweber, 1961, p. 168), since according to (2.8) and (2.12) we have

$$[\tilde{Z}_{\ell,m}/(2\pi)^3]\,\Delta_x^{(+)}(\zeta';\zeta) \xrightarrow[\ell\to+0]{} \Delta_x^{(+)}(q'-q)\,, \tag{2.14a}$$

$$\Delta_x^{(+)}(q'-q) := -(2\pi)^{-3} i \int_{V_m^+} \exp[-i(q'-q)\cdot k]\; d\Omega_m(k)\,. \tag{2.14b}$$

Furthermore, in the next chapter we shall see the *bona fide* functions in (2.13b) play a similar role of 2-point functions for quantum-geometric fields.

On the other hand, in view of the general interpretation of (3.7.14), and of the manner in which quantum frames are operationally defined, one can assign to the functions in (2.13b) the interpretation of *local quantum metric fluctuation amplitudes* (cf. (5.14)) that affect the Riemann normal coordinates generated, in a suitably small neighborhood $\mathcal{N}_x$ of $x \in \mathbf{M}$, by the exponential maps (cf. Straumann, 1984, Part I, §5.3)

$$\exp_x:\ \boldsymbol{q} = q^\mu \boldsymbol{e}_\mu(x) \ \mapsto\ y \in \mathcal{N}_x\,, \qquad \boldsymbol{q} \in T_x\mathbf{M}\,, \tag{2.15}$$

related to the orthonormal frame $\{\boldsymbol{e}_i(x) |\, i = 0,1,2,3\} \in L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$. Indeed, we have seen, in the course of discussing in Sec. 2.6 the significance of the strong equivalence principle, that an operational meaning can be assigned in CGR to such coordinates: they are inertial coordinates in those sufficiently small neighborhoods $\mathcal{N}_x$ in which second-order derivatives of the metric tensor can be neglected (cf. Misner *et al.*, 1973, §§8.6 and 11.6). Hence, in that neighborhood, and to that order, the probability densities

$$\rho_m^{\boldsymbol{u}(x)}(\boldsymbol{\Psi};\zeta) = \left|\left\langle \boldsymbol{\Phi}_{m;\zeta}^{\boldsymbol{u}(x)} \,\middle|\, \boldsymbol{\Psi}\right\rangle\right|^2\,, \qquad \zeta = q - i\ell v\,, \tag{2.16}$$

supply for any local quantum state vector $\boldsymbol{\Psi} \in \mathbf{F}_x$ the relative probabilities of measurement outcome values q^μ for the normal coordinates of $y \in \mathcal{N}_x$, as well as of values v^i for the stochastic 4-velocity components with respect to the local quantum frame soldered to the Poincaré frame $\{(\mathbf{0},\boldsymbol{e}_i(x)) |\, i = 0,1,2,3\} \in P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$. (Naturally, this latter frame is identifiable with the Lorentz frame $\{\boldsymbol{e}_i(x) |\, i = 0,1,2,3\}$, and, as explained in Sec. 2.7, the latter has to belong to an inertial Lorentz moving frame for some timelike geodesic γ in order to represent free-fall conditions.) In particular, if we choose $\boldsymbol{\Psi}$ equal to the state vectors of a quantum frame element in (2.13b), then we arrive at the conclusion that

$$\rho_m^{\boldsymbol{u}(x)}(\boldsymbol{\Phi}_{m;\hat{\zeta}}^{\boldsymbol{u}(x)};\zeta) = \left|\Delta_x^{(+)}(\zeta;\hat{\zeta})\right|^2\,, \qquad \hat{\zeta} = -i\ell \boldsymbol{e}_0\,, \tag{2.17}$$

supplies locally the *relative* probability densities for stochastic fluctuations in the measurement verifications of the spacetime location of the origin of that quantum frame.[3]

Expressions for the generalized coherent states representing quantum frame elements for arbitrary chosen integer-valued spin j can be deduced from (3.7.11), and corresponding

[3] By an adaptation (Prugovečki, 1982a) to the quantum-geometric framework of Wigner's (1972) measurement-theoretical treatment of the time-energy uncertainty relation, it can be shown that these fluctuations manifest themselves equally well in timelike as in spacelike directions – as is to be expected in the relativistic regime.

expression for half-integer values of spins can be obtained by means of Wigner-type representations of the covering group $\mathrm{ISL}(2,\mathbf{C})$ of $\mathrm{ISO}_0(3,1)$. However, for the case of spin-1/2, which is of central importance to quantum electrodynamics (cf. Chapters 6 and 7), it is more convenient to use Dirac-type bispinor representations.

The phase space theory of these representations (Prugovečki, 1980) leads to quantum bundles whose typical fibres $\mathbf{F}$ decompose into positive and negative energy subspaces $\mathbf{F}^{(\pm)}$ of bispinor-valued wave functions that satisfy the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\Psi^{(\pm)}(\zeta) = 0 \ , \qquad \Psi^{(\pm)} \in \mathbf{F}^{(\pm)} \ , \qquad \partial_\mu := \partial/\partial q^\mu \ , \tag{2.18}$$

and in which the respective inner products are given by

$$\left\langle \Psi_1^{(\pm)} \middle| \Psi_2^{(\pm)} \right\rangle = \pm \int \overline{\Psi}_1^{(\pm)}(\zeta)\Psi_2^{(\pm)}(\zeta)\, d\Sigma(\zeta) \ , \qquad \overline{\Psi}_1^{(\pm)} := \Psi_1^{(\pm)*}\gamma^0 . \tag{2.19}$$

A corresponding form of these inner products holds in the respective subfibres of the resulting Dirac quantum bundle, but in each of its fibres those inner products can be written in the alternative form,[4]

$$\langle \Psi_1 | \Psi_2 \rangle = \int \overline{\Psi}_1(\zeta)\gamma^\mu \Psi_2(\zeta)\, d\sigma_\mu(q) d\hat{\Omega}(v) \ , \qquad d\hat{\Omega}(v) := 2\hat{Z}_{\ell m}\, d\Omega(v) \ , \tag{2.20}$$

which is common to both subspaces $\mathbf{F}^{(\pm)}$, but incorporates the normalization constant in (2.5) into the renormalized measure on the forward 4-velocity hyperboloid.

The elements of the corresponding Dirac quantum frames are given, for each value of the bispinorial index $s = 1,2,3,4$, by one-column matrix-valued functions with components

$$\boldsymbol{\Phi}^{\mathbf{u}(x)}_{m;\pm\zeta,s}(\pm\zeta',s') = \tfrac{1}{2}\left[\left(1-(i/m)\gamma^k \partial/\partial q^k\right)\gamma^0\right]_{s's} \boldsymbol{\Phi}^{\mathbf{u}(x)}_{m;\pm\zeta}(\pm\zeta') \ , \qquad \pm v, \pm v' \in \boldsymbol{V}^{\pm} \ . \tag{2.21}$$

Strictly speaking, these local Dirac quantum frames are not soldered directly to the elements $\boldsymbol{u}(x)$ of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, but rather to the corresponding elements $\mathbf{u}(x)$ of the affine spin-frame bundle $S\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, whose spin-frames (cf., e.g., Wald, 1984, Sec. 13.2) are represented by spinors associated in a standard manner to the Lorentz frames in each $\boldsymbol{u}(x)$ (cf. Sec. 6.1). According to a theorem of Geroch (1968), the existence of $S\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ implies that $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ is trivial. Consequently, from now on we shall assume that $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ as well as $S\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ possess cross-sections representing *global* moving frames.

In view of (2.13b), the Dirac quantum frame elements in (2.21) give rise locally to spinorial quantum fluctuation amplitudes represented by the 4×4 matrix-valued functions

$$S_x^{(\pm)}(\zeta;\zeta') := -\left(\tfrac{1}{2} + \tfrac{i}{2m}\gamma^k \partial/\partial q^k\right)\Delta_x^{(\pm)}(\zeta;\zeta') \ , \qquad \Delta_x^{(-)}(\zeta;\zeta') := -\Delta_x^{(+)}(\zeta';\zeta) \ . \tag{2.22}$$

The coordinate wave functions of the projections onto the positive and negative energy subfibres of a Dirac fibre $\mathbf{F}_x$ can be expressed as follows:

$$\Psi_x^{(\pm)}(\zeta) = \mp i \int S_x^{(\pm)}(\zeta;\zeta')\Psi_x(\zeta')\, d\Sigma(\zeta') \ , \qquad \Psi_x \in \mathbf{F} . \tag{2.23}$$

Taken in conjunction with (2.19), this leads to the following reproducibility properties:

[4] All the results on Dirac quantum bundles and frame bundles described in the remainder of this section are proved in (Prugovečki, 1980), and are discussed in greater detail in Chapter 6 of (Prugovečki, 1992).

$$S_x^{(\pm)}(\zeta';\zeta'') = \mp i\int S_x^{(\pm)}(\zeta';\zeta)\,S_x^{(\pm)}(\zeta;\zeta'')\,d\Sigma(\zeta) \tag{2.24}$$

Moreover, on account of (2.13) and (2.14) we also have

$$[2m\tilde{Z}_{\ell,m}/(2\pi)^3]\,S_x^{(\pm)}(\zeta;\zeta') \xrightarrow[\ell\to+0]{} S_x^{(\pm)}(q-q')\ , \tag{2.25a}$$

$$S_x^{(\pm)}(q-q') := -\left(m + i\gamma^k\partial/\partial q^k\right)\Delta_x^{(\pm)}(q-q')\ , \tag{2.25b}$$

where (2.25b) corresponds to the well-known 2-point functions for anti-commutators of Dirac creation and annihilation operators in conventional quantum field theory (cf., e.g., Schweber, 1961, p. 226).

The analyticity properties of the coherent-state wave functions representing the elements of any of the above considered quantum frames ensures the *local* informational completeness of those frames, in the sense that any local quantum state represented by some $\Psi \in \mathbf{F}_x$ can be, at least in principle, uniquely determined by measurements over the "almost flat" neighborhoods $\mathcal{N}_x$ discussed in conjunction with (2.15) and (2.16). For example, according to (1.9a) and (2.11), we can write

$$\left\langle \Psi \middle| \Phi_{\zeta'}^{u(x)} \right\rangle \left\langle \Phi_{\zeta}^{u(x)} \middle| \Psi \right\rangle = \tilde{Z}_\ell^{-2}\int \exp\left[i(\zeta\cdot u - \overline{\zeta}'\cdot u')\right]\tilde{\Psi}(u)\tilde{\Psi}^*(u')\,d\Omega(u)d\Omega(u')\ . \tag{2.26}$$

This expression is clearly analytic in the complex variables $\zeta = q - iv$ and $\overline{\zeta}' = q' + iv'$ corresponding to values of v and v' from some neighborhood of the 4-velocity hyperboloid $\mathbf{V}^+$ in $\mathbf{R}^4$ supplied with the Minkowski metric η. Consequently, the values of the left-hand side of (2.26) are completely determined by the values it assumes when these two variables are set equal, with $q = q' \in T_x\mathbf{M}$ belonging to *any* given neighborhood of the point of contact between $T_x\mathbf{M}$ and $\mathbf{M}$, and $v = v' \in \mathbf{R}^4$ belonging to *any* given neighborhood of $\mathbf{V}^+$. But these latter values are exactly the values in (2.16) that can be measured, at least in principle, with arbitrary accuracy if the neighborhood of the point of contact between $T_x\mathbf{M}$ and $\mathbf{M}$ is so small that the use of the exponential map at x, which appears in (2.15), can be related to an operationally viable procedure for the given strength of the gravitational field in the corresponding neighborhood $\mathcal{N}_x$ of $x \in \mathbf{M}$.

4.3. Connections on Affine Frame Bundles and Associated Bundles

The fact that in the semi-classical regime the base Lorentzian manifold $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$ is given *a priori* implies that, in the course of the propagation of quantum objects within the quantum bundles formulated in the preceding two sections, the external classical gravitational field represented by the Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ influences the motion of those quantum objects, but it is not in turn influenced by them. Thus, the propagation of those quantum objects (which, in the quantum field theoretical regime of the next chapter, we shall identify as quantum-geometric "excitons" rather than as "particles") displays the same general features as that of classical test particles in CGR, whose presence is also assumed not to affect the metric. In particular, according to the geodesic postulate discussed in Sec. 2.6, such test particles are assumed to follow in free-fall the geodesics of the Levi-Civita connection determined by $\boldsymbol{g}^{\mathrm{L}}$ without, however, influencing the values assumed by that metric field.

To combine this idea with the path integral approach to quantum propagation we need, first of all, to extend the Levi-Civita connection from the Lorentz frame bundle $L\mathbf{M}_0(\boldsymbol{g}^L)$ to the Poincaré frame bundle $P\mathbf{M}_0(\boldsymbol{g}^L)$. Hence we shall first outline the standard mathematical theory dealing with the extension of connections from linear to affine principal frame bundles, and after that we shall discuss the physical relevance of that theory.

According to the general theory of affine connections,[5] to every linear connection form ω on $GL\mathbf{M}$ can be assigned various *generalized affine connection forms* $\omega^{(\varphi)}$ on the general affine frame bundle $GA\mathbf{M}$, which consists of the affine frames $\{(\boldsymbol{a},\boldsymbol{e}_i)|\ i = 0,1,2,3\}$ obtained by translating each local linear frame $\{\boldsymbol{e}_i \mid i = 0,1,2,3\} \in GL\mathbf{M}$ by all the possible amounts described by the 4-vectors $\boldsymbol{a}$ within the tangent space within which that linear frame lies. These affine connection forms are related to the linear connection forms ω by means of the pull-back generated by an injective mapping ι from $GL\mathbf{M}$ into $GA\mathbf{M}$, which maps all the linear frames $\{\boldsymbol{e}_i \mid i = 0,1,2,3\}$ into their affine counterparts $\{(\mathbf{0},\boldsymbol{e}_i)|\ i = 0,1,2,3\}$. The outcome of this pull-back operation determined by ι is given by

$$\iota^*\omega^{(\varphi)} = \omega + \varphi\ , \qquad \iota\ :\ \boldsymbol{u}\ \mapsto\ (\mathbf{0},\boldsymbol{u}) \in GA\mathbf{M}\ , \quad \boldsymbol{u} \in GL\mathbf{M}\ , \tag{3.1}$$

where φ is a tensorial 1-form on $GL\mathbf{M}$. If φ is the canonical form θ defined in (1.5), then the corresponding connection form $\omega^{(\theta)}$ on $GA\mathbf{M}$ is called simply an *affine connection*. A one-to-one correspondence then exists between all the linear connections on $GL\mathbf{M}$, and all the thus defined affine connections on $GA\mathbf{M}$, and various theorems can be derived relating their curvature and torsion forms, the corresponding curvature and torsion tensors, as well as other key geometric properties (cf. Kobayashi and Nomizu, 1963, Chapter III).

In the presence of a Lorentzian metric $\boldsymbol{g}^L$, this mathematical procedure can be used to extend all linear connections which are compatible with that Lorentzian metric. Since such connections can be restricted to the principal bundle $L\mathbf{M}_0(\boldsymbol{g}^L)$, this procedure results in the unique assignment to each linear connection on th Lorentz frame bundle $L\mathbf{M}_0(\boldsymbol{g}^L)$ of an affine connection on the Poincaré frame bundle $P\mathbf{M}_0(\boldsymbol{g}^L)$.

Let us consider now any vector bundle $(\mathbf{E},\ \pi,\ \mathbf{M},\ \mathbf{F})$ which is associated, in the sense described in Sec. 2.4, with the principal bundle $P\mathbf{M}_0(\boldsymbol{g}^L)$ via some given representation $U(g)$: $\mathbf{F} \to \mathbf{F}$, $g \in \mathrm{ISO}_0(3,1)$, of the Poincaré group. In order to compute the covariant derivative of any vector field Ψ which assumes its values Ψ_x within the total space $\mathbf{E}$, we shall proceed as in Sec. 2.5, by first selecting a cross-section of $P\mathbf{M}_0(\boldsymbol{g}^L)$,

$$\boldsymbol{s}\ :\ x\ \mapsto\ (\boldsymbol{a}(x),\boldsymbol{u}(x)) \in \boldsymbol{\Pi}^{-1}(x)\ \subset P\mathbf{M}_0(\boldsymbol{g}^L)\ , \qquad x \in \mathbf{M}\ , \tag{3.2}$$

which provides a global Poincaré moving frame, representing a choice of Poincaré gauge. For any affine connection, with connection form $\omega^{(\theta)}$ given on $P\mathbf{M}_0(\boldsymbol{g}^L)$, we can repeat the steps in (2.5.15)-(2.5.17). We then arrive at the following expression for the Poincaré covariant derivative of the vector field Ψ (cf. (2.6.2), as well as Drechsler, 1984),

$$\tilde{\nabla}_X \Psi_x = [\partial_X + \tilde{\theta}^i_s(X)\tilde{\mathbf{P}}_{i;s} + \tfrac{1}{2}\tilde{\omega}^{sr}_{jk}(X)\tilde{\mathbf{M}}^{jk}_{sr}]\Psi_x\ , \qquad \tilde{\omega}^{sr}_{jk} = \omega^{sr}_{jk}\ , \tag{3.3a}$$

[5] Cf. Sec. III.3 in (Kobayashi and Nomizu, 1963). Note that in this general fibre-theoretical approach a natural and straightforward terminology is used: a linear connection is a connection of a linear frame bundle, and an affine connection is a connection on an affine frame bundle – as opposed to the terminology encountered in physics literature, which refers to the covariant differentiation operator form in (2.5.18) as an "affine connection."

$$\partial_X \Psi_x = \left[X^\mu \partial_\mu \left(\Psi^\rho_{x;s}\right)\right]\tilde{\mathbf{V}}_\rho(s)\ , \qquad \tilde{\boldsymbol{\theta}}^i_s(X) = \boldsymbol{\theta}^i_s(X) + (\nabla_X a)^i\ , \tag{3.3b}$$

in which the infinitesimal generators of spacetime translations and Lorentz transformations, that occur next to the connection 1-forms, are to be computed in relation to the axes of the Lorentz frame $\{\boldsymbol{e}_i \mid i = 0,1,2,3\}$ that constitutes part of the local Poincaré frame $\boldsymbol{s}(x) = \{(\boldsymbol{a},\boldsymbol{e}_i) \mid i = 0,1,2,3\}$ above the considered point $x \in \mathbf{M}$.

In accordance with (1.5) and (3.1), this affine covariant derivative contains the soldering forms introduced by Drechsler (1977b, 1982), that are explicitly displayed in (3.3b). This type of soldering forms incorporates the components of the covariant derivative $\nabla_X \boldsymbol{a}$ for the corresponding linear connection, in addition to the 1-forms represented by the coframes of the Lorentzian components $\{\boldsymbol{e}_i \mid i = 0,1,2,3\}$ within the affine frames $\{(\boldsymbol{a},\boldsymbol{e}_i) \mid i = 0,1,2,3\}$ belonging the chosen moving frame in (3.2). Hence, if that linear connection is the Levi-Civita connection, and if we choose a moving frame whose affine frames are given by $\{(\mathbf{0},\boldsymbol{e}_i) \mid i = 0,1,2,3\}$, so that the i-th components $(\nabla_X \boldsymbol{a})^i$ of the field $\nabla_X \boldsymbol{a}$ in relation to the axes of the linear frames $\{\boldsymbol{e}_i \mid i = 0,1,2,3\}$ vanish identically, then

$$\tilde{\boldsymbol{\theta}}^i_s(X) = \boldsymbol{\theta}^i_s(X)\ , \qquad \tilde{\boldsymbol{\omega}}^s_{jk}(X) = \boldsymbol{\omega}^s_{jk}(X)\ . \tag{3.4}$$

Thus, we see that even for cross-sections of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ identifiable with cross-sections of $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, the covariant derivative in (3.3a) does not coincide with the covariant derivative in (2.6.2) for the Levi-Civita connection. Instead, the resulting covariant derivative for vector bundles associated with $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ in general exhibits non-zero torsion, resulting in a non-vanishing torsion operator in (2.6.3a).

Torsion that is viewed as the antisymmetric part of a connection compatible with a metric on a manifold, was first introduced by Cartan (1923, 1924), who believed that it led to a possible generalization of CGR (Debever, 1979), in which the torsion in spacetime could be coupled to the intrinsic angular momentum of a *classically* conceived depiction of matter in free fall. Many researchers subsequently contributed to the investigation of the ensuing Riemann-Cartan spacetimes, determined by the presence of a Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$, and of a Riemann-Cartan connection compatible with that metric, but generically displaying non-zero torsion (cf. Hehl *et al.*, 1976, Sec. I.D). The formulation of such classical spacetimes by means of the modern differential geometry techniques has been carried out by Trautman (1970, 1972, 1973, 1975). However, in such Riemann-Cartan spacetimes the presence of non-zero torsion leads to a type of free-fall behavior of test particles (Adamowicz and Trautman, 1975) for which there is no experimental evidence in the classical context. Indeed, matter does not display intrinsic angular momentum at the macroscopic level, so that classically "the very notion of a spinning test particle [is] obscure" (Hehl *et al.*, 1976, Sec. V.D3). On the other hand, the possibility remains open that torsion might play a role for the quantum particles with spin, and, in fact, the consideration of Riemann-Cartan spacetimes in the quantum regime has been strongly advocated in a series of papers by Drechsler (1977b-1985, 1988, 1990).

General relativity is predicated, however, on the assumption that the connection in any spacetime manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ is not only compatible with the metric $\boldsymbol{g}^{\mathrm{L}}$, but that in addition its torsion tensor vanishes identically. As discussed in Sec. 2.6, that singles out

the Levi-Civita connection on $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ as the only one acceptable in CGR, so that when the extension to $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ of that linear connection is performed, the only possibility acceptable in CGR is the generalized affine connection with $\boldsymbol{\varphi} \equiv 0$ in (3.1), i.e., with connection form $\boldsymbol{\omega}^{(0)}$. The covariant derivatives to which this generalized affine Levi-Civita connection gives rise are therefore obtained by removing all the $\boldsymbol{\theta}^i$ 1-forms from (3.3a), so that the resulting soldering forms in (3.3b) then contain only $(\nabla_{\boldsymbol{X}}\boldsymbol{a})^i$. However, such an extension of the Levi-Civita connection from $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ to $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ does not contribute any physically new insights, or any mathematically interesting features, to any general relativistic models of spacetime, or to CGR in general.

This situation changes radically in the quantum regime, as can be most easily seen in the case of the Lorentz frame bundle $L\mathbf{M}_0(\boldsymbol{\eta})$ defined as in Sec. 2.3, that has the Lorentz group $\mathrm{SO}_0(3,1)$ as its structure group, and the Minkowski space M^4 as its base manifold.

In such a bundle the Levi-Civita connection determines path-independent parallel transports of frames, and their outcomes coincide with those resulting from parallel translation in $\mathbf{R}^4$. Indeed, for the cross-section $\boldsymbol{s}(\mathcal{L})$ of $L\mathbf{M}_0(\boldsymbol{\eta})$ that was depicted in (2.3.9), the values assumed by all the connection coefficients of the Levi-Civita covariant derivative $\nabla_{\boldsymbol{X}}$ in (2.6.2) vanish identically. Hence, the components v^i of such a vector as the 4-velocity $\boldsymbol{v}$ of a classical particle do not change under parallel transport, as they do not change under parallel translation in relation to the global Lorentz frames $\mathcal{L}$, with which $\boldsymbol{s}(\mathcal{L})$ can be identified. This is totally in keeping with the fact that the velocity of a classical particle that is free, and therefore moves by inertia, does not change with the passage of time. In particular, if that classical particle is at rest in relation to the inertial frame that constitutes the spatial part of $\mathcal{L}$, then it remains at rest, and its 3-velocity stays zero at all times.

On the other hand, as is evident from (3.7.7), as well as from (3.6.15), *any* wave function of a *quantum* particle undergoes changes under *any* spacetime translation. These changes reflect the fundamental fact that, according to Heisenberg's uncertainty principle, no quantum particle can be in a *deterministic* state of rest in relation to any kind of frame of reference. In fact, its state vector changes with time even if the particle is free, and if it is *stochastically* at rest in relation to some inertial frame of reference, so that the distribution of its 3-velocity probabilities is time-independent, and has a zero mean value with respect to that inertial frame. Actually, as is well known already from the nonrelativistic regime, *any* configuration space wave function of a free quantum particle evanesces with the passage of time from *any* given bounded region in space. Hence, all these fundamental facts have to be taken into account by the formulation of parallel transport in the quantum regime.

In the case of all the gauges represented by cross-sections $\boldsymbol{s}(\mathcal{L})$ of $L\mathbf{M}_0(\boldsymbol{\eta})$, this can be achieved by noting that, according to (2.3.9), (1.1) and (1.6), the local wave functions

$$\varphi_{x;\mathcal{L}}(\zeta) = \varphi(q - x, v) \ , \qquad \zeta = (q^k - iv^k)\boldsymbol{e}_k(x) \ , \quad x = x^k \boldsymbol{e}_k(O) \ , \tag{3.5}$$

provide the elements of a cross-section $\boldsymbol{\Psi}$ of the Klein-Gordon quantum bundle $\mathbf{E}$ associated to the quantum frame bundle $Q\mathbf{M}_0(\boldsymbol{\eta})$. In view of (3.7.7) and (1.13), the global quantum field on $M^4 = (\mathbf{R}^4, \boldsymbol{\eta})$ defined, in accordance with (2.11), by the map

$$\boldsymbol{\varphi} : \ x \mapsto \boldsymbol{\varphi}_x = \varphi^{\zeta}_{x;\mathcal{L}} \boldsymbol{\Phi}_{\zeta}^{\boldsymbol{s}(\mathcal{L};x)} \ , \qquad \varphi^{\zeta}_{x;\mathcal{L}} := \varphi_{x;\mathcal{L}}(\zeta) = \left\langle \boldsymbol{\Phi}_{\zeta}^{\boldsymbol{s}(\mathcal{L};x)} \middle| \boldsymbol{\varphi}_x \right\rangle \ , \tag{3.6}$$

corresponds to a special relativistic state vector $\varphi \in \mathbf{P}_{\eta}\omega L^2(\Sigma)$ of the kind defined in Sec. 2.8, and represents the same state as φ, but described from a general relativistic point of view, which admits only local Lorentz frames of reference.

The infinitesimal generators of spacetime translations for the unitary representation in (3.7.7) represent 4-momentum operators, which we shall denote by P_μ. It is then easily seen that, in any of the moving Lorentz frames represented by some $\boldsymbol{s}(\mathcal{L})$, the coordinate wave functions of the quantum field in (3.6) satisfies the following equations at all $x \in \mathbf{R}^4$,

$$\partial\varphi_{x;\mathcal{L}}(\zeta)/\partial x^\mu = -\partial\varphi_{x;\mathcal{L}}(\zeta)/\partial q^\mu = iP_\mu\varphi_{x;\mathcal{L}}(\zeta) \ , \qquad U(a,I) = \exp(iP_\mu a^\mu) \ . \tag{3.7}$$

The only generalized affine connection that corresponds, in accordance with (3.1), to the Levi-Civita connection on $P\mathbf{M}_0(\boldsymbol{\eta})$, and for which the field in (3.6) satisfies the equation $\nabla\boldsymbol{\varphi} \equiv 0$, so that the values $\boldsymbol{\varphi}_x$ that it assumes at various base locations $x \in M^4$ are the parallel transports of each other, is the affine connection with connection form $\boldsymbol{\omega}^{(\boldsymbol{\theta})}$. Indeed, this connection gives rise to the covariant derivatives in (3.3), so that (3.6) then becomes equivalent to

$$\tilde{\nabla}_{\boldsymbol{X}}\boldsymbol{\varphi}_x = [\partial_{\boldsymbol{X}} + \boldsymbol{\theta}^\mu_{\boldsymbol{s}}(\boldsymbol{X})\tilde{\mathbf{P}}_{\mu;\boldsymbol{s}}]\boldsymbol{\varphi}_x = \mathbf{0} \ , \qquad \boldsymbol{\theta}^\mu_{\boldsymbol{s}}(\boldsymbol{X}) = X^\mu \ , \qquad \tilde{\mathbf{P}}_{\mu;\boldsymbol{s}} \equiv iP_\mu \ , \tag{3.8}$$

in any of the Poincaré gauges that give rise to some cross-section $\boldsymbol{s}(\mathcal{L})$ of $L\mathbf{M}_0(\boldsymbol{\eta})$ of the form (2.3.9). Naturally, since the quantum geometric framework provided in this chapter is Poincaré gauge invariant, these equations remain true in their generic form,

$$[\partial_{\boldsymbol{X}} + \tilde{\boldsymbol{\theta}}^i_{\boldsymbol{s}}(\boldsymbol{X})\tilde{\mathbf{P}}_{i;\boldsymbol{s}} + \tfrac{1}{2}\tilde{\boldsymbol{\omega}}^{\boldsymbol{s}}_{jk}(\boldsymbol{X})\tilde{\mathbf{M}}^{jk}_{\boldsymbol{s}}]\boldsymbol{\varphi}_x = \mathbf{0} \ , \qquad \partial_{\boldsymbol{X}}\boldsymbol{\varphi}_x = \left[X^\mu\partial_\mu\left(\varphi^\zeta_{x;\boldsymbol{s}}\right)\right]\boldsymbol{\Phi}^{\boldsymbol{s}(x)}_\zeta \ , \tag{3.9a}$$

when the transition is made, in accordance with (1.11) and (1.12), to any global Poincaré gauge, represented by an arbitrary cross-sections $\boldsymbol{s}$ of $P\mathbf{M}_0(\boldsymbol{\eta})$, so that

$$\varphi^{\zeta'}_{x;\boldsymbol{s}} = U(b(x),\Lambda(x))\varphi^\zeta_{x;\mathcal{L}} \ , \qquad \boldsymbol{s}(x) = \boldsymbol{s}(\mathcal{L};x)\cdot(b(x),\Lambda(x)) \ , \qquad x \in \mathbf{M} \tag{3.9b}$$

In view of the fact that the state vectors $\varphi \in \mathbf{P}_{\eta}\omega L^2(\Sigma)$ correspond to zero spin, we see that, in the quantum regime, the presence within the covariant derivatives in (3.8) and (3.9a) of the soldering forms in (3.3b) does not reflect the physical presence of any kind of quantum torsion coupled to the intrinsic spin of the quantum entity that undergoes parallel transport, but merely reflects a very basic feature of quantum propagation, which is a direct corollary of the uncertainty principle: no quantum object can exist in a state of deterministic rest in relation to any physical frame of reference.

The application of the strong equivalence principle to quantum bundles over generic Lorentzian manifolds $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ therefore requires the adoption in those bundles of the *affine Levi-Civita connection* with connection form $\boldsymbol{\omega}^{(\boldsymbol{\theta})}$ on $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$. Indeed, the propagation of quantum state vectors governed over infinitesimal spacetime distances by that connection will then coincide with their propagation in the above considered special relativistic regime.

The next step on the road leading to the formulation of a quantum-geometric mode of propagation in the various quantum bundles described in Secs. 4.1 and 4.2 requires the extension of this physical idea to propagation over arbitrary spacetime distances.

4.4. Connections and Parallel Transport in Quantum Bundles

The extrapolation of the strong equivalence principle to quantum bundles over classical spacetime manifolds $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ singles out the affine Levi-Civita connection form $\boldsymbol{\omega}^{(\theta)}$ on $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ as the one governing parallel transport in a quantum frame bundle $Q\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ along any piecewise smooth curve γ, with initial-point $x'\in\mathbf{M}$ and end-point $x''\in\mathbf{M}$. The horizontal lifts (cf. Sec. 2.5) of γ from the base space $\mathbf{M}$ to the total space of the Poincaré frame bundle $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ gives rise to operators for parallel transport,

$$\tilde{\boldsymbol{\tau}}_\gamma(x'',x') \;:\quad \boldsymbol{u}(x')=(\boldsymbol{a}',\boldsymbol{e}_i') \;\mapsto\; \boldsymbol{u}(x'')=(\boldsymbol{a}'',\boldsymbol{e}_i'')\;, \tag{4.1}$$

representing diffeomorphisms between the fibres of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ above the initial and final point of γ. Since the quantum frames of $Q\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ are soldered to the Poincaré frames in $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, this gives rise to corresponding operators for parallel transport,

$$\boldsymbol{\tau}_\gamma(x'',x') \;:\quad \boldsymbol{\Phi}_\zeta^{\boldsymbol{u}(x')} \;\mapsto\; \boldsymbol{\Phi}_\zeta^{\boldsymbol{u}(x'')}\;, \tag{4.2}$$

between the fibres of $Q\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ above the initial and final points of γ. In turn, this produces the Poincaré gauge invariant operators for parallel transport

$$\boldsymbol{\tau}_\gamma(x'',x') \;:\quad \boldsymbol{\Psi}'=\Psi^\zeta\boldsymbol{\Phi}_\zeta^{\boldsymbol{u}(x')} \;\mapsto\; \boldsymbol{\Psi}''=\Psi^\zeta\boldsymbol{\Phi}_\zeta^{\boldsymbol{u}(x'')}\;, \tag{4.3}$$

between the fibres $\mathbf{F}_{x'}$ and $\mathbf{F}_{x''}$ of any of the quantum bundles $\mathbf{E}$ described in Secs. 4.1 and 4.2, that are associated to $Q\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$.

For any quantum field with domain $\mathcal{D}$ in $\mathbf{M}$, that assumes values in $\mathbf{E}$, and therefore represents a section $\{\boldsymbol{\Psi}_x \mid x\in\mathcal{D}\}$ of $\mathbf{E}$, we can follow the procedure discussed in Sec. 2.5, and define as in (2.5.13) its covariant derivative along $\boldsymbol{X}\in T_x\mathbf{M}$, at a base location $x\in\mathcal{D}$,

$$\nabla_{\boldsymbol{X}}\boldsymbol{\Psi}_x = \mathrm{s}\text{-}\lim_{t\to 0}\frac{1}{t}\big[\boldsymbol{\tau}_\gamma(x,x(t))\boldsymbol{\Psi}_{x(t)}-\boldsymbol{\Psi}_x\big]\;,\quad x(0)=x\in\mathbf{M}\;,\quad \dot{x}^\mu(0)\partial_\mu=\boldsymbol{X}\,, \tag{4.4}$$

in any Poincaré gauge specified by a cross-section $\boldsymbol{s}$ of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$. A technical distinction from the case of the associated bundles with finite-dimensional fibres that was considered in Chapter 2, is that the above strong limit, uniquely determined by the requirement that

$$\lim_{t\to 0}\Big\|t^{-1}\big[\boldsymbol{\tau}_\gamma(x,x(t))\boldsymbol{\Psi}_{x(t)}-\boldsymbol{\Psi}_x\big]-\nabla_{\boldsymbol{X}}\boldsymbol{\Psi}_x\Big\|=0\;, \tag{4.5}$$

does not exist on the entire fibre $\mathbf{F}_x$, but only a dense subspace of the Hilbert space $\mathbf{F}_x$.

To pinpoint that subspace, and to obtain an expression for the covariant derivative in (4.4) that would be analogous to (3.3a), we introduce the Poincaré transformations

$$\Big(b_\gamma^{\boldsymbol{s}}(x,x(t)),\Lambda_\gamma^{\boldsymbol{s}}(x,x(t))\Big) \;:\; \boldsymbol{s}(x) \;\mapsto\; \tilde{\boldsymbol{\tau}}_\gamma(x,x(t))\boldsymbol{s}(x(t))\;, \tag{4.6}$$

which are uniquely determined by the operators for parallel transport in (4.1). We can then take advantage of the soldering maps in (1.3) to write:

$$\begin{aligned}&\boldsymbol{\sigma}_x^{\boldsymbol{s}}\big(\boldsymbol{\tau}_\gamma(x,x(t))\boldsymbol{\Psi}_{x(t)}-\boldsymbol{\Psi}_x\big)\\&\quad=\big(\boldsymbol{\sigma}_{x(t)}^{\boldsymbol{s}}\boldsymbol{\Psi}_{x(t)}-\boldsymbol{\sigma}_x^{\boldsymbol{s}}\boldsymbol{\Psi}_x\big)+\Big[U\Big(b_\gamma^{\boldsymbol{s}}(x,x(t)),\Lambda_\gamma^{\boldsymbol{s}}(x,x(t))\Big)-\mathbf{1}\Big]\boldsymbol{\sigma}_{x(t)}^{\boldsymbol{s}}\boldsymbol{\Psi}_{x(t)}\,.\end{aligned} \tag{4.7}$$

The smoothness of the section $\{\Psi_x \mid x \in \mathcal{D}\}$ of $\mathbf{E}$ then ensures the existence of the limit

$$\partial_X \Psi_x = (\sigma_x^s)^{-1} \mathrm{s}\text{-}\lim_{t\to 0} t^{-1}\left(\sigma^s_{x(t)}\Psi_{x(t)} - \sigma_x^s \Psi_x\right) = [X^\mu \partial_\mu(\Psi_x^\zeta)]\,\Phi_\zeta^{s(x)} \ , \qquad (4.8a)$$

which contributes the first term in the quantum analogue of (3.3a).

To establish under what kind of circumstances the second limit

$$(\sigma_x^s)^{-1}\mathrm{s}\text{-}\lim_{t\to 0} t^{-1}\left[U\left(b^s_\gamma(x,x(t)), \Lambda^s{}_\gamma(x,x(t))\right) - \mathbf{1}\right]\sigma^s_{x(t)}\Psi_{x(t)} \qquad (4.8b)$$

exists, let us consider in $\mathbf{F}_x$ the following unitary representation of the Poincaré group:

$$U_{s(x)}(b,\Lambda) = (\sigma_x^s)^{-1} U(b,\Lambda)\sigma_x^s \ , \qquad (b,\Lambda) \in \mathrm{ISO}_0(3,1) \ . \qquad (4.9)$$

All the infinitesimal generators of this representation can be expressed in terms of ten self-adjoint operators, which in case of quantum bundles $\mathbf{E}$ corresponding to spin zero and rest mass m can be expressed exclusively in terms of the operators

$$P_{j;s(x)} = i\,\partial/\partial q^j \ , \qquad Q^j_{s(x)} = q^j - i\,\partial/\partial p_j \ , \qquad p_j = m\eta_{jk}v^k \ . \qquad (4.10)$$

These eight operators give rise to a representation of the relativistic canonical commutation relations in (3.8.3). The first four are self-adjoint in $\mathbf{F}_x$, and represent the 4-momentum operators (cf. (3.5.14)-(3.5.15)) corresponding to the infinitesimal generators of spacetime translations along the axes of the local Lorentz frame within $\boldsymbol{s}(x) = \{(\boldsymbol{a},\boldsymbol{e}_i) \mid i = 0,1,2,3\}$:

$$P_{j;s(x)} = -i \lim_{\varepsilon\to 0} \varepsilon^{-1}\left[U_{s(x)}\left(a_{(j)}(\varepsilon), I\right) - \mathbf{1}\right] \ , \qquad a^i_{(j)}(\varepsilon) = \varepsilon\,\delta^i_j \qquad (4.11)$$

The remaining four operators in (4.10) involve off-shell differentiation, but when taken in conjunction with the 4-momentum operators, they give rise to the infinitesimal generators

$$M^{jk}_{s(x)} = Q^j_{s(x)}P^k_{s(x)} - Q^k_{s(x)}P^j_{s(x)} \ , \qquad P^j_{s(x)} = \eta^{jk}P_{k;s(x)} \ , \qquad (4.12)$$

of rotations around each of the spatial axes $\boldsymbol{e}_i$, $i = 1,2,3$, parametrized by their rotation angles, and of Lorentz boosts in the direction of those same axes, parametrized by a variable ϑ that imparts to them the formal appearance of Lorentz "rotations." For example, in the case of the Lorentz boosts in the direction of $\boldsymbol{e}_3$, we can set

$$(0,\Lambda^{(3)}_\omega): \quad (\boldsymbol{a},\boldsymbol{e}_0,\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3) \ \mapsto \ (\boldsymbol{a},\boldsymbol{e}'_0(\omega),\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}'_3(\omega)) \ , \qquad (4.13a)$$

$$\boldsymbol{e}'_0(\omega) = \boldsymbol{e}_0\cosh\omega + \boldsymbol{e}_3\sinh\omega \ , \qquad \boldsymbol{e}'_3(\omega) = \boldsymbol{e}_0\sinh\omega + \boldsymbol{e}_3\cosh\omega \ , \qquad (4.13b)$$

and the corresponding infinitesimal generator in (4.12) is then given by:

$$M^{21}_{s(x)} = -M^{12}_{s(x)} = i\lim_{\omega\to 0}\omega^{-1}\left[U_{s(x)}\left(0,\Lambda^{(3)}_\omega\right) - \mathbf{1}\right] . \qquad (4.14)$$

All of the above infinitesimal generators are unbounded self-adjoint operators, whose domains of definition are dense in $\mathbf{F}_x$, but cannot be extended to all of $\mathbf{F}_x$ on account of the Hellinger-Toeplitz theorem, which states that *any* self-adjoint operator that is defined everywhere on a Hilbert space is *necessarily* bounded (cf. Prugovečki, 1981, p. 195). Further-

more, other basic theorems in functional analysis demonstrate that such limits as those in (4.11) and (4.14) exist only on the dense domains of the corresponding unbounded self-adjoint operators (*ibid.*, p. 288). Consequently, in a given Poincaré gauge, specified by some cross-section $\boldsymbol{s}$ of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, the strong limit in (4.8b) does not exist for an arbitrary section of $\mathbf{E}$. Rather, it exists if and only if the elements of the quantum field $\{\boldsymbol{\Psi}_x \mid x \in \mathcal{D}\}$ belong to the domains of the respective self-adjoint operators that are densely defined by the following strong limits (i.e., limits in the norm of $\mathbf{F}_x$):

$$\Big(\tilde{\boldsymbol{\theta}}^i(\boldsymbol{X})P_{i;\boldsymbol{s}(x)} - \tfrac{1}{2}\tilde{\boldsymbol{\omega}}_{jk}(\boldsymbol{X})M^{jk}_{\boldsymbol{s}(x)}\Big)\boldsymbol{\Psi}_x$$
$$= i\,\mathrm{s}\text{-}\lim_{t\to 0} t^{-1}\Big[U_{\boldsymbol{s}(x)}\Big(b^{\boldsymbol{s}}_{\gamma}(x,x(t)),\Lambda^{\boldsymbol{s}}{}_{\gamma}(x,x(t))\Big) - \mathbf{1}_x\Big]\boldsymbol{\Psi}_x \ . \qquad (4.15)$$

The domain of definition of this operator contains the dense linear span of all the generalized coherent states in (2.12) that belong to the quantum frame corresponding to the Poincaré frame $\boldsymbol{u}(x)$ from the adopted cross-section $\boldsymbol{s}$ of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$. On account of (1.8), (1.12) and (4.9), this linear space generated within a quantum fibre $\mathbf{F}_x$ by all the coherent states in (2.12) is actually frame-independent. We shall denote it by $\mathbf{C}_x$, since it provides within $\mathbf{F}_x$ a common core[6] for all the operators in (4.15), and therefore also a core for the covariant differentiation operator defined by (4.4) (cf. Prugovečki and Warlow, 1989b, Theorem A.2). In view of (4.7), (4.8) and (4.15), that operator is seen to assume the following form on that dense core:

$$\nabla_{\boldsymbol{X}}\boldsymbol{\Psi}_x = \Big(\partial_{\boldsymbol{X}} - i\tilde{\boldsymbol{\theta}}^i_{\boldsymbol{s}}(\boldsymbol{X})P_{i;\boldsymbol{s}(x)} + \tfrac{i}{2}\tilde{\boldsymbol{\omega}}^{\boldsymbol{s}}_{jk}(\boldsymbol{X})M^{jk}_{\boldsymbol{s}(x)}\Big)\boldsymbol{\Psi}_x \ , \qquad \boldsymbol{\Psi}_x \in \mathbf{C}_x \ . \qquad (4.16)$$

In order to be able to deal in subsequent chapters with the mathematical fact that the covariant derivatives of quantum fields in a quantum bundle $\mathbf{E}$ are not defined everywhere within their infinite-dimensional fibres $\mathbf{F}_x$, we shall introduce the concept of *coherent section* of $\mathbf{E}$: by definition, a section $\{\boldsymbol{\Psi}_x \mid x \in \mathcal{D}\}$ of $\mathbf{E}$ is coherent if for some choice of Poincaré gauge, provided by a cross-section $\boldsymbol{s}$ of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, and for some fixed value of ζ, we have $\boldsymbol{\Psi}_x = \boldsymbol{\Psi}_{\zeta;x}$ at all $x \in \mathcal{D}$, where $\boldsymbol{\Psi}_{\zeta;x}$ are elements of the quantum frames within the section of $Q\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ corresponding to $\boldsymbol{s}$. Coherent sections lie within the domain of definition of covariant derivatives for *any* choice of Poincaré gauge (cf. Prugovečki and Warlow, 1989b, Theorem A.2), and in a Poincaré gauge given by a section $\boldsymbol{s}$ with respect to which they are coherent, the first term on the right-hand side of (4.16) vanishes, so that:

$$\Big(\nabla_{\boldsymbol{X}} + i\tilde{\boldsymbol{\theta}}^i_{\boldsymbol{s}}(\boldsymbol{X})P_{i;\boldsymbol{s}(x)} - \tfrac{i}{2}\tilde{\boldsymbol{\omega}}^{\boldsymbol{s}}_{jk}(\boldsymbol{X})M^{jk}_{\boldsymbol{s}(x)}\Big)\boldsymbol{\Psi}_{\zeta;x} = \mathbf{0} \ , \qquad \boldsymbol{\Psi}_{\zeta;x} = \boldsymbol{\Phi}^{\boldsymbol{s}(x)}_{m;\zeta} \ . \qquad (4.17)$$

To deal with the domain problem in the case of covariant derivatives of arbitrary sections of a quantum bundle $\mathbf{E}$, we first transfer to Poincaré frame bundles $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ the concept of moving frame adapted to a smooth curve γ in $\mathbf{M}$, defined in Sec. 2.7: a Poincaré moving frame $\boldsymbol{s} = \{(\boldsymbol{a}(x),\boldsymbol{e}_i(x)) \mid x \in \mathbf{M}^{\boldsymbol{s}}\}$ is *adapted to the curve* γ if the corresponding

[6] A linear set $\boldsymbol{C}$ within the domain of definition of a self-adjoint operator A is a *core* for that operator if the restriction $A_{\boldsymbol{C}}$ of A to $\boldsymbol{C}$ is an *essentially* self-adjoint operator, i.e., if $A_{\boldsymbol{C}}$ has A as its unique self-adjoint extension. Cores can be defined for closable operators in general, and play a key role in the mathematically rigorous treatment of operators governing time-evolution in quantum mechanics (cf., e.g., Prugovečki, 1981, p. 366).

Lorentz moving frame $\{\boldsymbol{e}_i(x) | x \in \mathbf{M}^s\}$ is adapted to γ, and if the components $a^i(x)$ of the field $\{\boldsymbol{a}(x) | x \in \mathbf{M}^s\}$ assume constant values with respect to $\boldsymbol{e}_i(x)$ at all $x \in \gamma \cap \mathbf{M}^s$ This last requirement implies that in a Poincaré gauge, given by a cross-section $\boldsymbol{s}$ of $P\mathbf{M}_0(\boldsymbol{g}^L)$ adapted to γ, all the Levi-Civita connection 1-forms in (4.16) that correspond to infinitesimal Lorentz transformations vanish at all $x \in \gamma$. Hence, if $\boldsymbol{X}$ is tangent to a curve γ with respect to which $\boldsymbol{s}$ is adapted, then it can be proved (cf. Prugovečki and Warlow, 1989b, Theorem A.4) that the corresponding covariant derivative in the direction $\boldsymbol{X}$ is given by

$$\nabla_{\boldsymbol{X}} \boldsymbol{\Psi}_x = \partial_{\boldsymbol{X}} \boldsymbol{\Psi}_x + \boldsymbol{\Psi}_x^{\zeta} \nabla_{\boldsymbol{X}} \boldsymbol{\Phi}_{m;\zeta}^{\boldsymbol{s}(x)} , \qquad \partial_{\boldsymbol{X}} \boldsymbol{\Psi}_x = (X^k \partial_k \boldsymbol{\Psi}_x^{\zeta}) \boldsymbol{\Phi}_{m;\zeta}^{\boldsymbol{s}(x)} \tag{4.18}$$

In accordance with (2.2.7), the first term on the right-hand side of the above relation is completely determined by the external derivatives

$$\partial_k \boldsymbol{\Psi}_x^{\zeta} := \lambda_k{}^{\mu} \partial_{\mu} \boldsymbol{\Psi}_x^{\zeta} = -\partial_k \boldsymbol{\Psi}_x^{\zeta} , \qquad \boldsymbol{e}_k = \lambda_k{}^{\mu} \partial_{\mu} , \quad \partial_{\mu} := \partial / \partial x^{\mu} , \quad \partial_k := \partial / \partial q^k . \tag{4.19}$$

Consequently, it is well-defined for *any* section $\boldsymbol{\Psi}$ of $\mathbf{E}$, and it is evidently independent of the choice of Poincaré gauge. The second term is also generically well-defined and gauge independent, since it involves only covariant derivatives of coherent sections of $\mathbf{E}$. Hence, we can adopt (4.18) as the definition of the covariant derivative of a section $\boldsymbol{\Psi}$ of $\mathbf{E}$ even in those cases where the limit in (4.15) does not exist.

In this manner we have arrived at a formulation of covariant differentiation of arbitrary sections $\boldsymbol{\Psi}$ of a quantum bundle $\mathbf{E}$ consisting of Klein-Gordon local state vectors of spin zero and rest mass m. We shall refer to the corresponding operator ∇ of covariant differentiation (i.e., to the Koszul connection on $\mathbf{E}$ – cf. Sec. 2.5), as the *quantum connection* on $\mathbf{E}$. In view of the total generality and gauge independence of its definition, we shall express that quantum connection in the following simplified notation:

$$\nabla = \boldsymbol{d} - i\tilde{\boldsymbol{\theta}}^i P_i + \tfrac{i}{2} \tilde{\boldsymbol{\omega}}_{jk} M^{jk}, \qquad \boldsymbol{d} := \boldsymbol{\theta}^k \partial_k . \tag{4.20}$$

The treatment of covariant differentiation on quantum bundles consisting of local state vectors with nonzero spin can be, in general, carried out in basically the same manner. However, the presence of spin affects the appearance of the infinitesimal generators for Lorentz transformations in the corresponding quantum connections. Furthermore, in the case of quantum bundles corresponding to half-integer spin values – such as the Dirac quantum bundles whose internal structure was treated in (2.18)-(2.24) – the structure group is no longer the Poincaré group $\mathrm{ISO}_0(3,1)$ itself, but rather its covering group $\mathrm{ISL}(2,\mathbf{C})$. This reflects the fact that such bundles are associated to the affine spin-frame bundle $S\mathbf{M}_0(\boldsymbol{g}^L)$ (cf. Sec. 4.2). As a consequence, in the case of Dirac quantum bundles, as well as other quantum bundles incorporating local quantum state vectors for half-integer spin, a choice of gauge consists of a choice of cross-section of the affine spin-frame bundle $S\mathbf{M}_0(\boldsymbol{g}^L)$, rather than of the Poincaré frame bundle $P\mathbf{M}_0(\boldsymbol{g}^L)$. As it will be seen in Sec. 6.4 from the treatment of connections and parallel transport in Dirac and Fock-Dirac quantum bundles, the operators in (4.10) then remain the same, but the infinitesimal generators in (4.12) have to be replaced by generators that incorporate spin. Modulo such changes, the ensuing quantum connection ∇ will retain the outward appearance of (4.20).

4.5. Quantum Tensorial Bundles and Quantum Metrics

The concept of covariant differentiation is required in CGR not only for sections of the tangent bundle $T\mathbf{M}$, to which the quantum bundles treated thus far bear a formal mathematical resemblance, but also for sections of various (r,s)-tensor bundles $T^{r,s}\mathbf{M}$. Similarly, in defining Fock bundles in the next chapter, we shall encounter various quantum tensorial bundles. We shall generically define them in this section by analogy with their classical counterparts, taking into account the fact that the fibres of quantum bundles are complex rather than real vector spaces. This means that the analogy with the classical case emerges most clearly in the spinorial approach to tensors in CGR (cf., e.g., Wald, 1984, p. 347).

Let us denote by $\mathbf{F}^\dagger$ the (topological) dual of the standard fibre $\mathbf{F}$ in (1.6). In accordance with the general definition of duals of topological vector spaces, $\mathbf{F}^\dagger$ consists of all complex-valued linear functionals on $\mathbf{F}$ which are continuous in the strong topology of $\mathbf{F}$, which is determined by the norm in $\mathbf{F}$. Since $\mathbf{F}$ is actually a Hilbert space, by Riesz' theorem each element $\Psi^\dagger$ of $\mathbf{F}^\dagger$ can be identified with a vector Ψ in $\mathbf{F}$ in such a manner that

$$\Psi^\dagger : \quad \Psi' \mapsto \langle \Psi | \Psi' \rangle \in \mathbf{C}^1 , \qquad \Psi, \Psi' \in \mathbf{F} , \qquad \Psi^\dagger \in \mathbf{F}^\dagger . \tag{5.1}$$

The resulting map $\Psi^\dagger \mapsto \Psi$ provides an antilinear transformation of $\mathbf{F}^\dagger$ onto $\mathbf{F}$ (cf., e.g., Prugovečki, 1981, pp. 184-186). Hence, we can assign to the unitary representation U of the Poincaré group $\mathrm{ISO}_0(3,1)$, obtained by restricting the operators in (3.7.7) to $\mathbf{F}$, a dual representation $U^\dagger$ in $\mathbf{F}^\dagger$, which acts from the left on the elements of $\mathbf{F}^\dagger$, and is defined by:

$$U^\dagger(a,\Lambda) : \quad \Psi^\dagger \mapsto (U(a,\Lambda)\Psi)^\dagger \in \mathbf{F}^\dagger , \qquad \Psi^\dagger \in \mathbf{F}^\dagger . \tag{5.2}$$

We can also introduce the complex-conjugate dual $\overline{\mathbf{F}}^\dagger$ of $\mathbf{F}$, defined by analogy with the aforementioned spinorial approach to tensors in CGR (cf. Wald, 1984, p. 347), namely as the space of all the continuous antilinear functionals on $\mathbf{F}$,

$$\overline{\Psi}^\dagger : \quad \Psi' \mapsto \langle \Psi | \Psi' \rangle^* = \langle \Psi' | \Psi \rangle \in \mathbf{C}^1 , \qquad \overline{\Psi}^\dagger \in \overline{\mathbf{F}}^\dagger . \tag{5.3}$$

We can introduce in this complex conjugate dual of $\mathbf{F}$ operators defined by

$$\overline{U}^\dagger(a,\Lambda)\overline{\Psi}^\dagger : \quad \Psi' \mapsto \left\langle \Psi' \middle| U^{-1}(a,\Lambda)\Psi \right\rangle \in \mathbf{C}^1 , \qquad \overline{\Psi}^\dagger \in \overline{\mathbf{F}}^\dagger . \tag{5.4}$$

These operators provide a representation of $\mathrm{ISO}_0(3,1)$ that is said to be contragradient to U (cf., e.g., Barut and Razcka, 1986, p. 139).

On account of the identification map in (5.1), the vector space $\mathbf{F}^\dagger$ dual to $\mathbf{F}$ is a Hilbert space with an inner product defined by the inner product in $\mathbf{F}$ under the inverse of that identification map. Hence, we can introduce on $\mathbf{F}^\dagger$ the continuous linear functionals

$$\overline{\Psi} : \quad \Psi'^\dagger \mapsto \left\langle \Psi^\dagger \middle| \Psi'^\dagger \right\rangle = \langle \Psi | \Psi' \rangle \in \mathbf{C}^1 , \qquad \overline{\Psi} \in (\overline{\mathbf{F}}^\dagger)^\dagger \cong \overline{\mathbf{F}} , \tag{5.5}$$

which give rise to the dual $(\overline{\mathbf{F}}^\dagger)^\dagger$ of $\overline{\mathbf{F}}^\dagger$, and can be regarded as providing a typical fibre $\overline{\mathbf{F}}$ that is the complex conjugate of $\mathbf{F}$. On this complex conjugate of $\mathbf{F}$, which is a Hilbert space on account of the above identification maps, we can define the unitary representation

$$\overline{U}(a,\Lambda) : \quad \overline{\Psi} \mapsto \left(U^\dagger(a,\Lambda)\Psi^\dagger\right)^\dagger \in (\overline{\mathbf{F}}^\dagger)^\dagger , \qquad \Psi^\dagger \in \mathbf{F}^\dagger , \tag{5.6a}$$

$$\overline{U}(a,\Lambda)\overline{\Psi}: \quad \overline{\Psi}' \mapsto \left\langle U^{-1}(a,\Lambda)\Psi \middle| \Psi' \right\rangle \in \mathbf{C}^1 , \qquad \overline{\Psi}' \in \overline{\mathbf{F}}^\dagger , \tag{5.6b}$$

which is said to be the conjugate of U (*ibid.*, p. 139).

Since all the above defined vector spaces actually are Hilbert spaces, we can introduce for any choice of non-negative integer values for r, s, r' and s' a Hilbert tensor product (cf., e.g., Prugovečki, 1981, pp. 144-147) consisting of r copies of $\mathbf{F}$ and s copies of $\mathbf{F}^\dagger$, as well as of r' and s' copies of the respective complex conjugates of these two spaces, in which those spaces are arranged in the following order:[7]

$$\mathbf{F}^{r,r';s,s'} = (\mathbf{F})^{\otimes r} \otimes (\overline{\mathbf{F}})^{\otimes r'} \otimes (\mathbf{F}^\dagger)^{\otimes s} \otimes (\overline{\mathbf{F}}^\dagger)^{\otimes s'} \tag{5.7}$$

Within this $(r,r';s,s')$-tensor product the following operators act from the left,

$$U^{r,r';s,s'}(a,\Lambda) = U(a,\Lambda)^{\otimes r} \otimes \overline{U}(a,\Lambda)^{\otimes r'} \otimes U^\dagger(a,\Lambda)^{\otimes s} \otimes \overline{U}^\dagger(a,\Lambda)^{\otimes s'} , \tag{5.8}$$

and provide a unitary representation $U^{r,r';s,s'}$ of the Poincaré group $\mathrm{ISO}_0(3,1)$. Hence, we can now considerably generalize the construction in (1.6), by introducing the $\mathbf{G}$-product

$$\mathbf{E}^{r,r';s,s'} = Q\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}}) \times_{\mathbf{G}} \mathbf{F}^{r,r';s,s'} , \qquad \mathbf{G} = \mathrm{ISO}_0(3,1) , \tag{5.9}$$

which defines the $(r,r';s,s')$-*tensorial quantum bundle* for each choice of values $r,r',s,s' = 0,1,2,\ldots$. This bundle is by its mode of construction associated to the standard quantum frame bundle $Q\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, and therefore also to the Poincaré frame bundle $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$.

We can now introduce within the fibres of these bundles the tensorial quantum frames that consist of the following state vectors:

$$\boldsymbol{\Phi}^{\boldsymbol{u}(x)\zeta'_1\cdots\zeta'_{s+s'}}_{\zeta_1\cdots\zeta_{r+r'}} = \boldsymbol{\Phi}^{\boldsymbol{u}(x)}_{\zeta_1} \otimes \cdots \otimes \overline{\boldsymbol{\Phi}}^{\boldsymbol{u}(x)}_{\zeta_{r+r'}} \otimes (\boldsymbol{\Phi}^{\boldsymbol{u}(x)}_{\zeta'_1})^\dagger \otimes \cdots \otimes (\overline{\boldsymbol{\Phi}}^{\boldsymbol{u}(x)}_{\zeta'_{s+s'}})^\dagger \in \mathbf{F}^{r,r';s,s'}_x \tag{5.10}$$

We can then expand any vector $\boldsymbol{\Psi}$ that belongs to one of these tensorial quantum fibres in terms of the elements of the tensorial quantum frame within that fibre, in a manner which generalizes (2.11), and which underlines the mathematical similarities with (2.2.12):

$$\boldsymbol{\Psi} = \Psi^{\zeta_1\cdots\zeta_{r+r'}}_{\zeta'_1\cdots\zeta'_{s+s'}} \boldsymbol{\Phi}^{\boldsymbol{u}(x)\zeta'_1\cdots\zeta'_{s+s'}}_{\zeta_1\cdots\zeta_{r+r'}} , \qquad \Psi^{\zeta_1\cdots\zeta_{r+r'}}_{\zeta'_1\cdots\zeta'_{s+s'}} := \left\langle \boldsymbol{\Phi}^{\boldsymbol{u}(x)\zeta'_1\cdots\zeta'_{s+s'}}_{\zeta_1\cdots\zeta_{r+r'}} \middle| \boldsymbol{\Psi} \right\rangle \tag{5.11}$$

In view of the fact that all the frames within the tensor product in (5.10) are soldered to the corresponding frame $\boldsymbol{u}(x)$ within the Poincaré frame bundle $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, the operator for parallel transport in (4.1) unambiguously determines the parallel transport of all these tensorial quantum frames. Thereby the parallel transport of all local tensorial quantum state vectors is also determined, since during any such parallel transport the coordinate wave functions provided by the coefficients of the expansions in (5.11) remain unchanged. Hence, the quantum covariant derivative in any of the above introduced $(r,r';s,s')$-tensorial quantum bundles can be defined by the method of Sec. 4.4. This definition leads to a quantum connection ∇ that has the same outward appearance as the one in (4.20), but

[7] We adopt an ordering convention which is somewhat different than the one used for spinorial tensors (cf., e.g., Wald, 1984, p. 347), so as to be able to identify the $(1,0;0,0)$-tensorial quantum bundle with the quantum bundle $\mathbf{E}$ in (1.6). This convention is consistent with action from the right on quantum frames, and with action from the left on quantum coordinate wave functions, for all the considered representations of the Poincaré group.

$$\overline{U}(a,\Lambda)\overline{\Psi}: \quad \overline{\Psi}'^{\dagger} \mapsto \left\langle U^{-1}(a,\Lambda)\Psi \middle| \Psi' \right\rangle \in \mathbf{C}^1 , \qquad \overline{\Psi}'^{\dagger} \in \overline{\mathbf{F}}^{\dagger} , \tag{5.6b}$$

which is said to be the conjugate of U (*ibid.*, p. 139).

Since all the above defined vector spaces actually are Hilbert spaces, we can introduce for any choice of non-negative integer values for r, s, r' and s' a Hilbert tensor product (cf., e.g., Prugovečki, 1981, pp. 144-147) consisting of r copies of $\mathbf{F}$ and s copies of $\mathbf{F}^{\dagger}$, as well as of r' and s' copies of the respective complex conjugates of these two spaces, in which those spaces are arranged in the following order:[7]

$$\mathbf{F}^{r,r';s,s'} = (\mathbf{F})^{\otimes r} \otimes (\overline{\mathbf{F}})^{\otimes r'} \otimes (\mathbf{F}^{\dagger})^{\otimes s} \otimes (\overline{\mathbf{F}}^{\dagger})^{\otimes s'} \tag{5.7}$$

Within this $(r,r';s,s')$-tensor product the following operators act from the left,

$$U^{r,r';s,s'}(a,\Lambda) = U(a,\Lambda)^{\otimes r} \otimes \overline{U}(a,\Lambda)^{\otimes r'} \otimes U^{\dagger}(a,\Lambda)^{\otimes s} \otimes \overline{U}^{\dagger}(a,\Lambda)^{\otimes s'} , \tag{5.8}$$

and provide a unitary representation $U^{r,r';s,s'}$ of the Poincaré group $\mathrm{ISO}_0(3,1)$. Hence, we can now considerably generalize the construction in (1.6), by introducing the $\mathbf{G}$-product

$$\mathbf{E}^{r,r';s,s'} = \mathcal{Q}\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}}) \times_{\mathbf{G}} \mathbf{F}^{r,r';s,s'} , \qquad \mathbf{G} = \mathrm{ISO}_0(3,1) , \tag{5.9}$$

which defines the $(r,r';s,s')$-*tensorial quantum bundle* for each choice of values $r,r',s,s' = 0,1,2,\ldots$. This bundle is by its mode of construction associated to the standard quantum frame bundle $\mathcal{Q}\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, and therefore also to the Poincaré frame bundle $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$.

We can now introduce within the fibres of these bundles the tensorial quantum frames that consist of the following state vectors:

$$\boldsymbol{\Phi}^{\boldsymbol{u}(x)\zeta'_1\cdots\zeta'_{s+s'}}_{\zeta_1\cdots\zeta_{r+r'}} = \boldsymbol{\Phi}^{\boldsymbol{u}(x)}_{\zeta_1} \otimes \cdots \otimes \overline{\boldsymbol{\Phi}}^{\boldsymbol{u}(x)}_{\zeta_{r+r'}} \otimes (\boldsymbol{\Phi}^{\boldsymbol{u}(x)}_{\zeta'_1})^{\dagger} \otimes \cdots \otimes (\overline{\boldsymbol{\Phi}}^{\boldsymbol{u}(x)}_{\zeta'_{s+s'}})^{\dagger} \in \mathbf{F}^{r,r';s,s'}_{x} \tag{5.10}$$

We can then expand any vector $\boldsymbol{\Psi}$ that belongs to one of these tensorial quantum fibres in terms of the elements of the tensorial quantum frame within that fibre, in a manner which generalizes (2.11), and which underlines the mathematical similarities with (2.2.12):

$$\boldsymbol{\Psi} = \Psi^{\zeta_1\cdots\zeta_{r+r'}}_{\zeta'_1\cdots\zeta'_{s+s'}} \boldsymbol{\Phi}^{\boldsymbol{u}(x)\zeta'_1\cdots\zeta'_{s+s'}}_{\zeta_1\cdots\zeta_{r+r'}} , \qquad \Psi^{\zeta_1\cdots\zeta_{r+r'}}_{\zeta'_1\cdots\zeta'_{s+s'}} := \left\langle \boldsymbol{\Phi}^{\boldsymbol{u}(x)\zeta'_1\cdots\zeta'_{s+s'}}_{\zeta_1\cdots\zeta_{r+r'}} \middle| \boldsymbol{\Psi} \right\rangle \tag{5.11}$$

In view of the fact that all the frames within the tensor product in (5.10) are soldered to the corresponding frame $\boldsymbol{u}(x)$ within the Poincaré frame bundle $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, the operator for parallel transport in (4.1) unambiguously determines the parallel transport of all these tensorial quantum frames. Thereby the parallel transport of all local tensorial quantum state vectors is also determined, since during any such parallel transport the coordinate wave functions provided by the coefficients of the expansions in (5.11) remain unchanged. Hence, the quantum covariant derivative in any of the above introduced $(r,r';s,s')$-tensorial quantum bundles can be defined by the method of Sec. 4.4. This definition leads to a quantum connection ∇ that has the same outward appearance as the one in (4.20), but

[7] We adopt an ordering convention which is somewhat different than the one used for spinorial tensors (cf., e.g., Wald, 1984, p. 347), so as to be able to identify the $(1,0;0,0)$-tensorial quantum bundle with the quantum bundle $\mathbf{E}$ in (1.6). This convention is consistent with action from the right on quantum frames, and with action from the left on quantum coordinate wave functions, for all the considered representations of the Poincaré group.

in the values (5.14) assumed by the quantum metric for those frames. Such renormalization factors in the quantum metric between pairs of elements of congruent quantum Lorentz frames do not affect, however, the (topologically) local probability density values in (2.17), since those values describe only the relative probabilities of events taking place within sufficiently small neighborhoods of the origin of the frames in question.

In complete analogy with the compatibility of the Levi-Civita connection with the classical Lorentzian metrics used in CGR, the quantum connection is compatible with the quantum metric $\boldsymbol{G}$. This follows immediately from the fact that all the transition probability amplitudes in (5.14) are preserved under parallel transport, so that $\nabla \boldsymbol{G} \equiv \boldsymbol{0}$. From this and (4.18) we can immediately deduce the quantum analogue of the Ricci identity (2.6.4),

$$\boldsymbol{X}\boldsymbol{G}(\boldsymbol{\Psi}, \boldsymbol{\Psi}') = \boldsymbol{G}(\nabla_{\boldsymbol{X}} \boldsymbol{\Psi}, \boldsymbol{\Psi}') + \boldsymbol{G}(\boldsymbol{\Psi}, \nabla_{\boldsymbol{X}} \boldsymbol{\Psi}') \ , \tag{5.15}$$

for any two sections $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}'$ of any quantum bundle $\mathbf{E}$.

All this provides strong support for the thesis that a quantum-geometric version of the path integration method described in Chapter 3 can be developed, that can be used to formulate a notion of quantum-geometric propagation which is not only equivalent to ordinary quantum propagation in the special relativistic regime, but also abides by the strong equivalence principle in the general relativistic regime.

4.6. Quantum-Geometric Propagation in Quantum Bundles

At the epistemological level, the basic idea of quantum-geometric propagation in a spacetime oriented and globally hyperbolic Lorentzian base manifold $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$ is very simple: it consists of adapting the path integration formulation of nonrelativistic and special relativistic quantum propagation to the general relativistic regime by replacing the straight lines of polygonal broken paths in Feynman's (1948, 1950) formulation with arcs of geodesics, and by replacing Feynman propagators with propagators for parallel transport between the end-points of those geodesic arcs. However, when a mathematically precise and physically well-grounded implementation of this epistemic idea is attempted, fundamental problems are encountered. Essentially, these problems are related to the foundational issue of general relativistic causality in the quantum regime, for whose complete resolution the general relativistic principles of strong equivalence and of general covariance do not suffice. For that reason, the formulation of quantum-geometric propagation adopted in this section (cf. Prugovečki, 1987b) is different from the strongly causal one in Sec. 5.4 of (Prugovečki, 1992): the latter is based on the averaging of contributions from the parallel transport of local states over *classical* causal paths, so that it abides by Einstein causality, that prohibits all propagation outside the causal future of any point $x \in \mathbf{M}$ (cf. Sec. 4.8); whereas, the present formulation is weakly causal, in the sense that it abides by the geometrodynamic time-ordering resulting in a foliation of $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$ of the type described in (2.7.8), but it otherwise follows Feynman's (1949, 1950) idea that quantum propagation can proceed along any continuous stochastic path – so that the Einstein causality that we observe macroscopically is then due to the superposition of probability amplitudes at points in classically forbidden regions, resulting in mutual cancelations within those regions.

In this section we shall formulate quantum-geometric propagation in this weakly

causal sense, and then discuss the foundational issues related to it in the next two sections.

Let $\mathbf{E}$ be any quantum bundle over a Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$, and let γ be any smooth curve joining two points $x' \in \mathbf{M}$ and $x'' \in \mathbf{M}$, so that $\gamma = \{x(\tau) |\, a \leq \tau \leq b\}$, where $x' = x(a)$ and $x'' = x(b)$. For any choice of Poincaré gauge, given by a cross-section $\boldsymbol{s}$ of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, we can define as follows the *propagator for parallel transport* from x' to x'' along γ by using the operator for parallel transport in (4.2):

$$K_\gamma(x'',\zeta'';x',\zeta') = \left\langle \boldsymbol{\Phi}^{\boldsymbol{s}(x'')}_{\zeta''} \middle|\, \boldsymbol{\tau}_\gamma(x'',x')\, \boldsymbol{\Phi}^{\boldsymbol{s}(x')}_{\zeta'} \right\rangle \ . \tag{6.1}$$

In view of (2.10) and (4.3), we then obtain that the coordinate wave function of a local quantum state in a Klein-Gordon quantum bundle, that was parallel transported along γ to x'', is related to the coordinate wave function of the original local quantum state at x' by this propagator for parallel transport:

$$\Psi^{\parallel}_{x''}(\zeta'') = \int K_\gamma(x'',\zeta'';x',\zeta')\, \Psi_{x'}(\zeta')\, d\Sigma(\zeta') \ , \qquad \boldsymbol{\Psi}^{\parallel}_{x''} = \boldsymbol{\tau}_\gamma(x'',x')\, \boldsymbol{\Psi}_{x'} \tag{6.2}$$

The propagators for parallel transport in arbitrary tensorial quantum frames can be constructed from the propagators for parallel transport in (6.1) by using the definition of a tensorial quantum frame in (5.10) in conjunction with the definitions provided in (5.1)-(5.9) for conjugate, contragradient and conjugate-contragradient representations, as well as for the respective fibres that carry them. The expansion in (5.11) can be then used to extend (6.2) to arbitrary tensorial bundles. Hence, the propagator for parallel transport in (6.1) determines, in essence, the outcome of the parallel transport of any local quantum state within all such bundles.

The family of all propagators for parallel transport in $\mathbf{M}$ defined by (6.1) displays properties that are essential to any type of propagator, namely the following ones:

$$\begin{aligned} K_\gamma(x'',\zeta'';x',\zeta') &= K^*_\gamma(x'',\zeta'';x',\zeta') \\ &= \int K_\gamma(x'',\zeta'';x,\zeta)\, K_\gamma(x,\zeta;x',\zeta')\, d\Sigma(\zeta) \end{aligned} \tag{6.3}$$

These relations can be easily verified by inserting $\tau_\gamma(x'', x') = \tau_\gamma(x'', x)\, \tau_\gamma(x, x')$ into the right-hand side of the inner product in (6.1), and by then using (2.10) at $x \in \gamma$.

The relations in (6.3) provide an extrapolation to the curved Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ of the special-relativistic relations (3.8.10) for propagators in Minkowski space. By iteration, they lead to the following segmental decomposition of the propagator in (6.1),

$$\begin{aligned} K_\gamma(x'',\zeta'';x',\zeta') = \lim_{\varepsilon \to +0} \int & K_\gamma(x_N,\zeta_N;x_{N-1},\zeta_{N-1}) \\ & \times \prod_{n=N-1}^{1} K_\gamma(x_n,\zeta_n;x_{n-1},\zeta_{n-1})\, d\Sigma(\zeta_n) \ , \end{aligned} \tag{6.4}$$

that holds good, as $\varepsilon = \max\,(\tau_n - \tau_{n-1}) \to +0$ for $N \to +\infty$, for any sequence of subdivisions of the interval from a and b given by $a = \tau_0 < \tau_1 < \cdots < \tau_N = b$. Due to the mode of construction of the Klein-Gordon quantum bundle $\mathbf{E}$ in (1.6), and to the definition of parallel transport within this bundle, for "infinitesimal" separations ε the propagators for

parallel transport on the right-hand side of the equation in (6.4) become related, via (4.20), to the special relativistic propagators in (3.8.6), in any Poincaré gauge that is adapted to γ (cf. Secs. 2.6 and 4.4). In particular, if γ is a geodesic and τ provides an affine parametrization, then, from the perspective of the local Lorentz frames along γ whose Poincaré counterparts belong to that gauge, the "infinitesimal" in (6.4) coincide with those in (3.8.6), due to the fact that all the connection coefficients next to the generators of infinitesimal Lorentz transformations in (4.17) vanish in any such gauge adapted to γ. This constitutes the basis for the extension of the strong equivalence principle from classical general relativistic particle propagation to quantum-geometric propagation.

In this context, it is important to note for future reference that the above observations apply not only to timelike geodesics, but remain valid even for null and spacelike geodesics, as long as τ represents an affine parametrization for those geodesics. Of course, according to Einstein's classical causality principle, the classical particle propagation along spacelike geodesics, or spacelike world lines in general, is strictly forbidden. That is, however, no longer the case with quantum propagation even in the special relativistic context, in which $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ is geometrically identified with the Minkowski space $(\mathbf{R}^4,\boldsymbol{\eta})$.

To see that first at a purely mathematical level, let us adopt in the case of $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})\equiv(\mathbf{R}^4,\boldsymbol{\eta})$ a Poincaré gauge that is provided by the cross-section $\boldsymbol{s}_0(\mathcal{L})$ of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, defined as in (1.13), and therefore mathematically identifiable with the global Lorentz frame $\mathcal{L}=\{\boldsymbol{e}_i(O)|\, i=0,1,2,3\}$ in $(\mathbf{R}^4,\boldsymbol{\eta})$. In view of the fact that the parallel transport between any two points $x'\in\mathbf{R}^4$ and $x''\in\mathbf{R}^4$ is path independent, and of the fact that its outcome actually coincides with that resulting from the parallel translation between those two points in $\mathbf{R}^4$, in this special kind of Poincaré gauge we shall have that

$$K_\gamma(x'',\zeta'';x',\zeta')=K^{(\ell)}(x''+q'',v'';x'+q',v')\,,\quad x=x^i\boldsymbol{e}_i(O)\,,\quad \zeta=q-i\ell v\,, \tag{6.5}$$

where the propagator on the right-hand side is the special relativistic one defined in (3.8.6). On the other hand, by iterating (3.8.10) we obtain as $\varepsilon=t_n-t_{n-1}=(t''-t')/N\to+0$,

$$K^{(\ell)}(x'',v'';x',v')=\lim_{\varepsilon\to+0}\int K^{(\ell)}(x_N,\zeta_N;x_{N-1},\zeta_{N-1})\times\prod_{n=N-1}^{1}K^{(\ell)}(x_n,v_n;x_{n-1},v_{n-1})\,d\Sigma(x_n,v_n)\,, \tag{6.6}$$

for any slicings of the Minkowski spacetime segment between the reference hyperplanes $\sigma(t')$ and $\sigma(t'')$, containing x' and x'', respectively, with hyperplanes $\sigma(t_n)$ corresponding to the Minkowski time values $t'=t_0<t_1<\cdots<t_N=t''$. The comparison of (6.6) with (3.5.12) shows that this formula is consistent with an interpretation of special relativistic quantum propagation that takes place along *all* broken polygonal paths in $(\mathbf{R}^4,\boldsymbol{\eta})$ whose straight-line segments join points x_{n-1} and x_n on adjacent reference hypersurfaces $\sigma(t_{n-1})$ and $\sigma(t_n)$ in Minkowski space, regardless of whether those points are at timelike, spacelike or null spacetime separations from each other.

It has to be emphasized that the formula in (6.6) represents a mathematically rigorous result for the same type of reasons as those discussed in Sec. 3.5, that the integrals on its right-hand side actually exist for any foliation of the Minkowski space $(\mathbf{R}^4,\boldsymbol{\eta})$ into maximal

spacelike slices $\sigma(t)$, and that its outcome is the same regardless of the adopted order of integration. On the other hand, in the sharp-point limit $\ell \rightarrow +0$, the propagators in (6.6) merge, after being subjected to the appropriate renormalizations in (3.8.8), into their conventional Feynman counterparts in (3.8.7). Although mathematical rigor is, of necessity, lost during such an infinite renormalization process, Feynman's (1948-1950) approach to quantum particle propagation is thereby recovered at the physical level.

In order to present a physically consistent recounting of Feynman's point of view towards special relativistic quantum propagation, quantum particles as well as antiparticles of a given rest mass and spin have to be simultaneously considered, and the state vectors of antiparticles have to be identified with negative-energy solutions of the equation of motion. As a result of this kind of identification, it becomes possible to regard the propagation of antiparticles into the future as being formally represented by the propagation of their state vectors into the past.[9] Hence, upon integrating these two types of propagation into a single picture, it becomes possible to visualize that, in the above described kind of limit for which $\varepsilon = t_n - t_{n-1} = (t'' - t')/N \rightarrow +0$, the broken paths used in the "summing up" procedure supplying the path-integral representation of a Feynman propagator in Minkowski space lead to continuous world lines weaving in all conceivable spacetime directions.

In a 1949 paper on the theory of electrons and positrons whose equation of motion is the Dirac equation with an external potential, Feynman used a "bombardier metaphor" to illustrate how, after performing the identification of positron wave functions with negative-energy solutions of that equation, such a pictorial image of quantum propagation can emerge: "Following the charge rather than the particles corresponds to considering this continuous world line as a whole rather than breaking it up into its pieces. It is as though a bombardier flying over a road suddenly sees three roads and it is only when two of them come together and disappear again that he realizes that he has simply passed over a long switchback in a single road." (Feynman, 1949, p. 749.)

As recalled in later years by Dyson, this new type of conceptualization of special relativistic quantum propagation was originally visualized by Feynman as follows: "The electron does anything it likes. It just goes in any direction at any speed, forward or backward in time, however it likes, and then you add up the amplitudes and it gives you the wave-function." (Dyson, 1980, p. 376.) And indeed, Feynman based his later treatment of solutions for the Klein-Gordon equation (cf. Feynman, 1950, pp. 453-455) on precisely this kind of image of indiscriminate diffusion in *all* directions in Minkowski space. On the other hand, when he discussed this same issue in an unpublished paper, he acknowledged that: "It may be that more problems and difficulties are produced than solved by such a point of view." (Cf. Schweber, 1986b, p. 489.)

The fact that there are physical difficulties with such a point of view in the case of quantum propagation in a curved spacetime manifold $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$, emerges already from the fact that any such manifold will contain singularities (Hawking and Ellis, 1973). This fact, established only well after Feynman's (1949) conceptualization was first stated in print, implies that some of the paths of indiscriminate diffusion in *all* directions in $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$ will

[9] This kind of identification will be actually implemented in the quantum-geometric framework, but only in its appropriate field-theoretical context, presented in Chapter 5, where quantum-geometric fields creating and annihilating particles as well as antiparticles are introduced.

end in black holes, or in the Big Bang initial singularity, so that the physical meaning associated with their "continuity" will be lost. Of course, it could be hoped that quantum gravity will eliminate such singularities, but, as we shall shortly discuss, the idea of a *mutual* interaction between "matter" and "geometry," required by any consistent general relativistic formulation of quantum gravity, is incompatible with propagation along stochastic paths that lead "backward in time."

At the purely technical level, it might appear that this problem can be resolved by the familiar transition to the "Euclidean regime," obtained by the analytic continuation of the Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ into a corresponding Riemannian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{R}})$ with "Euclidean metric" $\boldsymbol{g}^{\mathrm{R}}$ (Hawking, 1979). And indeed, in the present quantum-geometric context, path-integral solutions of the Klein-Gordon equation in curved spacetime manifolds can be then obtained by making the transition to such a "Euclidean regime," and using Itô's (1962, 1975) and Dynkin's (1968) treatment of diffusion processes, that has inspired the development of stochastic differential geometry methods (Daletskii, 1983; Belopolskaya and Dalecky, 1990). In this treatment, the stochastic parallel transport in tensor bundles over a Riemannian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{R}})$ with a connection compatible with $\boldsymbol{g}^{\mathrm{R}}$ can be formulated as a limit-in-probability of ordinary parallel transports that take place along broken paths consisting of geodesic arcs. Such stochastic parallel transport can be then used to provide a mathematically satisfactory alternative to Feynman's treatment of the Klein-Gordon equation (Prugovečki, 1991b; 1992, Sec. 5.6). However, in addition to the fact the transition to the Euclidean regime is not always mathematically feasible (DeWitt, 1984b), this procedure basically provides solutions to the diffusion equation in the Riemannian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{R}})$. Therefore, all these solutions have to be subsequently continued analytically to the Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$, so that the following question remains open: which are the paths that are actually *physically* allowed, if such propagation is viewed as a physical process which takes place in *real* spacetime?

The fact that a physically satisfactory answer cannot be provided by the paths of an indiscriminate diffusion process becomes very evident as soon as the interaction between the underlying geometry and quantum fields, representing matter and nongravitational radiation, is considered from a general relativistic perspective. Indeed, on one hand, such indiscriminate diffusion obeys no ordering in *real* time; whereas, on the other hand, if the *physical* propagation of matter fields is not subject to time-ordering, then the physical geometry of spacetime cannot *evolve* as a result of their propagation. In other words, as a result of an indiscriminately diffusive propagation, events *throughout* spacetime would be instantaneously and perpetually affected, so that there would not be a causal past or future in relation to any of them, and the very concept of spacetime would lose all physical meaning. Such a conclusion might be acceptable from the standpoint of a purely instrumentalist approach to quantum physics, steeped in the production of numerical "predictions" with no regard towards their deeper physical meaning and consistency, but not from the point of view of any attempt to describe physical reality as it actually manifests itself in nature, where a time-arrow is certainly present (Omnès, 1992).

If an epistemic approach that takes for granted the existence of *quantum* reality is adopted towards quantum gravity, then the existence of time-ordering of events becomes an essential feature of any concept of quantum spacetime. A modification of the geometro-

dynamic conceptualization of the canonical approach to CGR (Wheeler, 1957-1968; Misner *et al.*, 1973), in which resulting "foliations" of a classical spacetime manifold are replaced by "segmentations" of the base manifold of a quantum spacetime, supplies the common denominator of all physically acceptable solutions of such time-orderings (cf. Chapter 8). In the present semi-classical regime, this conceptualization can be related to the algorithmic scheme described in Sec. 2.7, whereby a classical spacetime manifold is envisaged as the outcome of general relativistic canonical developments within base-segments $\mathbf{S}_n$, $n = 0, \pm 1, \pm 2, \ldots$, represented by differential manifolds corresponding to respective spans $[t_{n-1}, t_n)$ in the values of a globally defined parameter t. As discussed in Sec. 3.7, a *natural* proper time is intrinsic to the notion of quantum frame, so that this parameter is supplied by the free-fall propagation of the origins of local quantum frames along the geodesics orthogonal to the inflow hypersurfaces $\Sigma_{t_{n-1}}$ of the respective base-segments $\mathbf{S}_n$. Hence, the values assumed by t represent *locally* the *maximal* proper time distance between the inflow hypersurface $\Sigma_{t_{n-1}}$ and the outflow hypersurface Σ_{t_n} of each base-segment $\mathbf{S}_n$.

The indiscriminately diffusive type of quantum propagation envisaged by Feynman as taking place in all possible spacetime directions can be thereby confined to those directions which lead, along geodesic arcs $\gamma(x_{n-1}, x_n)$ for the Levi-Civita connection associated with $\boldsymbol{g}^{\mathrm{L}}$, from all the points x_{n-1} on the inflow hypersurface $\Sigma_{t_{n-1}}$ to all the points x_n on the outflow hypersurface Σ_{t_n} of each base-segment $\mathbf{S}_n$. This results in the following extrapolation of (6.6) into a definition of the *quantum-geometric propagator* from $x' \in \Sigma_{t'}$ to $x'' \in \Sigma_{t''}$, that is carried out in the limit $\varepsilon = t_n - t_{n-1} = (t'' - t')/N \rightarrow +0$,

$$K(x'', \zeta''; x', \zeta') = \lim_{\varepsilon \to +0} \int K_{\gamma(x_{N-1}, x_N)}(x_N, \hat{\zeta}_N; x_{N-1}, \hat{\zeta}_{N-1}) \times \prod_{n=N-1}^{1} K_{\gamma(x_{n-1}, x_n)}(x_n, \hat{\zeta}_n; x_{n-1}, \hat{\zeta}_{n-1})\, d\Sigma(x_n, \upsilon_n) \,, \tag{6.7a}$$

$$\hat{\zeta}_0 = \zeta' \,, \quad \hat{\zeta}_N = \zeta'' \,, \quad \hat{\zeta}_n = -\alpha(x_n) - i\ell\upsilon_n \,, \quad n = 1, 2, \ldots, N-1 \,, \tag{6.7b}$$

where the 4-tuples $-\alpha(x_n)$ obviously represent the coordinates of the point of contact between $T_{x_n}\mathbf{M}$ and $\mathbf{M}$ with respect to the local Lorentz frame $\{\boldsymbol{e}_i(x_n)\}$ within the moving frame $\boldsymbol{s} = \{\boldsymbol{\alpha}(x), \boldsymbol{e}_i(x) | \, x \in \mathbf{M}^g\}$, representing the chosen Poincaré gauge.

The propagator in (6.7) can be used, as was the case in (6.2), to compute the coordinate wave function at x'' of the local quantum state vector that results from the quantum-geometric propagation of a local Klein-Gordon quantum state originally located at x':

$$\Psi_{x''}(\zeta'') = \int K(x'', \zeta''; x', \zeta')\, \Psi_{x'}(\zeta')\, d\Sigma(\zeta') \,. \tag{6.8}$$

Quantum-geometric propagation within arbitrary tensorial quantum frames can be then treated as indicated earlier, in the case of propagation by parallel transport.

For a curved Lorentzian manifold $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$, these two forms of propagation will have very different outcomes, despite the fact that both quantum-geometric propagation as well as propagation by parallel transport are Poincaré gauge invariant. This is a consequence of the path dependence of parallel transport in a curved manifold: since the "summing up" in (6.7a) takes place over stochastic paths in $\mathbf{M}$, a different outcome can be expected than that

resulting from the "summing up" procedure in (6.4), which takes place over paths within the tangent bundle $T\mathbf{M}$ that are located *above* the points of a fixed curve in $\mathbf{M}$. It is only in the case where the base space $\mathbf{M}$ is a Minkowski space that, as a result of the identification of the tangent spaces in $T\mathbf{M}$ with $\mathbf{M}$ itself, these latter paths can be "lowered" into $\mathbf{M}$, so that the two forms of propagation can be shown to produce identical results.

It is in this respect[10] that the physical nature of the general relativistic propagation of "particles" emerges as fundamentally different in the classical and in the quantum case: in the latter case the superposition principle mediates a "summing up" of probability amplitudes, which is totally meaningless classically even in the case of Brownian motion, or other diffusion processes, that rely on stochastic parallel transport (Itô's, 1962; Dynkin, 1968; Belopolskaya and Dalecky, 1990). For, in the classical theory of stochastic processes it is only probabilities that are being "summed up," and they *always* reinforce each other; whereas, in the formally analogous present formulation of *quantum* stochastic processes, it is complex probability amplitudes that are being "summed up," and they can not only reinforce, but also attenuate each other.

4.7. The Physical Meaning of Quantum-Geometric Propagation

The basic fact that, as opposed to the classical treatment of Brownian motion and diffusion processes, quantum propagation does not allow for any *bona fide* probability measure over the physical paths of quantum propagation in *real* spacetime is reflected in the physical interpretation of the outcome of quantum-geometric propagation.

To illustrate this point, let us consider, as the simplest possible example, the quantum-geometric propagation of a local Klein-Gordon quantum state prepared at $x(t_0) \in \Sigma_{t_0}$,

$$\Psi_{x(t_0)} \mapsto \Psi_{x(t)} \in \mathbf{F}_{x(t)} \subset \mathbf{E} , \qquad x(t) \in \Sigma_t , \quad t > t_0 , \tag{7.1a}$$

$$\Psi_{x(t)}(\zeta) = \int K(x(t),\zeta;x(t_0),\zeta')\,\Psi_{x(t_0)}(\zeta')\,d\Sigma(\zeta') , \tag{7.1b}$$

within the geometrodynamic future of the point $x(t_0)$ where the operation of state preparation was originally performed. The physical meaning of this propagation emerges from a most straightforward extrapolation of the interpretation of the probability measure in (3.7.14), as well as from a global extension of the interpretation of (2.16) as a local relative probability density. Thus, for any choice of quantum moving frame, corresponding to some cross-section $\boldsymbol{s}$ of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, we interpret

$$\Psi(x(t),v) = \boldsymbol{G}\left(\Phi^{\boldsymbol{s}(x(t))}_{\zeta(x(t))}, \Psi_{x(t)}\right) , \qquad \hat{\zeta}(x(t)) = -\,a(x(t)) - i\ell v , \tag{7.2}$$

to be a *quantum-geometric wave function*, whose values along any base reference hypersurface Σ_t provide, for a quantum object *locally* prepared in the quantum state $\Psi_{x(t_0)}$, all the *relative* probability amplitudes for the detection of the stochastic phase space values

[10] Moreover, in the present semi-classical regime quantum-geometric propagation is not invariant under diffeomorphisms, since a fixed Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ had to be adopted, and the foliation of the manifold $\mathbf{M}$ had to be implemented accordingly. However, the diffeomorphism invariance will be recovered in the quantum gravity framework presented in Chapter 8, where the interaction between "matter" and "geometry" is mutual.

$$(x(t),v) \in \Sigma_t := \bigcup_{x \in \Sigma_t} V_x^+ \subset TM , \qquad v = v^i e_i(x(t)) , \tag{7.3}$$

for that quantum object.[11] In other words, the following positive-definite measure

$$\int_B |\Psi(x(t),v)|^2 \, d\Sigma(x,v) , \qquad B \subset \Sigma_t , \tag{7.4}$$

provides the conditional probabilities that, if a local state $\Psi_{x(t_0)}$ of a quantum object of rest mass $m = 1$ and zero spin were prepared at $x(t_0) \in \Sigma_{t_0}$, then that quantum object will be found by measurements of stochastic position and momentum, performed along the reference hypersurface Σ_t with test bodies of the same spin and rest mass, to be located within the (Borel) subsets B of the hypersurfaces Σ_t in the general relativistic stochastic phase space over the base manifold $\mathbf{M}$.

Naturally, the cases of various rest masses and spins for either the system or the test body can be treated by using the corresponding propagators and systems of covariance – as will be explicitly demonstrated for the spin 1/2 case in the field theoretical framework presented in the next chapter. However, even in the present context, note should be taken of the fact that in such considerations the roles of "system" and "test body" are interchangeable. This is in full concordance with a purely quantum theory of measurement, which does not draw a sharp demarcation line between "system" and "apparatus."

In the case of a base manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ which is identifiable with the Minkowski space $(\mathbf{R}^4,\boldsymbol{\eta})$, the present interpretation of (7.4) coincides with the one provided in Sec. 3.7 for the probability measure in (3.7.14). Indeed, in that case parallel transport becomes path independent, so that the quantum-geometric propagator between any two points in $(\mathbf{R}^4,\boldsymbol{\eta})$ coincides with the propagator for parallel transport between those same two points, and, for the latter, (6.5) holds in those Poincaré gauges for which $\boldsymbol{s}_0(\mathcal{L})$ in (1.13) corresponds to a choice $\mathcal{L}$ of global Lorentz frame.

This agreement between the outcome of quantum-geometric propagation and the ordinary type of quantum propagation in relativistic phase space considered in the preceding chapter, confirms the physical veracity of the interpretation of (7.4) in those spacetime regions of a generic base manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ where curvature effects can be neglected.

For those spacetime regions where such curvature effects are not negligible, but the prevalent stochastic velocities as well as the strength of the gravitational field are such that nonrelativistic approximations become applicable, the physical correctness of the present interpretation has been confirmed (Prugovečki, 1987a) by formulating the idea of quantum-geometric propagation within a Newton-Cartan spacetime framework (cf. Misner *et al.*, 1973, §12.3). At the classical level, a Newton-Cartan spacetime provides a fully geometrized version of a corresponding Newtonian theory of gravity (Trautman, 1963, 1966; Künzle, 1972-1988; Duval *et al.*, 1977-1991), so that it supplies a natural arena (Kuchař, 1980) for testing ideas and methods that pertain to quantum propagation in a curved spacetime manifold in a setting where the transition to a physically equivalent

[11] In the quantum field theoretical framework of the next chapter, we shall refer to this "object" as a *geometro-stochastic exciton*, since it will be viewed as an "excitation" of the *local* vacuum at $x(t_0)$. We avoid the term "particle" since, as will be discussed later in this section as well as in the next chapter, the conventional concept of particle in curved spacetime is, first of all, "nebulous," and, second, at odds with the present conceptualization.

formulation in a flat Newtonian spacetime is possible by introducing a Newtonian gravitational potential in that Newtonian spacetime.

The quantum-geometric propagators within quantum bundles over a Newton-Cartan spacetime manifold $\mathbf{M}$ can be formulated in *exactly* the same manner as in (6.7), namely as free-fall propagation along stochastic paths obtained from future-pointing arcs of geodesics – except that those geodesics correspond now to a Cartan connection in a Newton-Cartan base manifold $\mathbf{M}$, and that the "future" of a point $x \in \mathbf{M}$ is one that corresponds to Newton's concept of an absolute time. It turns out that the quantum-geometric propagation within quantum bundles over a Newton-Cartan spacetime manifold corresponding to a gravitational potential V leads, under a physical interpretation corresponding to that in (7.4), to the same outcome as the quantum evolution governed by the Schrödinger equation with that same external potential V. Due to the presence of the fundamental length ℓ, that evolution has to be formulated in the stochastic phase space framework described in Sec. 3.4 (cf. Prugovečki, 1992, Chapter 4); but, in the sharp-point limit $\ell \to 0$, the outcome of quantum-geometric propagation totally coincides with that provided by the Schrödinger equation with the external gravitational potential V within the orthodox framework for quantum mechanics. In fact, in that case quantum-geometric propagation can be formulated directly in terms of a geometrized version of Feynman propagators for point particles, and the outcome can be proved to coincide with that provided by the conventional Feynman method of path integration in the presence of the external potential V (De Bièvre, 1989a).

Naturally, the fibres of quantum Newton-Cartan bundles are soldered to a Galilei frame bundle over a base Newton-Cartan manifold $\mathbf{M}$, and therefore all light-cone structure is absent. This implies that, albeit in both the nonrelativistic and the relativistic regimes quantum-geometric propagation takes place along all possible geodesic arcs that connect pairs points $x_{n-1} \in \Sigma_{t_{n-1}}$ and $x_n \in \Sigma_{t_n}$ on the inflow and outflow hypersurfaces of each base-segment $\mathbf{S}_n$ of the respective Newton-Cartan and Lorentzian manifolds, in the Newton-Cartan case those arcs always belong to timelike geodesics. Indeed, in the nonrelativistic regime of a Newton-Cartan spacetime, the tangents $\boldsymbol{X} \in T_{x_n}\mathbf{M}$ to timelike geodesics cover, at any point $x_n \in \Sigma_{t_n}$, the entire tangent space $T_{x_n}\mathbf{M}$ (which is isomorphic to $\mathbf{R}^4$) minus a subspace of spacelike directions tangential to Σ_{t_n}, which is isomorphic to $\mathbf{R}^3$ – and therefore is of lower dimensionality than $T_{x_n}\mathbf{M}$. On the other hand, in the Lorentzian case the tangents $\boldsymbol{X} \in T_{x_n}\mathbf{M}$ to timelike geodesics cover only the interior of the light cone at x_n. This means that, as opposed to the strongly causal formulation presented in (Prugovečki, 1992), Sec. 5.4, in the present formulation of quantum-geometric propagation the same subset of directions in $T_{x_n}\mathbf{M}$ is accessible to relativistic quantum-geometric propagation as it would be in the corresponding nonrelativistic quantum-geometric propagation. Nevertheless, a form of general relativistic causality is retained that is weaker than the Einstein causality in CGR, but it is physically meaningful from a quantum-geometric point of view.

This is possible on account of the fact that quantum-geometric propagation is stochastic only by virtue of the nature of the paths along which it proceeds, and by virtue of the macroscopic manifestations of stochastic values for position and momenta in the context of measurement processes, rather than by virtue of associating with those paths an intrinsic probability measure over those paths – as is the case in the context of classical stochastic processes. This is, therefore, an altogether different conceptualization of stochasticity than,

for example, in Nelson's (1967, 1986) stochastic mechanics – which had unsuccessfully[12] attempted to reduce quantum propagation to a stochastic dynamics of classical point particles, whose behavior is influenced by a hypothetical background field. Thus, from the point of view of Nelson's stochastic mechanics, quantum dynamics describes only *ensembles* of quantum particles, whose time evolution is governed by a classical diffusion process, whose mathematical description can be related to a Schrödinger equation (Guerra, 1981; Nelson 1967, 1986). This means that each single quantum particle actually follows a single stochastic path, but that the collective behavior is probabilistically described by an entire family of stochastic paths, with various *additive probabilities* being assigned to each measurable family of paths by some suitable model of a diffusion process. Hence, as stated by Nelson himself, his "interpretation [of quantum interference in the two-slit experiment] flatly contradicts the customary discussion of interference" presented in (Feynman and Hibbs, 1965), since, according to him, "(I) each particle must go through either the top slit or the bottom slit, and (II) the probability of arrival at a given point is the sum of two parts, [i.e.] the probability of arrival coming through the top slit and the probability of arrival coming through the bottom slit." (Nelson, 1986, pp. 92-93).

In contradistinction, no *local* quantum state vector propagates quantum-geometrically over a *single* path, but simultaneously over all the paths geometrodynamically available to it. This means that the quantum entities propagating in this manner embody the classical attributes of "waves" as well as of "particles" in the very nature of their mathematical description, thus removing the dichotomy between "waves" and "particles" displayed by the orthodox conceptualization of quantum mechanics. Indeed, the *proper* state vectors of such quantum entities are represented by *internal* wave functions belonging to quantum fibres $\mathbf{F}_x$. On the other hand, such *proper* state vectors propagate *externally*, i.e., within the base manifold $\mathbf{M}$, along stochastic paths in a manner *formally* analogous to that of classical particles in diffusion processes. However, whereas the concept of superposition of point *particles* located *at* some location $x \in \mathbf{M}$ has absolutely no physical meaning, the superposition of *exciton* states above the *base* location $x \in \mathbf{M}$ has a perfectly clear-cut physical meaning (cf. Sec. 5.6) – which, of course, in classical physics is associated only with the behavior of waves. However, whereas classical waves are normally envisaged as being smoothly spread out over regions in $\mathbf{M}$, as we shall see in the next chapter, the propagation of local quantum state vectors can be envisaged as an ever-going process of *local* vacuum excitation and deexcitation, so that the quantum entities undergoing it can be most appropriately referred to as *geometro-stochastic excitons*. Moreover, since these geometro-stochastic excitons are localized in $\mathbf{M}$ in a geometric rather than a topological sense (cf. Footnote 8), no direct conflict with Einstein causality arises from the assumption that their propagation can follow also the arcs of those spacelike geodesics which are future oriented from the geometrodynamic point of view, discussed in Sec. 2.7.

Thus, the *mathematical* similarities between the classical and quantum notions of

[12] As proved by Wallstrom (1989), Nelson's stochastic mechanics displays very fundamental difficulties which preclude its agreement with orthodox quantum mechanics at a foundational level, since they gives rise to stochastic states that possess absolutely no quantum mechanical counterparts. In the special relativistic realm, there have been attempts by Dohrn and Guerra (1978), and by Guerra and Ruggiero (1978), to adapt Feynman's (1950) idea of relativistic quantum particle propagation to Nelson's (1967, 1986) formulation of stochastic mechanics, but they did not result in a relativistically covariant theory.

stochasticity are somewhat deceptive when the quantum-geometric approach of Sec. 4.6 is adopted: in the classical case one deals with propagators based on probabilistic *mean values*; whereas, in the quantum case, one deals with a genuine *superposition of probability amplitudes* which has no relation to the computation of mean values. It is only in those cases where analytic continuations to the Euclidean regime can be carried out that the thus analytically continued quantum propagators emerge as mean values over the same type of stochastic paths, namely, as mentioned earlier, of continuous paths that can be approximated by broken paths consisting of arcs of geodesics.

On the other hand, no agreement is possible between quantum-geometric propagation and the conventional formulation (cf., e.g., Birell and Davies, 1982; Fulling, 1989) of particle propagation in a curved spacetime manifold since, as we have seen in Sec. 1.5, the latter is *inherently* ambiguous, due to its reliance on topologically global rather than on geometrically local methods. For example, in the case of spin-zero particles, this ambiguity is manifested in the lack of a *physically* unique manner of splitting the space of all solutions of the covariant Klein-Gordon equation with minimal coupling to an external gravitational field, obtained by replacing in (3.6.13) partial derivatives with corresponding covariant derivatives, into subspaces representing positive-energy and negative-energy solutions, that would represent counterparts of those in Sec. 3.6. Thus, this ambiguity is a direct consequence of the adoption of a global point of view that is foreign to the spirit of general relativity, whereby globally defined solutions of a mathematical extrapolation of the Klein-Gordon equation from the special relativistic to the general relativistic regime are viewed as representing quantum states even *before* the problem of evolution of such states has been posed in a physically and mathematically suitable geometrodynamic manner, that would reflect the local nature of the interaction between a quantum "particle" and the external gravitational field. "Therefore, in a general curved spacetime where no timelike Killing field is present, it should not be surprising that there is no natural, universally applicable procedure for defining particles and that, in a given spacetime, different constructions (such as constructions using affine parameter or Killing parameter on the horizon of a black hole) will define different notions of particles." (Wald, 1984, p. 416.)

In addition to its limited scope and its intrinsic ambiguity, the conventional method of formulating the propagation of quantum particles in a curved spacetime gives rise to measurement-theoretical difficulties related to *ex nihilo* particle production.[13] Hence, with considerable justification, it has been concluded by some researchers that: "The study of DeWitt-style particle detectors has exposed the nebulousness of the particle concept and suggests that it should be abandoned completely." (Davies, 1984, p. 76.)

None of the quantum bundles introduced in this chapter share *any* of these problems, since their internal structure is *geometrically* local. As we have seen earlier (cf. Footnote 8), this means that the quantum state vectors constituting these bundles are local in the sense that, as mathematical entities, they exist in fibres *above* the Lorentzian base manifold $(\mathbf{M}, \boldsymbol{g}^{L})$. Hence, the formulation of positive-energy and negative-energy local quantum states, respectively describing geometro-stochastic excitons and antiexcitons, is totally

[13] Cf. (Unruh, 1976), (Unruh and Wald, 1984, 1989). Alternative methods of monitoring quantum phenomena in the presence of gravity can be developed through the use of *quantum* gravimeters based on the behavior of quantum particles in gravitational fields (Anandan and Stodolski, 1983; Anandan, 1984, 1986).

independent of the presence of timelike Killing fields, or of other global features of that manifold. Moreover, the influence of the external gravitational field on these quantum states is geometrically local, since it is exerted only on account of the fact that each quantum fibre is *locally* soldered to the base manifold $\mathbf{M}$ at some point x. This means that the generators of spacetime translations can be represented by the geometrically local operators in (4.11), rather than by the Hamiltonian operators $H(t)$ that are in the conventional approach globally associated to reference hypersurfaces Σ_t, so that they can govern quantum evolution only in the direction of a global timelike Killing field to which the foliation of $\mathbf{M}$ has been adjusted.

Due to this geometrically local nature of quantum-geometric propagation in $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$, the physical interpretation of quantum-geometric wave functions does *not* require any probability measures that are globally defined and normalized over Σ_t, and would therefore have to be conserved during the quantum-geometric evolution of quantum states. This is in keeping with the fact that energy and momentum are defined in CGR locally, by means of a stress-energy tensor – and this is also the case in the quantum-geometric approach to QGR, as will become apparent in the field-theoretical context studied in the next chapter. Hence, no global Hamiltonians are required either as operators for the total energy of the system, or as generators of time translations. In fact, the latter make sense physically only when the base manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ is identifiable with the Minkowski space $(\mathbf{R}^4,\eta)$, and the present quantum-geometric free-fall evolution becomes equivalent to the free evolution governed by a global Hamiltonian, in the manner described in Sec. 3.7.

Of course, a partial identification of this type can be carried out in a generic $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ if there are regions where gravitational curvature effects are negligible, so that the effects of quantum-geometric propagation in free-fall become experimentally indistinguishable from those of the free quantum evolution formulated in Sec. 3.7. The presence of the fundamental length ℓ then gives rise to causal "spillovers" outside the chronological future $I^+(x(t_0))$ of the base points $x(t_0)$ within such regions (cf. Sec. 4.8). The probability amplitudes for the occurances of these "spillovers" are, however, exceedingly small as one moves in spacelike directions beyond proper distances of several orders of magnitude larger than ℓ, and rapidly approach zero as t tends to infinity (cf. Greenwood and Prugovečki, 1984). Consequently, if the fundamental length ℓ is actually of the order of Planck's length, it would be very difficult for experimental tests carried out in a terrestrial setting to detect and measure such minute departures from strict Einstein causality.

Violations of Einstein causality might be, however, much more pronounced in strong gravitational fields. Their most clear-cut manifestation is presented by the phenomenon of black hole evaporation, where they can be pictorially described as a form of "tunnelling" through the event horizon of a black hole, which takes place along those stochastic paths that are approximated by broken piecewise smooth curves incorporating spacelike geodesics that pass through that horizon. This possibility has been actually considered already by Hawking as a physical depiction of particle emission predicted by his well-known derivation of black hole evaporation (Hawking, 1975a; Wald, 1975). In fact, he stated the following in the abstract of a review paper on this subject: "Black holes are often defined as areas from which nothing, not even light, can escape. There is good reason to believe, however, that particles can get out of them by 'tunneling'." (Hawking, 1977, p.34.)

Hence, if black hole radiation is actually observed, that would represent a spectacular

experimental proof that, at the quantum level, *strict* Einstein causality is violated in nature. However, such violations are actually very common even in conventional relativistic quantum mechanics and quantum field theory in Minkowski space. We shall describe some of the most striking examples in the next section.

4.8. Relativistic Causality of Classical and Quantum Propagation

At the general philosophical level, the meaning and manifestations of causality are still a subject of debate and analysis (cf., e.g., Russell, 1948; Bunge, 1970). However, in classical physics, the scope and meaning of this term has been considerably restricted already in the context of Newtonian mechanics, where the "causal" conceptualization of the time evolution of a classical system provides a very sharp contrast to the "teleological" conceptualization of the same time evolution (cf., e.g., Barrow and Tipler, 1986).

The "causal" conceptualization is reflected in the common attitude that the presence of various fields "causes" classical particles to move along trajectories determined by laws of motion, regardless of whether those laws are given in the original Newtonian form, or in their Hamiltonian form. Furthermore, despite the manifest invariance of those laws under time reversals (which assign to each solution of an equation of motion another solution by means of the substitution $t \mapsto -t$), all solutions of the equations of motion are deemed to be future-oriented in time, so that by classical Newtonian causality, "present" conditions and circumstances are deemed to determine future events, without influencing past events.

The "teleological" conceptualization of classical time evolution is in evidence only in variational approaches to classical propagation. The epistemic origins of such approaches lie in the action principles first formulated by Maupertuis in the mid-seventeenth century on the basis of *teleological* arguments rooted in a doctrine of "final causes" (cf. Barrow and Tipler, 1986, p. 66). Thus, according to the teleological conceptualization (which can be traced in Greek antiquity to Aristotle's philosophy – cf., e.g., Russell, 1945), a classical particle supposedly *chooses* to follow, amongst all kinematically acceptable paths, the one which is, within the Newtonian framework, dynamically correct *because* its *purpose* is to minimize its own action within a given external field. In other words, the motion of the particle is influenced not only by its "present" circumstances, but also by *potential* future events, which it somehow "senses" as it makes its "decision" as to which path to follow.

It should be noted for future reference that, if one does not adopt such a teleological attitude, then there are no compelling epistemological grounds for considering variationally-based action principles as being fundamental. This becomes most evident in the Lagrangian approach to quantum mechanics originally formulated by Dirac (1933), which, in the form later developed by Feynman (1948), has removed all the teleological undertones from action principles. In fact, after Feynman was first exposed to the action principles of classical mechanics, he found their teleological undertones to be physically totally unacceptable, despite some of their obvious computational advantages (cf. Gleick, 1992, pp. 60 and 88). Subsequently, his formulation of path integration grew out of his attempt to remove all these teleological features from action principles, by arguing that a *quantum* particle actually follows *all* the paths available to it, but that, on account of the superposition principle, only the paths that are adjacent to the classical ones contribute significantly to the outcome of that

propagation. And indeed, as we saw in Sec. 3.5, in Feynman's path integral formulation of quantum mechanics, *all* kinematically acceptable broken paths are used in the computation of a path integral for a quantum propagator.

Actually, even in CGR, variationally based action principles historically emerge as secondary. Thus, the first CGR action principle was independently discovered by Einstein and Hilbert in 1915, only *after* the geometric foundations of the theory had been already worked out by Einstein over a period of many years, and the Palatini variant followed four years later – cf. Chapter 21 in (Misner *et al.*, 1973), and the references cited therein. Moreover, in three spacetime dimensions, the aforementioned Einstein-Hilbert action turns out to be equal to the Chern-Simons action for the Poincaré group ISO(2,1) (Witten, 1989). Taken in conjunction with these facts, the recent formulation by Witten (1988) of topological quantum field theories with no classical analogues provides a strong indication that actions based on field Lagrangians are only secondary entities, actually determined by symmetries (Montano and Sonnenschein, 1989).

All this indicates that in general relativity predominance should be assigned to those formulations of causality that are based on *geometric* principles that embody conservation laws, rather than on those which embody teleological laws in the guise of variational arguments. For example, it is perfectly true that, by the geodesic postulate, a classical particle in free fall follows a timelike geodesic in a Lorentzian spacetime manifold, and that therefore, amongst all smooth curves connecting any two given points on that geodesic, its actual world line maximizes the length between those two points – which has the physical meaning of lapsed proper time. This means that variational methods based on the teleological principle that a particle chooses, amongst all the trajectories potentially available to it, the one of the longest or the shortest length, indeed yield the same result as the one obtained by regarding the same geodesic as an autoparallel curve for the Levi-Civita connection. However, if the adopted connection on the same Lorentzian spacetime manifold were of a generic Riemann-Cartan type, the "autoparallel" and the "extremal" curves would no longer coincide, and the trajectories of a test particles in free fall that would be obtained from equations of motion based on the conservation of energy and momentum would *not* coincide with either one of these two types of curves (cf. Hehl *et al.*, 1976, p. 398). Thus, the causal and the teleological points of view would then lead to conflicting results, and the causal view would be supported by the best established conservation law in Nature, namely the one for energy and 3-momentum; whereas, the variationally formulated teleological approach would violate that conservation law.

In view of all this, in the quantum-geometric formulations of quantum field propagation, Lagrangians are not *postulated* entities but, as we shall see in the next chapter, they are *derived* from expressions for stress-energy tensors, that embody the law of conservation of energy and momentum in a local form. This fact, in conjunction with the complete Poincaré gauge invariance of the present framework, will reveal a more diverse Lagrangian structure than the one present in conventional quantum field theory in Minkowski space, where only ordinary Poincaré invariance under the change of global Lorentz frames is assumed.

To analyze the meaning and significance of relativistic causality of propagation in the classical and quantum context, let us first recall that in the context of classical fields, the

universally accepted concept of general relativistic causality is that of *Einstein causality*. According to that formulation of causality, no classical field in a classical spacetime manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ can propagate outside the causal future $J^+(x_0)$ of any point $x_0 \in \mathbf{M}$, and no massive classical test particle can propagate outside its chronological future $I^+(x_0)$. In this context, the *chronological future* $I^+(x_0)$ of a point x_0 in a time orientable Lorentzian manifold $\mathbf{M}$ consists of all points $x \in \mathbf{M}$ which can be connected with x_0 by means of a future-directed timelike smooth curve – i.e., a curve whose tangent at any point lies within the interior of the future light cone at that point. Similarly, the *causal future* $J^+(x_0)$ of the same point x_0 consists of all points $x \in \mathbf{M}$ which can be connected with x_0 by means of a future-directed causal smooth curve – i.e., a curve whose tangent at any point lies within the interior of the future light cone at that point, or on the future light cone itself. The definitions of the *chronological past* $I^-(x_0)$ and of the *causal past* $J^-(x_0)$ of the point x_0 can be then obtained by replacing in the above respective definitions the term "future" with the term "past". Note should be also taken of the fact that, if $x \in J^+(x_0)$ but $x \notin I^+(x_0)$, then x and x_0 can be connected only with null causal curves, and the same holds true if $x \in J^-(x_0)$ but $x \notin I^-(x_0)$ (cf., e.g., Wald, 1984, p. 191), so that massive *classical* particles indeed cannot propagate between such points.

Let us retain this terminology and notation also in the special relativistic context, where $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ is identified with the Minkowski space $M^4 = (\mathbf{R}^4, \boldsymbol{\eta})$, so that $I^+(x_0) \cup I^-(x_0)$ coincides with the interior of the light cone at a point x_0. If we view the special relativistic Klein-Gordon equation in (3.6.13) as an equation for a classical field $\varphi_{\mathrm{cl}}(x)$ in $(\mathbf{R}^4, \boldsymbol{\eta})$, then its relativistically invariant solutions that satisfy Einstein causality for initial data given along a maximal spacelike hypersurface σ_{t_0} in that Minkowski space are provided by the following integrals over that hypersurface,

$$\varphi_{\mathrm{cl}}(x) = \int \Delta_R(x-x')\, \overleftrightarrow{\partial}'_\mu\, \varphi_{\mathrm{cl}}(x')\, d\sigma^\mu(x') \ , \qquad \partial'_\mu := \partial/\partial x'_\mu \ , \tag{8.1}$$

which incorporate the retarded Green function $\Delta_R(x-x')$ for the Klein-Gordon equation:

$$\Delta_R(x-x') = -\theta(x^0 - x'^0)\Delta(x-x') \ , \qquad \Delta(x) = \Delta^{(+)}(x) + \Delta^{(-)}(x) \ , \tag{8.2a}$$

$$\Delta^{(\pm)}(x-x') = \mp i \int_{V_m^+} \exp[\mp i(x-x')\cdot k]\, d\Omega_m(k) \ , \qquad x, x' \in \mathbf{R}^4 \ . \tag{8.2b}$$

However, the positive-energy solutions (3.6.10) of the Klein-Gordon equation in (3.6.13) do not belong to this class of causal solutions, since they can be expressed by means of the propagator $i\Delta^{(+)}(x-x')$ in (8.2b) in the following manner,

$$\hat{\varphi}(x) = -\int \Delta^{(+)}(x-x')\, \overleftrightarrow{\partial}'_\mu\, \hat{\varphi}(x')\, d\sigma^\mu(x') \ , \tag{8.3}$$

and this propagator does not vanish outside the causal future $J^+(x')$ of any point $x' \in M^4$; rather, it decreases exponentially in all spacelike directions – cf., e.g., (Itzykson and Zuber, 1980, Sec. 1-3-1), or (Schweber, 1961, Sec. 7c).

The same is true of the Feynman propagator in (3.8.7a), since that propagator can be expressed as follows (cf. Itzykson and Zuber, 1980, p. 34; Schweber, 1960, p. 444):

$$K_F(x-x') = i\Delta_F(x-x') \ , \qquad i\Delta_F(x) = \theta(x^0)\Delta^{(+)}(x) - \theta(-x^0)\Delta^{(-)}(x) \ . \tag{8.4}$$

The Feynman propagator is therefore seen *not* to vanish outside the causal future $J^+(x')$ or the causal past $J^-(x')$ of x' – despite the fact that it is sometimes referred to as "causal," since it serves as the basis of perturbative computations in conventional "local" quantum field theory, which is *presumed* to be "microcausal" on account of the fact that conventional quantum fields are assumed to satisfy a "local commutativity" postulate. However, as we shall see in Chapter 5, the local commutativity of such fields has not, first of all, actually been proven to be related to any quantum version of *strict* Einstein causality, and, second, all the indicators gathered from studies in constructive quantum field theory point to the fact that consistent models of quantum fields interacting in the Minkowski space and obeying the local commutativity postulate actually do not exist in a mathematically rigorous sense.

All the above types of violations of Einstein causality become even more pronounced and universal in the case of the relativistically invariant Bargmann-Wigner (1948) equations

$$[\beta^\mu \partial_\mu + m - ie\beta^\mu A_\mu(x)]\psi(x) = 0 \ , \tag{8.5}$$

for which the prototype is the Dirac equation – which corresponds to the choice $\beta^\mu = -i\gamma^\mu$ in (2.18). These equations provide generalizations of the Dirac equation to all the spin values that are equal to or larger than one (cf., e.g., Barut and Raczka, 1986). Hence, their solutions of the form

$$\psi(x) \ = \ \psi^{\text{in}}(x) + ie\int S_R(x,x';A)\beta^\mu A_\mu(x')\,\psi(x')\,d^4x' \ , \tag{8.6}$$

are meant to describe the behavior of quantum particles or fields with such spins, which are minimally coupled to electromagnetic or other types of external fields described by 4-potentials. They came under close scrutiny (Wightman, 1971, 1973) after the discovery by Velo and Zwanziger (1969) that the retarded Green functions in (8.6), which satisfy

$$[\beta^\mu \partial_\mu + m - ie\beta^\mu A_\mu(x)]S_R(x,x';A) = \delta^4(x-x')I \ , \tag{8.7a}$$

$$S_R(x,x';A) = 0 \ , \qquad x^0 < x'^0 \ , \qquad x,x' \in \mathbf{R}^4 \ , \tag{8.7b}$$

and were therefore previously *presumed* to vanish when x and x' are spacelike separated in M^4, are not actually generically Einstein-causal in the presence of external fields; rather, these Green functions, which are "causal" at the perturbative level, merely obey a kind of "weak retardedness" – defined to mean that, as one proceeds to infinity in any spacelike direction, they approach zero faster than some power of $(x-x')^2$. Thus, it turns out that the non-perturbative "solutions to [these] familiar spin equations propagate acausally, whereas the perturbative solution is causal in every order" (Velo and Zwanziger, 1971, p. 9). Consequently, these mathematically rigorous studies reveal the basic unreliability of conclusions reached exclusively through the use of perturbative methods. In Wightman's words, "this is a kind of instability of relativistic wave equations not anticipated by earlier work by physicists: *the particles described by the wave equation move faster than light in regions where the external perturbation is nonvanishing*." (Wightman, 1973, p. 454.)

Similar types of problems are encountered all by the treatments of the Stueckelberg (1941) equation with minimal coupling to an external electromagnetic field,

$$i\partial_\tau \varphi_\tau(x) = \tfrac{1}{2M}\eta^{\mu\nu}(\partial_\mu - ieA_\mu(x))(\partial_\nu - ieA_\nu(x))\varphi_\tau(x) \quad , \tag{8.8}$$

which was used by Feynman (1950) during his treatment of solutions of the Klein-Gordon equation by path integral method. Some of its advocates (Horwitz and Piron, 1973; Horwitz 1984, 1992; Horwitz *et al.*, 1988; Saad *et al.*, 1989) view this equation as governing the propagation of a relativistic quantum particle that possesses no precise rest mass, but displays an unsharp mass spectrum around the value M. In so doing, they allow the possibility of quantum propagation along worldlines which become spacelike as they undergo the transition from the "forward" to a "backward" orientation in Minkowski space (cf., e.g., Arshansky *et al.*, 1983, p. 1172). Such departures from Einstein causality is required by them to represent the transition from particle propagation to antiparticle propagation that leads to pair annihilation. This gives rise, however, to the question as to the physical meaning of the non-Minkowski dimension required by the variable τ in (8.8) – which in that context is dubbed "historical time" (Horwitz, 1984, 1992). It turns out (Horwitz *et al.*, 1988, 1991) that as long as one restricts oneself to timelike world lines, one can interpret τ as the proper time of the center of the wave packet $\varphi_\tau(x)$ that represents a quantum particle in this approach. However, that kind of interpretation of τ obviously cannot be retained along the null or spacelike segments of those worldlines.

In quantum-geometric propagation the role of τ is played by the affine parameter of the geodesic arcs that approximate a given stochastic path. As is well known, the affine parameter is well defined, modulo a multiplicative factor and an additive term (cf., e.g., Spivak, 1979), for the geodesics of any connection in any manifold $\mathbf{M}$. Hence, in the quantum-geometric context τ is defined, modulo those two constants, by the Levi-Civita connection in the Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$, which constitutes the base manifold of quantum bundles. In turn, those constants can be fixed by the requirement that, for any particular choice of initial-data reference hypersurface Σ_{t_0} in the treatment of the quantum-geometric evolution described in (7.1), τ should coincide with proper time t_0 for those timelike geodesics that are normal to Σ_{t_0}, and that it should consecutively assume the values t_n along the respective hypersurfaces Σ_{t_n}, as the quantum-geometric evolution is followed along all the timelike geodesics that are normal to those reference hypersurfaces. The continuity properties of the paths of quantum-geometric propagation then uniquely determine the values of τ along all other paths in such a manner that τ represents proper time and proper spatial distance along all the smooth geodesic arcs of broken paths which are timelike and spacelike, respectively. Thus, in the present semi-classical context, the metric field $\boldsymbol{g}^{\mathrm{L}}$ is treated in the manner in which all classical potentials corresponding to nongravitational forces are treated in orthodox quantum theory; namely, as a potential barrier which can be penetrated, thus giving rise to "tunnel effects," whereby a geometrostochastic exciton "tunnels" through those base spacetime regions which are absolutely forbidden in the classical regime. Since the probability amplitudes for such "acausal" propagation vanish rapidly as one moves away from the light cone in those regions where $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ is nearly flat (Greenwood and Prugovečki, 1984), no macroscopically observable conflict with Einstein causality arises in the presence of weak gravitational fields where the linearized gravity approximation can be employed – such as is the case under terrestrial conditions.

According to the geometrodynamic point of view (Wheeler, 1962, 1968), in reality the very "object which is central to all of classical general relativity, the four-dimensional spacetime geometry, simply does not exist, except in a classical approximation" (Misner *et al.*, 1973, p. 1183). In other words: "The dynamic object is not spacetime. It is space. The configuration of space changes with time." (*Ibid.*, p. 1181.) Consequently, although the quantum-geometric approach to quantum gravity adopted in Chapter 8 does not rely on the superspaces of 3-geometries postulated by the geometrodynamic approach, this fundamental physical and epistemological idea, which can be traced to Dirac's (1950-1959) canonical approach CGR, is central to that approach.

And indeed, *classical* general relativity postulates the possibility of a description of the actual spacetime in existence by means of an equivalence class of diffeomorphic four-dimensional Lorentzian manifolds $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$, which, despite the dynamical nature of the continuous mutual interaction between "matter" and "geometry" that is reflected by the Einstein field equations in (2.7.1), already *exists* in its entirety. However, in *reality* the geometrodynamic evolution embodied in the canonical formulation (2.7.18) of the Einstein equations (2.7.1) is still taking place, and relativistic observers are *actively partaking* in that evolution. Thus, the CGR point of view is epistemologically compatible only with a *strict global determinism*, that envisages the entire physical spacetime as *predetermined in its entirety* at the very instant of birth of our universe. Since matter and geometry interact, that means that the evolutions of all matter fields, as well as of all non-gravitational radiation fields, also had to be determined, in their minutest details, at the instant of creation of the universe. Consequently, all the fundamental quantum epistemological notions, which are intrinsically based on the *intervention* of observers in the quantum measurement process, and which are therefore fundamentally stochastic in nature, have no place at all in such a conceptualization. In fact, the very act of "measurement", as opposed to mere "observation" in the sense of a passive registration of physical events, is *not consistent* with such a strictly deterministic point of view, namely, a determinism which postulates the actual existence, *in toto*, of a unique classical spacetime, whose geometry is nevertheless dynamically influenced by all states of motion of the matter present in it, including that which in part constitutes the equipment of "observers."

For that reason, the semi-classical point of view adopted in the present chapter, as well as in the next three chapters that deal with a geometric version of quantum field theory in curved spacetime, is only an *approximation* which cannot, in the ultimate analysis, be physically self-consistent. This lack of physical self-consistency is reflected by the fact that in this semi-classical regime the diffeomorphism invariance of CGR is absent, since a global base manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ is chosen from the outset, so that the evolution of the quantum fields describing matter and nongravitational radiation has neither the opportunity, nor the means, of influencing the Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ of that base manifold. Thus, in the semi-classical regime one is not dealing with a true quantum-geometric evolution, that would represent a quantum counterpart of the geometrodynamic evolution of a classical spacetime, but merely with a foliation of an already existing Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$. As a consequence, the evolution of all quantum fields is dependent on the mode of foliation of $\mathbf{M}$ into reference hypersurfaces. Hence, although Poincaré gauge invariance is retained as a gauge invariance of the first kind, which is *geometrically* local (cf. Footnote 8), no

form of diffeomorphism invariance, which is intrinsically global, can be present in the semi-classical regime.

On the other hand, according to the formulation of the *quantum* spacetime presented in Chapter 8, such a spacetime is mathematically described by a *superfibre* bundle that constitutes an infinite-dimensional supermanifold. Such a supermanifold can be foliated in an infinity of ways that give rise to physically equivalent quantum-geometric developments describing a universe in the process of evolution, in which "matter" interacts dynamically and *creatively* with geometry. This results in a *fundamental* irreversibility of quantum-geometric evolution, which is in agreement with long-standing conjectures by Prigogine (1980, 1989) about the role of irreversible dynamics in Nature. It is also in agreement with the long-standing conjecture of Penrose (1987) that such irreversibility is essential to the consistent integration of quantum theory and general relativity – as reflected by his emphatic declaration that "*the true quantum gravity must be a time-asymmetrical theory.*" (Penrose, 1987, p.37.) It is, however, in total contrast with the situation in canonical quantum gravity, where "it is generally believed that 'time' is hidden among the canonical variables of general relativity and must be isolated prior to the imposition of the inner product, . . . [but that] for the physical interpretation of the framework and particularly for the quantum measurement theory, . . . it may suffice to have only an approximate notion of time." (Ashtekar, 1991b, p. 170.)

The reason why "'time' is hidden" in the canonical approach to CGR is that, as can be seen from the outline provided in Sec. 2.7, the presence of local Lorentz frames, which is so essential to the concept of "time" in general relativity, remains totally hidden in the conventional formulation of that approach. Since the values of the ADM lapse and shift functions, as well as of the ADM *parameter t*, can be changed by means of diffeomorphisms, there is no *intrinsic* concept of time in that formulation. This means that "the familiar notions of time and evolution can be introduced only *after* one has solved the field equations and singled out a solution" (*ibid.*, p. 194). But in quantum gravity that is not possible, since the end result of evolution is obviously not a classical spacetime. As a consequence, in the canonical framework the debate over "The Issue of Time in Quantum Gravity" is still going on unabated (cf. Ashtekar and Stachel, 1991, pp. 123-296).

On the other hand, in quantum-geometric gravity "time" is embedded into the very fabric of a *quantum* spacetime by virtue of the fact that the concept of such a spacetime is intrinsically based on the notion of *quantum* frame, and that all such frames automatically keep track of a *natural* time, in accordance with a quantum-geometric adaptation of de Broglie's epistemic idea that originated wave mechanics. In the quantum-geometric framework, the main departure from de Broglie's (1923, 1924) original idea is that, as we saw in Sec. 3.7, in the presence of the fundamental length ℓ this natural time is actually a *mean* proper time, so that the fluctuations in the quantum metrics, which "blur" the light-cone structure (cf., e.g., Pauli, 1956), take place in timelike as well as spacelike directions. This mean proper time manifests itself during the parallel transport of the proper state vectors of geometro-stochastic excitons that gives rise to their quantum-geometric propagation, as it keeps track of the maximum value of the lapsed proper time between the reference hypersurfaces associated with that propagation.

In this context it has to be stressed again that, after the gravitational field is quantized,

the mathematical choice of reference hypersurfaces is no longer dictated by a choice of a single Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$, as is the case in the semi-classical regime adopted in this and the next three chapters. Indeed, as we shall see in Chapter 8, any quantum-geometric evolution that incorporates quantum gravity gives rise to a variety of base Lorentzian manifolds $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$, provided by the Lorentzian metrics that are the mean values of a *quantum* gravitational field for various superlocal states. These superlocal states are interrelated by a quantum gravitational gauge group which incorporates diffeomorphism invariance, and can therefore interrelate various modes of foliation of a given manifold $\mathbf{M}$. Their geometrodynamic evolutions provide the physical basis for the formulation of general relativistic quantum causality which stipulates that the time-ordering of those evolutions is to be obeyed by all quantum processes. Thus, propagation along *all* spacelike geodesic arcs that conform to that time-ordering is allowed by such a quantum-geometric causality criterion. On the other hand, the time-reversal of such propagation is ruled out, thus reflecting the fact that cause always precedes effect. Hence, any *formal* propagation of geometro-stochastic excitons along portions of geodesics that are oriented "backward" in time has to be interpreted as actually representing a propagation of *anti*excitons that takes place "forward" in time.

The treatment of such exciton-antiexciton propagation is best suited to quantum field theoretical methods, to whose formulation within the quantum-geometric framework we turn in the next chapter.

Chapter 5

Massive Quantum-Geometric Boson Fields

The construction of Fock quantum bundles from some of the tensorial quantum bundles introduced in the preceding chapter mediates the formulation of quantum frame fields. These fields generate second-quantized boson frames within the Fock fibres of those bundles. This leads to a quantum-geometric framework for quantum field theory (QFT) over curved spacetime manifolds which, in contradistinction to the conventional QFT framework in such manifolds, does not require the presence of timelike Killing fields, and can be actually formulated for arbitrary geometrodynamic evolutions giving rise to those manifolds. Moreover, the propagation of quantum-geometric fields over such base Lorentzian manifolds is governed by a parallel transport that abides by the strong equivalence principle, so that it does not produce any of the Bogolubov transformations that give rise to *ex nihilo* pair creation in the conventional framework.[1] This quantum-geometric propagation incorporates the principle of local conservation of energy-momentum, and the Lagrangian densities in its action integrals emerge as geometrically derived quantities.

5.1. Microcausality vs. Local Commutativity of Quantum Fields

As we have seen from the brief historical review in Secs. 1.3 and 1.4, the framework for quantum field theory (QFT) founded by Dirac (1927) was plagued by divergencies from the outset. During the course of many decades following the foundation of QFT, Dirac himself, as well as Bohr, Born, Heisenberg, Pauli, and other founders of quantum theory, called for revisions of the conventional ideas on spacetime structure in order to cope with those divergencies and related problems. One of the central ideas proposed as necessary for such a modification was that of a fundamental length. As mentioned in Chapter 1, that idea was championed very strongly by Heisenberg (1938, 1943) and Born (1938, 1949), but it had been publicly supported also by Dirac (1949). On the other hand, Dirac's (1949) most straightforward proposal as to how to incorporate a fundamental length in QFT later proved unfeasible, due to the very restrictive nature of the postulate of the "local commutativity" of quantum fields that describe bosons, and of the corresponding postulate regarding the "local anticommutativity" of quantum fields describing fermions.

This local (anti)commutativity *postulate* was nevertheless introduced and retained over the years due to a widespread *belief* that it could be identified with the physical idea of "microcausality," which had emerged during the development of S-matrix theory in the

[1] Cf. Sec. 1.5 for a brief review. An in-depth analysis of this and other problems encountered by the conventional formulation of quantum field theory in curved spacetime can be found in (Prugovečki, 1992), Secs. 7.1-7.3.

1950s. However, a thoroughly researched history of this subject arrives at the following conclusion: "The *exact* meaning of this commutator [or anticommutator] relation is not clear. As we mentioned earlier, this is underscored in the 1957 LSZ paper in which an operator $A(x)$ satisfying $[A(x),A(y)] = 0$ for spacelike separations [of x and y] is *defined* as a *causal* operator – without discussing here (or anywhere in that paper) the physical interpretation (Lehmann, Symanzik, and Zimmermann, 1957, p. 323). It seems *plausible* that it may have something to do with first-signal-principle [i.e., Einstein causality], but the equivalence of these two definitions of causality is not established. The main justification of this new criterion for a causal theory is that it produces a desired result (dispersion relations), while not obviously conflicting with a more direct notion of causality." (Cushing, 1990, p. 214.)

We shall see shortly that not only have there not been, during all the decades since 1957, any measurement-theoretical developments in conventional quantum field theory that would provide any evidence in favor of that equivalence, but that all foundational evidence indicates that "local (anti)commutativity" is merely a convenient *mathematical* assumption, which is actually unrelated to any *physically* well-grounded notion of microcausality. However, even *if* the physical interpretation of local (anti)commutativity as a reflection of microcausality were correct, "[m]isgivings could arise concerning the experimental justification of this postulate: we have no special reason for supposing that the measurement of a component of a Hermitian field at some point has no influence on the value of the components of this field at another point separated from the first point by an arbitrarily small spacelike interval." (Bogolubov *et al.*, 1990, p. 373.) It is only after several other key *assumptions* are made (such as the existence of a *global* vacuum state), that it turns out that the property of local (anti)commutativity can be derived from the apparently much weaker *postulate* that quantum fields (anti)commute only at sufficiently large spacelike separations. "It is therefore no surprise that the attempts to introduce 'non-locality in the small' at the same time require a rejection of some other requirements of the Wightman formalism, for example, 'renormalizability'." (*Ibid.*, p. 373.)

The mathematically rigorous framework for QFT that has become best known as the Wightman axiomatic formalism (cf. Streater and Wightman, 1964; Bogolubov *et al.*, 1975, 1990) has emerged from an early attempt of Wightman (1956) to cope with the very basic *mathematical fact* that, although the framework of conventional QFT is routinely formulated in terms of quantum fields defined at individual points x in Minkowski space, such fields *do not exist as bona fide operators* in a Hilbert space (cf. Bogolubov *et al.*, 1975, Sec. 10.4). This basic observation is true of *all* conventional quantum fields, and its neglect represents the main mathematical cause of the divergencies exhibited by conventionally formulated quantum field theoretical interactions. However, this very plain fact is clearly evident already in the case of free quantum fields defined in a Fock space.

To see that, let us consider the simple example of the free scalar quantum field that describes neutral quantum particles of rest mass m and zero spin, which have no distinct antiparticles. The second-quantization procedure requires the introduction of the Fock space

$$\tilde{\mathcal{F}} = \bigoplus_{n=0}^{\infty} \tilde{\mathcal{F}}_n \ , \qquad \tilde{\mathcal{F}}_n = \tilde{\mathcal{F}}_1 \underset{S}{\otimes} \cdots \underset{S}{\otimes} \tilde{\mathcal{F}}_1 \ , \qquad \tilde{\mathcal{F}}_1 \cong L^2(\boldsymbol{V}_m^+, d\Omega_m) \ , \tag{1.1}$$

which is the direct sum of the symmetric tensor products of $n = 1,2,...$ single-particle Hilbert space in (3.6.4), and of the vacuum sector for $n = 0$. The formulation of a free quantum field can be achieved within such a Fock space (cf., e.g., Bogolubov *et al*, 1975, 1990) by first constructing the creation and annihilation operators for a quantum particle in a generic state f represented by a wave function in (3.6.10). These creation and annihilation operators can be defined in an unambiguous manner by the following respective actions on the momentum space representatives of the n-particle wave functions belonging to the state vectors of the Fock space in (1.1):

$$\left(a^*(f)\tilde{\Psi}_n\right)_{n+1}(k_1,...,k_{n+1}) = (n+1)^{-1/2}\sum_{j=1}^{n+1}\tilde{f}(k_j)\,\tilde{\Psi}_n(k_1,...,k_{j-1},k_{j+1},...,k_{n+1}), \quad (1.2a)$$

$$\left(a(f)\tilde{\Psi}_n\right)_{n-1}(k_1,...,k_{n-1}) = n^{1/2}\int_{V_m^+}\tilde{f}^*(k)\,\tilde{\Psi}_n(k,k_1,...,k_{n-1})\,d\Omega_m(k)\ . \quad (1.2b)$$

Indeed, this definition can be extended by linearity to all finite linear combinations of various n-particle states, and produces operators that are unbounded but closable, so that they are well-defined on maximal dense domains of the Fock space defined in (1.1).

The conventional scalar quantum fields are then usually defined as equal to

$$\phi(x) = \sum_{\alpha=1}^{\infty}[f_\alpha(x)a(f_\alpha) + f_\alpha^*(x)a^*(f_\alpha)]\ , \quad (1.3)$$

for any choice of orthonormal basis $\{f_1, f_2, ...\}$ in the single-particle Hilbert space (3.6.4). However, the above operators are actually *not* mathematically well-defined by the infinite series in (1.3), since the second part of that series diverges. In fact, as we mentioned earlier, it has been *proved* (Bogolubov *et al*, 1975, Sec. 10.4) that no "local" quantum field operators are well-defined at the points x of Minkowski space M^4, so that some form of "smearing" with test functions is required to make the expression in (1.3) well-defined.

Wightman (1956) had originally proposed a "smearing" with test functions f from the Schwartz space $\mathcal{S}(\mathbf{R}^4)$ of all functions which are infinitely many times differentiable, and which vanish at infinity in $\mathbf{R}^4$, together with all their partial derivatives, faster than any polynomial (cf. Bogolubov *et al.*, 1975). Indeed, if such "smearings" are performed on each term in the series in (1.3) prior to taking the limit that is implicit in computing the sum of such infinite series, then the "smeared" quantum field

$$\phi(f) = \int_{\mathbf{M}} f(x)\phi(x)d^4x := \phi^{(-)}(f) + \phi^{(+)}(f)\ , \quad (1.4)$$

is well-defined in the Fock space (1.1), since both of the following weak limits,[2]

$$\phi^{(-)}(f) = \text{w-}\lim_{n\to\infty}\int d^4x\ f(x)\sum_{\alpha=1}^{n} f_\alpha(x)\ a(f_\alpha)\ , \quad (1.5a)$$

$$\phi^{(+)}(f) = \text{w-}\lim_{n\to\infty}\int d^4x\ f(x)\sum_{\alpha=1}^{n} f_\alpha^*(x)a^*(f_\alpha)\ , \quad (1.5b)$$

[2] The weak limit of a sequence of operators $A_1, A_2,...$ in a Hilbert space is determined by the weak limits of the corresponding sequences $A_1 f, A_2 f,...$ for all vectors f in their domains. In turn, a vector h is the weak limit of a sequence of vectors $h_1, h_2,...$ in a Hilbert space if $\langle g|h\rangle$ is the limit of $\langle g|h_1\rangle, \langle g|h_2\rangle,...$ for *any* vector g in that space. The existence of a strong limit implies that of the corresponding weak limit, but the converse is not true – cf. (Prugovečki, 1981), pp. 229-231, for further discussion of this and other types of limits in Hilbert spaces.

exist on a dense set in that Fock space, and determine there an operator-valued distribution in $f \in \mathcal{S}(\mathbf{R}^4)$ – sometimes referred to as a "smeared" quantum field.

Despite the mathematical fact that no relativistic quantum fields are well-defined at single points x in Minkowski space as *bona fide* operators in Fock space, some other basic expressions, such as the following ones for the relativistic total 4-momentum operators and the relativistic total angular momentum operators,

$$\hat{P}_\mu = \int :T_{\mu\nu}[\phi(x)]: d\sigma^\nu(x) \ , \qquad \hat{M}_{\mu\nu} = \int :x_\mu T_{\nu\lambda} - x_\nu T_{\mu\lambda} : d\sigma^\lambda(x) \ , \tag{1.6}$$

can be formally expressed by inserting such quantum fields at a point into the classical version of the stress-energy tensor for a scalar field,

$$T_{\mu\nu} = \eta_{\mu\lambda} \frac{\partial \mathcal{L}}{\partial \phi_{,\lambda}} \phi_{,\nu} - \eta_{\mu\nu} \mathcal{L} = \phi_{,\mu} \phi_{,\nu} + \tfrac{1}{2} \eta_{\mu\nu} \left(m^2 \phi^2 - \eta^{\kappa\lambda} \phi_{,\kappa} \phi_{,\lambda} \right) \ , \tag{1.7}$$

$$\mathcal{L}[\phi, \phi_{,\lambda}] = \tfrac{1}{2} \left(\eta^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - m^2 \phi^2 \right) \ , \qquad \phi_{,\lambda} := \partial \phi / \partial x^\lambda \ , \tag{1.8}$$

and then using the following conventional definition of normal ordering:

$$:\phi^{(-)}(x) \phi^{(+)}(x): \equiv :\phi^{(+)}(x) \phi^{(-)}(x): \equiv \phi^{(+)}(x) \phi^{(-)}(x) \ , \tag{1.9}$$

We observe that the operators in (1.6) represent also the infinitesimal generators of spacetime translations and of Lorentz transformations for the unitary representation of the Poincaré group to which the single-particle representation in (3.6.3) gives rise in the Fock space defined in (1.1) from single-particle Hilbert spaces.

At first sight, all this makes plausible the conjecture that the test functions f used in the formulation of smeared quantum fields are related to the well-known Bohr and Rosenfeld (1933, 1950) results on quantum field localization, which proved that no such localization can be established with *classical* test bodies over *sharply* delineated regions R of spacetime. This has led to the postulation of C^*-algebras of "observables" $\mathcal{A}(R)$ (Haag and Kastler, 1964; Haag, 1992), which in the Wightman formalism would be generated by all the values assumed by quantum fields smeared with test functions with supports within such spacetime regions R. However, no interpretation of the operators within these C^*-algebras as representing *measurable* quantities was ever provided. Rather, the essence of this entire approach to QFT "observables" is very clearly and succinctly summarized in the following paragraph, whose validity has remained unchanged since its publication in 1975:

"Haag's algebraic formulation of quantum field theory is based on the general idea that, since any physical measurement is performed in a finite space-time domain, the whole theory must be formulated in terms of local observables. It is in the spirit of quantum theory to assume that all the possible operations in a given space-time region generate an algebra. In spite of the apparent simplicity of this idea it is so unusual from the point of view of the conventional formalism of quantum field theory that up to now it has not been carried out in a pure form. The difficulty stems from the fact that the operators $U(a,\Lambda)$ of the representation of the Poincaré group and their generators – for example, the energy and the momentum (or their spectral projections) – are not local observables. The same is true

of all conserved quantities such as the total angular momentum and the electric charge of the system." (Bogolubov *et al.*, 1975, pp. 595-596.)

Thus, the 4-momentum and angular momentum observables in (1.6) are *not* "local observables," although they are essential to conventional QFT. However, it was obvious from the start that, since the particle number as well as the energy and momentum were amongst the most basic of observables in the traditional quantum mechanical sense, they had to be somehow represented within the C^*-algebra of "observables" $\mathcal{A}(R)$ for bounded spacetime regions R of arbitrarily small dimensions if the idea that "the whole [QFT] theory must be formulated in terms of local observables" had any physical validity. On the other hand, already in the case of the free field in (1.3), the measurement of the particle number and of the momenta *within* R must entail knowledge of the simultaneous values of the position and momenta of the particles created by that field in a given Fock state. Hence, foundational questions related to the uncertainty principle had to be answered at both the physical as well as the mathematical level in order to substantiate the claim that some self-adjoint operators in $\mathcal{A}(R)$ can actually represent observables confined to any bounded region R in Minkowski space.

A systematic effort (Prugovečki, 1964-1969) to substantiate all these *conjectures* was launched soon after they were formulated. It produced results which shed some light on the problem of treating simultaneous measurements of position and momentum in the nonrelativistic regime, but in the relativistic QFT regime it proved incapable of answering the questions about the physical significance of $\mathcal{A}(R)$ even in the simple case of the smeared quantum field in (1.4) (cf. Prugovečki, 1969a). No further attempts at answering such basic questions seem to have followed, as witnessed by the fact that even the most recent surveys of algebraic quantum field theory (Horuzhy, 1990; Haag, 1992) never address the question of what might be the precise *physical* significance of the operators belonging to an algebra of "observables" generated by a smeared quantum field. Hence, setting aside the purely mathematical interest of this entire subject, from the physics point of view Wigner's (1976) negative verdict as to its potential to answer questions pertaining to the actual measurability of "local" quantum fields remains as valid nowadays as when it was originally passed.

Indeed, after discussing in his review article "Interpretation of Quantum Mechanics" the foundational difficulties pertaining to the conventional notion of quantum particle localization, E. P. Wigner concludes that article with the following significant statement: "All the preceding discussion is based on the usual quantum-mechanical theory, as taught in classes, not on the more advanced axiomatic quantum field theory. The question therefore arises whether quantum field theory avoids the difficulties mentioned. . . [In this respect] my opinion is negative also. The best known discussion of the measurement of field strengths, that given by Bohr and Rosenfeld postulates an electric test charge with arbitrarily large charge and arbitrarily small size", and, therefore, "it does not resolve our problems [with regard to quantum field localization]." (Wigner, 1976, p. 313.)

This very same measurement-theoretical fact had been actually *emphasized* already by Bohr and Rosenfeld in their two classic papers on the subject of quantum field localization. For example, in the first of these two papers, they stated the following: "[W]e want to stress once more that the fundamental difficulties which confront the consistent utilization

of field theory in the atomic theory remain entirely untouched by the present investigation. Indeed, consideration of the atomistic constitution of all measuring instruments would be essential for an assessment of the connection between these difficulties and the well-known paradoxes of the measurement problem in relativistic quantum mechanics." (Cf.Wheeler and Zurek, 1983, pp. 481-482.)

All these difficulties, which arise from the foundational problem of measurability of conventional "local" quantum fields and the algebras of operators which they generate in arbitrarily small classical spacetime regions, and which purportedly represent observables, are greatly exacerbated in a curved spacetime manifold. Moreover, in the presence of quantum fluctuations of the metric, the local commutativity postulate loses even the last of the potential physical significance intended by its advocates (cf. Haag *et al.*, 1984). Hence, after we present in the next section the most basic features of the solution offered by the quantum-geometric formulation of QFT, we shall return to the discussion of all these difficulties, which will reveal why, after all, "the apparent simplicity of this idea" that "the whole [of quantum field] theory must be formulated in terms of local observables" is very deceptive: it hinges on a series of *postulates* (cf. Haag, 1992, pp. 105-107) about the concept of quantum "locality," which might appear very "natural" (*ibid.*, p. 297) at first sight, but possess no measurement-theoretical support in the quantum relativistic regime.

5.2. Quantum Frame Fields and Microcausality

The original formulation by Parker (1966, 1968) of quantum fields in a classical curved spacetime manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ was very formal, and dealt only with the special case of a spatially flat Robertson-Walker metric expressed in the form in which its components are in their outward appearance very similar to those for the Minkowski metric (cf. Wald, 1984). However, it was soon noted by Fulling (1973) that such a construction was inherently nonunique. Since that time it has been often claimed that this intrinsic ambiguity of the extrapolation of the canonical second-quantization method from Minkowski space to curved spacetime originates from the fact that "*the notion of a vacuum or no-particle state in a curved spacetime is inherently ambiguous*" (Gibbons, 1979, p. 639).

As we have seen in Secs. 1.5 and 4.7, this ambiguity actually occurs at the single-particle level, and therefore, by inference it extends to the vacuum state. In the quantum geometry approach it is removed by introducing, *above* each $x \in \mathbf{M}$, a *local* vacuum state vector Ψ_{0x}, which is the counterpart of the Fock vacuum state Ψ_0 within the Fock space

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n \quad , \qquad \mathcal{F}_n = \mathbf{F} \underset{S}{\otimes} \cdots \underset{S}{\otimes} \mathbf{F} \, , \qquad \mathbf{F} = \mathbf{P}_{\eta(\iota)} L^2(\Sigma_m) \, , \tag{2.1}$$

constructed from the mass-renormalized (cf. (4.2.12)) typical fibre $\mathbf{F}$ of the Klein-Gordon quantum bundle $\mathbf{E}$ in (4.1.6). The unitary irreducible representation of $\mathrm{ISO}_0(3,1)$, obtained by restricting to $\mathbf{F}$ the representation U in (3.7.7), gives rise in this Fock space to the following unitary (reducible) representation of the Poincaré group $\mathrm{ISO}_0(3,1)$:

$$\boldsymbol{U}(b,\Lambda) = \bigoplus_{n=0}^{\infty} [U(b,\Lambda)\mathbf{P}_{\eta(\iota)}]^{\otimes n} \, , \qquad (b,\Lambda) \in \mathrm{ISO}_0(3,1) \, . \tag{2.2}$$

This representation leaves the Fock vacuum state invariant, and acts on $\mathcal{F}$ from the left.

Therefore, we can construct from this typical Fock fibre the following *Fock bundle*:

$$\mathcal{E} = P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}}) \times_{\mathbf{G}} \mathcal{F} \ , \qquad \mathbf{G} = \mathrm{ISO}_0(3,1) \ . \tag{2.3}$$

The n-boson subbundle $\mathcal{E}_n$ of this quantum bundle $\mathcal{E}$ is a symmetrized $(n,0;0,0)$-tensorial bundle that can be constructed, in accordance with the general definition in Sec. 3.5, directly from the n-boson typical fibre $\mathcal{F}_n$. Consequently, a *boson creation operator* acts on a state vector of $\mathcal{E}_n$ by creating an $(n+1)$-boson state vector whose additional boson gives rise to a Poincaré gauge-independent local quantum metric fluctuation amplitude,

$$\begin{aligned}\left(\varphi^{(+)}(x;\zeta)\,\Psi_{n;x}\right)_{n+1}(\zeta_1,\ldots,\zeta_{n+1}) \\ = i(n+1)^{-1/2}\sum\nolimits_{j=1}^{n+1}\Delta_x^{(+)}(\zeta_j;\zeta)\,\Psi_{n;x}(\zeta_1,\ldots,\zeta_{j-1},\zeta_{j+1},\ldots,\zeta_{n+1}) \ ,\end{aligned} \tag{2.4}$$

in accordance with (4.2.13b) and (4.5.14). The domains of definition of these boson creation operators can be extended, by using linearity and standard procedures for constructing closures of operators (cf., e.g., Prugovečki, 1981, p. 210), to that of closed operators in each Fock fibre $\mathcal{F}_x$ of the Fock bundle $\mathcal{E}$. Thus, albeit unbounded, the boson creation operators are *bona fide* densely defined operators. Their mathematical nature is therefore fundamentally different from that of the corresponding particle creation operators in conventional QFT, which are formally defined in an analogous manner, but have to be smeared with test functions $f \in \mathcal{S}(\mathbf{R}^4)$ in order to acquire a legitimate mathematical meaning.[3]

The boson creation operators defined by (2.4) have well-defined adjoints, which turn out to represent *boson annihilation operators* by virtue of the fact that, on account of (4.2.13a), they transform local n-boson state vectors into local $(n-1)$-boson state vectors:

$$\begin{aligned}\left(\varphi^{(-)}(x;\zeta)\,\Psi_{n;x}\right)_{n-1}(\zeta_1,\ldots,\zeta_{n-1}) &= n^{1/2}\,\Psi_{n;x}(\zeta,\zeta_1,\ldots,\zeta_{n-1}) \\ &= in^{1/2}\int\Delta_x^{(+)}(\zeta;\zeta_n)\,\Psi_{n;x}(\zeta_1,\ldots,\zeta_n)\,d\Sigma(\zeta_n)\end{aligned} \tag{2.5}$$

When superimposed, these two mutually adjoint boson creation and annihilation operators produce, within each Fock fibre $\mathcal{F}_x$, the *Klein-Gordon quantum frame field*,

$$\varphi(x;\zeta) = \varphi^{(+)}(x;\zeta) + \varphi^{(-)}(x;\zeta) \ , \qquad \varphi^{(+)}(x;\zeta) = \varphi^{(-)*}(x;\zeta) \ . \tag{2.6}$$

On account of (3.7.5), (4.2.8) and (4.2.12), this quantum frame field satisfies the Klein-Gordon equation in the following geometrically local sense:

$$(\partial^i\partial_i + m^2)\varphi(x;\zeta) = 0 \ , \qquad \partial_i := \partial/\partial q^i \ , \quad \partial^i := \eta^{ij}\partial_j \tag{2.7}$$

Moreover, it follows directly from the definitions (2.4) and (2.5) of boson creation and annihilation operators, respectively, and the second defnition in (4.2.22), that these operators satisfy within each Fock fibre $\mathcal{F}_x$ the following commutation relations:

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[3] Cf. (1.5b), and then compare (2.4) with eqs. (104) and (108) on p. 172 of (Schweber, 1961), but note that the creation operators, conventionally denoted there by $\phi^{(-)}$, act on "ket" vectors *outside* the Fock space in question.

$$\left[\varphi^{(\mp)}(x;\zeta),\varphi^{(\pm)}(x;\zeta')\right] = i\Delta_x^{(\pm)}(\zeta;\zeta') \ , \qquad \left[\varphi^{(\pm)}(x;\zeta),\varphi^{(\pm)}(x;\zeta')\right] = 0 \ . \tag{2.8}$$

Consequently, the Klein-Gordon quantum frame field in (2.6) obeys the following geometrically local canonical commutation relations:

$$\left[\varphi(x;\zeta),\varphi(x;\zeta')\right] = i\Delta_x(\zeta;\zeta') \ , \qquad \Delta_x(\zeta;\zeta') = \Delta_x^{(+)}(\zeta;\zeta') + \Delta_x^{(-)}(\zeta;\zeta') \ . \tag{2.9}$$

It should be emphasized that the present *geometrically* local commutation relations bear no direct relationship to the topologically local conventional canonical commutation relations satisfied by the smeared field in (1.4), despite the formal similarity in their appearance – cf., e.g. (Schweber, 1961), p. 177. Indeed, as is the case in classical general relativity with all (r,s)-tensor fields $\boldsymbol{T}$ with $r+s > 0$, quantum frame field values over distinct points in the base manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ are not algebraically related, since they exist in distinct fibres. It is only when $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ equals the Minkowski space $M^4 = (\mathbf{R}^4,\boldsymbol{\eta})$, and when one adopts a Poincaré gauge provided by a cross-section $\boldsymbol{s}_0(\mathcal{L})$ of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ defined as in (4.1.13), that such field values can be meaningfully compared and related *after* the identification of $\boldsymbol{s}_0(\mathcal{L})$ with the global Lorentz frame $\mathcal{L} = \{\boldsymbol{e}_i(O)|\, i = 0,1,2,3\}$.

Indeed, once such an identification has been executed, one of the two variables x and q in (2.4)-(2.9) becomes redundant, since we then have

$$\varphi^{(\pm)}(x;\boldsymbol{\zeta}) = \varphi^{(\pm)}((x^i+q^i)\boldsymbol{e}_i(O);-i\ell v^i\boldsymbol{e}_i(O)) \ , \qquad x = x^i\boldsymbol{e}_i(O) \ , \quad \zeta = q - i\ell v \ . \tag{2.10}$$

This result follows from the equivalence of the special and general relativistic descriptions of boson states obtained by extending to the Poincaré gauge $\boldsymbol{s}_0(\mathcal{L})$ the identification carried out in (4.3.5) in the corresponding Lorentz gauge provided by the cross-section $\boldsymbol{s}(\mathcal{L})$ of $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ defined in (2.3.9). If we then set everywhere in the Minkowski space M^4

$$\varphi^{(\pm)}(\zeta) = \varphi^{(\pm)}(x;-i\ell v^i\boldsymbol{e}_i(O)) \ , \qquad x = q^i\boldsymbol{e}_i(O) \ , \quad \zeta = q - i\ell v \ , \tag{2.11}$$

the canonical commutation relations in (2.9) assume the form

$$\left[\varphi(\zeta),\varphi(\zeta')\right] = i\Delta(\zeta;\zeta') := i\Delta_O(\boldsymbol{\zeta};\boldsymbol{\zeta}') \ , \qquad \varphi(\zeta) = \varphi^{(+)}(\zeta) + \varphi^{(-)}(\zeta) \ . \tag{2.12}$$

Consequently, a physically meaningful comparison with conventional QFT can be now carried by observing that, in accordance with (4.2.14), we have

$$[\tilde{Z}_{\ell,m}/(2\pi)^3]\,\Delta(q-i\ell v;q'-i\ell v') \xrightarrow[\ell\to 0]{} \Delta(q-q') \ , \tag{2.13}$$

so that, at least formally, the commutation relations in (2.12), rewritten in the form

$$\left[\phi(\zeta),\phi(\zeta')\right] = i(\tilde{Z}_{\ell,m}/8\pi^3)\,\Delta(\zeta;\zeta') \ , \qquad \phi(\zeta) := (\tilde{Z}_{\ell,m}/8\pi^3)^{1/2}\varphi(\zeta) \ , \tag{2.14}$$

merge in the sharp point limit $\ell \to 0$ into their conventional QFT counterparts:

$$\left[\phi(q),\phi(q')\right] = i\Delta(q-q') = (2\pi)^{-3}\int_{V_m} \varepsilon(k_0)\exp[i(q'-q)\cdot k]\,d\Omega_m(k) \ . \tag{2.15}$$

A number of conclusions can be drawn from this very elementary comparison, which

throw additional light on the reasons underlying the actual lack of the *desired* physical significance of the "local" commutativity of quantum fields. Before drawing them, let us first review, however, the main points of this contentious issue, which is still clouded by many false impressions of what has been *actually* established in key papers on the subject.

As was discussed in the previous section, it is conventionally *assumed* that the commutativity of fields localized within spacelike separated regions R and R' in the Minkowski space M^4 reflects a concept of "microcausality" patterned after Einstein causality. However, whereas Einstein causality actually pertains to the *propagation* of *classical* particles and fields, this concept of "microcausality" postulates that the *measurements* of the quantum fields localized in R and R', respectively, cannot influence each other. The fact that in *non*relativistic quantum mechanics the compatibility of observables is identified with the commutativity of the self-adjoint operators representing those observables makes that assumption, at first sight, *appear* very plausible. On the other hand, in order to actually *justify* it physically, it is necessary to *prove* that conventional quantum fields can produce *observables* not merely as *operators* belonging to some algebra generated by smeared quantum fields, but also in the traditional sense of *measurable* quantities that are *strictly* localized within a *arbitrariily small* regions R in the Minkowski space M^4. However, as was pointed out in the previous section, barring one unsuccessful effort in that direction (Prugovečki, 1964-1969), this is not what is normally done. Rather, it is often simply *asserted* that "(1) local observables [in arbitrarily small space-time regions] exist and (2) local observables relative to space-like separated regions commute. Experimental verification of this [locality] postulate are only indirect since we do not possess any apparatus comparable to nuclear and subnuclear sizes." (Itzykson and Zuber, 1980, p. 106.)

On the other hand, the actual experimental verifications carried out in practice involve observations pertaining to S-matrix expansions based on formal computations with Feynman diagrams. In the configuration representation Feynman rules for such computations (cf. Schweber, 1961, p. 471) involve only Feynman propagators that are not the Green functions directly produced by the commutators of quantum fields. For example, in the spin-0 case considered thus far, the Feynman propagator is the one in (3.8.7a) and (4.8.4), and it is related to the time-ordered product of the vacuum expectation value of quantum field in (1.3) in the following manner (Itzykson and Zuber, 1980, p. 124),

$$K_F(x-x') = i\langle 0|T\phi(x)\phi(x')|0\rangle = i\Delta_F(x-x') \ , \tag{2.16}$$

$$T\phi(x)\phi(x') := \theta(x^0-x'^0)\phi(x)\phi(x') + \theta(x'^0-x^0)\phi(x')\phi(x) \ . \tag{2.17}$$

Consequently, this propagator is quite distinct from the Green function in (2.15), given by

$$\Delta(x) = \Delta_A(x) - \Delta_R(x) \ , \qquad \Delta_R(x) = -\theta(x^0)\Delta(x) \ , \quad \Delta_A(x) = \theta(-x^0)\Delta(x) \ , \tag{2.18}$$

as can be seen from the following alternative expressions (cf. Schweber, 1961, Sec. 13d):

$$\Delta_F(x) = \theta(x^0)\Delta^{(+)}(x) - \theta(-x^0)\Delta^{(-)}(x) \ , \qquad \Delta(x) = \Delta^{(+)}(x) + \Delta^{(-)}(x) \ . \tag{2.19}$$

Moreover, while such Green functions as the ones in (2.17) are "zero outside the light cone, this is not the case for $G_F\,[\equiv K_F]$, which has an exponential tail for negative x^2 "

(Itzykson and Zuber, 1980, p. 35); namely for the Feynman propagator K_F in (2.16), which therefore does *not* vanish for spacelike separations of points in the Minkowski space M^4 All this implies that the scattering experiments performed to verify the correctness of numerical predictions obtained by means of Feynman rules of computation of renormalized terms in perturbation series of S-matrices in conventional QFT actually do not have a bearing on the local commutativity postulate. Of course, a relationship would be obtained *if* those S-matrices resulted from quantum fields interacting in M^4 for which the actual existence of asymptotic states was established by means of formulae of the type (4.8.6) (such as, e.g., the Yang-Feldman equations provided by the LSZ formalism), containing retarded or advanced Green functions that were causal; however, as we shall see in Sec. 5.5, all mathematically rigorous efforts in that direction have failed. Moreover, we saw in Sec. 3.6 that the problem of the special relativistic localizability of quantum particles, such as those produced by LSZ asymptotic fields, has proved to be insoluble.

In the absence of a consistent special relativistic concept of localizability of the quantum particles produced by conventional quantum fields, it is sometimes contended that the Bohr and Rosenfeld (1933, 1950) analysis of the measurability of quantum fields with *classical* test bodies provides a justification for introducing smeared fields, and regarding C^*-algebras of operators generated by them as "algebras of observables." However, that analysis pertains only to the measurement of field and charge-current *averages* over regions R that accommodate such classical test bodies, and whose boundaries are *not* sharply delineated, so that the fundamental difficulty of extending that analysis to microregions remains unresolved. In fact, in their first paper on the subject, Bohr and Rosenfeld *stress* that "consideration of the atomistic constitution of all measuring instruments would be essential for an assessment of the connection between these difficulties and the well-known paradoxes of the measurement problem in relativistic quantum mechanics" (cf.Wheeler and Zurek, 1983, p. 482.); whereas, in the second paper, they qualify this statement as follows: "The disregard of all limitations . . . which may originate in the atomic constitution of matter, is, however, entirely justified when dealing with quantum electrodynamics in the initial stage of approximation. In fact, at this stage, the formalism is essentially independent of space-time scale . . ." (*ibid.*, p. 530). However, as one probes *beyond* this *initial approximation*, the quantum nature of *all* test bodies obviously has to be taken into account, so that Bohr and Rosenfeld conclude their second (and last) paper on the subject with the following observation concerning all these *foundational* measurement-theoretical problems: "The further exploration of such problems may, however, demand a radical revision of the foundation for the application of the dual concepts of fields and particles." (*Ibid.*, p. 533.)

As we saw in Sec. 1.4, according to Schwinger, "the observational basis of quantum electrodynamics is self-contradictory. . . . A convergent theory cannot be formulated consistently within the framework of present space-time concepts." (Schwinger, 1958, p. xvi.) And, of course, the same remains true of all other conventional QFT models of interacting fields, since they all display divergences. On the other hand, even after all the intervening years since all these observations were made, the following remains true: "The main justification of [the local commutativity] criterion for a causal theory is that it produces a desired result (dispersion relations), while not obviously conflicting with a more direct notion of causality." (Cushing, 1990, p. 214.)

The comparison of (2.14) and (2.15) provides the first clue as to why the reliance laid by conventional QFT on the local commutativity postulate might be misguided. Indeed, in the conventional formulation of the S-matrix for interacting quantum field theories, those fields become asymptotically free. Hence, after renormalization, they are surmised to create and annihilate quantum particles in the manner of free fields (cf., e.g., Schweber, 1961, Sec. 17d). However, a quantum field such as that in (1.3) is not mathematically well-defined. After its "smearing" with one of the test functions $f \in S(\mathbf{R}^4)$, that field becomes the mathematically well-defined quantum field in (1.4), which creates and annihilates quantum particles in "states" described by that test function. However, the physical significance of such "states" is determined neither by field-theoretical measurement considerations, nor by a consistent theory of relativistic quantum localization – which actually does not exist by virtue of a lack of a conserved and Poincaré covariant probability current, as well as a lack of conventional Poincaré covariant probability measures in Minkowski space. Hence, such a "smearing" merely produces a mathematical framework of no physical relevance.

On the other hand, the quantum frame field in (2.12) creates and annihilates bosons whose spacetime localizability is based on the conserved and Poincaré covariant *bona fide* probability currents in (3.7.16), and on the Poincaré covariant *bona fide* probability measures in (3.7.14). Therefore, as we shall see in more detail in Sec. 5.6, it has a well-defined physical interpretation, and albeit it does not satisfy the local commutativity postulate, its renormalized version in (2.14) does formally satisfy that postulate in the sharp-point limit $\ell \to 0$. However, the physical interpretation of the renormalized quantum fields in (2.14) cannot be retained in that sharp-point limit, since according to (3.8.4b) and (4.2.6a) the normalization constant $\tilde{Z}_{\ell,m}$ diverges in that limit, so that there is no smooth transition from the quantum-geometric to the conventional regime as $\ell \to 0$. Moreover, the normalization constant $\tilde{Z}_{\ell,m}$ cannot be adjusted at will, since it is intrinsic to the reproducibility properties (3.8.10) of quantum-geometric propagators – i.e., its value plays as key a role to the definition of such propagators as that played by the proportionality constant $1/A = (m/2\pi i t)^{3/2}$ in the Feynman propagator in (3.5.1), on whose determination Feynman based his own initial progress beyond the results previously derived by Dirac (1933). Hence, the divergence of $\tilde{Z}_{\ell,m}$ in the limit $\ell \to 0$ underlines the physical inconsistency of that limit. It also ties in with the impossibility of implementing Dirac's (1949) suggestion of modifying the "local commutativity" postulate by simply setting the commutator in (2.15) different from zero for spacelike separations of q and q' of the order of magnitude of ℓ.

Indeed, the Wightman functions determined by the vacuum expectation values of products of Wightman quantum fields are holomorphic in the Cartesian product of forward tubes (cf. Bogolubov *et al.*, 1975, p. 267). However, analyticity is a mathematically very restrictive property: its imposition implies that the assumption of a zero value on an arbitrarily small neighborhood is compatible only with a *globally* zero value. But such a property is not at all consistent with any pragmatic approach to the theory of quantum measurement, which has to take into account the fact that sufficiently small values might become experimentally indistinguishable from zero on account of *ever-present* margins of measurement-errors, and that such "error" margins cannot be reduced indefinitely due to intrinsic limitations on the sizes of samples, apparatuses, etc. It is also *not* in the spirit of the Bohr and Rosenfeld (1933, 1950) semi-classical theory of measurement of quantum

fields, since, as was seen in the earlier presented quotations, this theory provides only an "initial approximation" for such measurements. Hence, the modification of the *theoretical assumption* of values which are *exactly zero* to values that are *sufficiently small* at spacelike separations to concur with experimental data provides a theoretically viable point of view. In the context of the quantum-geometric framework for QFT in curved spacetime presented in this chapter, all the physically desirable aspects of conventional QFT can be retained when this point of view adopted, by removing only such physically extraneous postulates as the conventional "local" (anti)commutativity postulate, and the axiom of the existence of a global vacuum state – which in the LSZ formalism is interrelated with the existence of *global* particle states that are produced by asymptotic fields.

5.3. Second-Quantized Boson Frames in Fock Bundles

The Fock bundle $\mathcal{E}$ defined in (2.3) equals, in a rather obvious manner, a Whitney sum (cf. Nakahara, 1990, p. 317) of the tensorial quantum bundles $\mathcal{E}_n$ of n-boson states. Hence, the following operator for parallel transport along a smooth curve γ,

$$\tau_\gamma(x'',x') \;:\; \mathcal{F}_{x'} = \bigoplus\nolimits_{n=0}^{\infty} \mathcal{F}_{n;x'} \;\;\rightarrow\;\; \mathcal{F}_{x''} = \bigoplus\nolimits_{n=0}^{\infty} \mathcal{F}_{n;x''} \;, \tag{3.1}$$

is a unitary operator uniquely determined by the unitary operators of parallel transport within those n-boson bundles $\mathcal{E}_n$, which, as described in Sec. 4.5, are in their turn uniquely determined by the operators in (4.4.2) for the parallel transport of single-boson quantum frames. The existence of a QFT connection

$$\boldsymbol{\nabla} = \boldsymbol{d} - i\tilde{\boldsymbol{\theta}}^i \boldsymbol{P}_i + \tfrac{i}{2}\tilde{\boldsymbol{\omega}}_{jk} \boldsymbol{M}^{jk} \;, \tag{3.2}$$

can be established exactly as in Sec. 4.4 for the case of the quantum connection (4.4.20) in the single-boson quantum bundle $\mathbf{E} \cong \mathcal{E}_1$. As a matter of fact, the connection coefficients in the operators for covariant differentiation corresponding to this QFT connection,

$$\boldsymbol{\nabla}_{\boldsymbol{X}} = \boldsymbol{\partial}_{\boldsymbol{X}} - i\tilde{\boldsymbol{\theta}}^i_{\boldsymbol{s}}(\boldsymbol{X}) \boldsymbol{P}_{i;\boldsymbol{s}(x)} + \tfrac{i}{2}\tilde{\boldsymbol{\omega}}^{\boldsymbol{s}}_{jk}(\boldsymbol{X}) \boldsymbol{M}^{jk}_{\boldsymbol{s}(x)} \;, \tag{3.3}$$

are the same as those that appear in (4.4.16); whereas, the generators of infinitesimal spacetime translations and Lorentz transformation are those of the representation

$$\boldsymbol{U}_{\boldsymbol{s}(x)}(b,\Lambda) = \bigoplus_{n=0}^{\infty} U_{\boldsymbol{s}(x)}(b,\Lambda)^{\otimes n} = (\boldsymbol{\sigma}^{\boldsymbol{s}}_x)^{-1} \boldsymbol{U}(b,\Lambda) \boldsymbol{\sigma}^{\boldsymbol{s}}_x \;, \tag{3.4}$$

within the fibre $\mathcal{F}_x$, where the above soldering map is the extension of the one in (4.1.7):

$$\boldsymbol{\sigma}^{\boldsymbol{\mu}}_x : \;\; \boldsymbol{\Psi} \mapsto \Psi \in \mathcal{F} \;, \qquad \boldsymbol{u} = (\boldsymbol{a}, \boldsymbol{e}_i) \in P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}}) \;, \qquad \boldsymbol{\Psi} \in \mathcal{F}_x \subset \mathcal{E} \;, \tag{3.5}$$

In order to establish a relationship between this purely geometric formulation of the parallel transport within the Fock bundle $\mathcal{E}$ and action-based functional integrals that are similar to those used in the various Lagrangian formulations of conventional quantum field theories, it is necessary that these generators be related to operator-valued stress-energy

tensors, whose expectation values in the parallel transported quantum frames would be in turn related to corresponding classical Lagrangian densities. As we shall see in the next section, in order to achieve this it is necessary that the parallel transported quantum frames be eigenvectors of the boson annihilation operators in (2.5). Although the tensorial quantum frames within the n-boson fibres $\mathcal{F}_{n;x}$, obtained from appropriately symmetrized counterparts of (4.5.10), do not provide such eigenvectors, particular superpositions of such tensorial quantum frames possess that property.

To arrive at such superpositions, let us introduce the creation operator of a boson in state $\mathbf{f}$, which is (densely) defined in $\mathcal{F}_x$ by the following Bochner integral,

$$\varphi^{(+)}(\mathbf{f}) = \int \varphi^{(+)}(x;\boldsymbol{\zeta})\mathbf{f}(\boldsymbol{\zeta})d\Sigma(\boldsymbol{\zeta}) \ , \qquad \mathbf{f} \in \mathbf{F}_x \ , \tag{3.6}$$

where the integration can be carried out, as (4.2.2) and any other similar cases, along any of the reference hypersurfaces $\Sigma = \sigma \times V^+$ in $T_x\mathbf{M}^{\mathbf{C}}$ corresponding to some maximal space-like hypersurface σ in $T_x\mathbf{M}$. We can then define a *second-quantized boson frame* at x,

$$Q_x^{\mathbf{F}} = \left\{ \boldsymbol{\Phi}_{\mathbf{f}} \middle| \ \mathbf{f} \in \mathbf{F}_x \equiv \mathcal{F}_{1;x} \right\} \ , \qquad x \in \mathbf{M} \ , \tag{3.7}$$

as a vector-valued functional of single-boson state vectors $\mathbf{f}$ above x, that assumes the following values within the Fock fibre $\mathcal{F}_x$:

$$\begin{aligned} \boldsymbol{\Phi}_{\mathbf{f}} &= \exp\left[-\tfrac{1}{2}\int|\mathbf{f}(\boldsymbol{\zeta})|^2 d\Sigma(\boldsymbol{\zeta}) + \varphi^{(+)}(\mathbf{f})\right] \boldsymbol{\Psi}_{0;x} \\ &= \exp\left(-\tfrac{1}{2}\|\mathbf{f}\|^2\right) \left(\sum_{n=0}^{\infty} \varphi^{(+)}(\mathbf{f})^n / n!\right) \boldsymbol{\Psi}_{0;x} \ . \end{aligned} \tag{3.8}$$

The above power series is strongly convergent, i.e., it is convergent in the Hilbert space norm of $\mathcal{F}_x$. Hence, by using (2.5) and (2.8), and also noting that according to (4.2.11)-(4.2.13) we have

$$\mathbf{f}(\boldsymbol{\zeta}) = i\int \Delta_x^{(+)}(\boldsymbol{\zeta};\boldsymbol{\zeta}')\mathbf{f}(\boldsymbol{\zeta}')d\Sigma(\boldsymbol{\zeta}') \ , \qquad \mathbf{f} \in \mathbf{F}_x \ , \tag{3.9}$$

$$\varphi^{(-)}(\mathbf{f}) := \int \varphi^{(-)}(x;\boldsymbol{\zeta})\mathbf{f}^*(\boldsymbol{\zeta})d\Sigma(\boldsymbol{\zeta}) = \varphi^{(+)*}(\mathbf{f}) \ , \tag{3.10}$$

we can easily establish that these second-quantized boson frames in (3.7) possess the following important property, which is shared by all types of Glauber (1963) coherent states:

$$\varphi^{(-)}(\mathbf{f})\boldsymbol{\Phi}_{\mathbf{g}} = \langle \mathbf{f} | \mathbf{g} \rangle \boldsymbol{\Phi}_{\mathbf{g}} \ , \qquad \mathbf{f}, \mathbf{g} \in \mathbf{F}_x \ . \tag{3.11}$$

Moreover, on account of (4.2.12), (4.2.13) and (2.4)-(2.5), we also have

$$\varphi^{(\pm)}(\boldsymbol{\Phi}_{m;\zeta}^{\boldsymbol{u}(x)}) = \varphi^{(\pm)}(x;\boldsymbol{\zeta}) \ , \quad \boldsymbol{\zeta} = q - i\ell v \ , \ \boldsymbol{\zeta} = \boldsymbol{a} + (q^k - i\ell v^k)\boldsymbol{e}_k, \ \boldsymbol{u}(x) = (\boldsymbol{a}, \boldsymbol{e}_k) \ , \tag{3.12}$$

so that the second-quantized frame elements in (3.7) possess the following property, which will turn out to be of key significance to the later derivation of action integrals from purely geometric considerations:

$$\varphi^{(-)}(x;\zeta)\Phi_{\mathbf{f}} = \mathbf{f}(\zeta)\Phi_{\mathbf{f}} \ , \qquad \mathbf{f} \in \mathbf{F}_x \ . \tag{3.13}$$

In this last context it is important to note that, although a similar relationship can be *formally* written for the conventional quantum fields in (1.3), not only are those fields not mathematically well-defined in the absence of the "smearing" in (1.4), but their corresponding "eigenvalue" $f(x)$ is also not well-defined for an *arbitrary* element of the single-particle Hilbert space obtained from the wave functions in (3.6.10), since the elements of that Hilbert space are actually equivalence classes of wave functions which can differ in value at single points x in Minkowski space (cf. the discussion of (3.2.9) in Sec. 3.2). On the other hand, the fields in (2.6) are totally well-defined, as they require no "smearing"; moreover, $\mathbf{f}$ is a smooth function of ζ, so that (3.13) contains no ambiguities. Mathematically, this is one of the fundamental reasons why many of the constructions carried out in the remainder of this chapter cannot be duplicated within the conventional formalism for QFT; whereas, physically, the reasons run deep, reaching all the way to the very notion of "locality" in relativistic quantum theory, which the analyses in Secs. 3.6, 3.7, 5.1 and 5.2 have indicated to be more subtle than it has been commonly assumed in the past.

A second-quantized frame, such as the one in (3.7), earns its name from the fact that it gives rise to a continuous resolution of the identity operator $\mathbf{1}_x$ within a Fock fibre $\mathcal{F}_x$,

$$\int_{\mathbf{F}_x} \left|\Phi_{\mathbf{f}}\right\rangle \mathbf{df}\,\mathbf{df}^* \left\langle\Phi_{\mathbf{f}}\right| \ = \ \mathbf{1}_x \ , \tag{3.14}$$

so that, in formal analogy with (4.2.11), any vector Ψ that belongs to that fibre $\mathcal{F}_x$ can be expanded as follows:

$$\Psi \ = \ \Psi^{\mathbf{f}}\Phi_{\mathbf{f}} := \int \mathbf{df}\,\mathbf{df}^* \left\langle\Phi_{\mathbf{f}}\middle|\Psi\right\rangle \Phi_{\mathbf{f}} \ , \qquad \Psi^{\mathbf{f}} := \left\langle\Phi_{\mathbf{f}}\middle|\Psi\right\rangle \ . \tag{3.15}$$

A precise mathematical meaning can be assigned to the above functional integrals by using a method developed to a large extent by Berezin (1966).

To apply this method, let us introduce in the typical fibre $\mathbf{F} \equiv \mathcal{F}_1$ a fixed orthonormal basis $\{w_1, w_2, \ldots\}$, so that[4]

$$\sum_{\alpha=1}^{\infty} \left|w_\alpha\right\rangle\left\langle w_\alpha\right| \ = \ \mathbf{1}_{\mathbf{F}} \ , \qquad \left\langle w_\alpha\middle|w_\beta\right\rangle = \delta_{\alpha\beta} \ , \qquad \alpha,\beta = 1,2,\ldots \ . \tag{3.16}$$

The inverse of the soldering map in (3.5) takes this basis into a corresponding orthonormal basis within $\mathbf{F}_x \equiv \mathcal{F}_{1;x}$:

$$(\boldsymbol{\sigma}_x^{\boldsymbol{u}})^{-1} : \ w_\alpha \ \mapsto \ \mathbf{w}_\alpha \in \mathbf{F}_x \ , \qquad \boldsymbol{u} \in \Pi^{-1}(x) \subset P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}}) \ , \qquad \alpha = 1,2,\ldots \ . \tag{3.17}$$

We can then expand any single-boson state vector $\mathbf{f} \in \mathbf{F}_x$ in this latter basis,

$$\mathbf{f}(\zeta) = \sum_{\alpha=1}^{\infty} z_\alpha \mathbf{w}_\alpha(\zeta) \ , \qquad z_\alpha = \left\langle \mathbf{w}_\alpha\middle|\mathbf{f}\right\rangle \ , \qquad \alpha = 1,2,\ldots \ . \tag{3.18}$$

The above series converges to $\mathbf{f} \in \mathbf{F}_x$ not only in the Hilbert space norm of $\mathbf{F}_x$, but it also

[4] For the sake of the construction that follows, an orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ could be introduced directly in the fibre $\mathbf{F}_x$, making it evident that this construction is Poincaré gauge invariant. However, the advantages of using a basis within the typical fibre will become evident in the case of fermion fields, considered in Sec. 6.2.

converges pointwise, and even uniformly, to the analytic wave function $\mathbf{f}(\zeta)$, as can be deduced from the fact that

$$\mathbf{f}(\zeta) = \left\langle \boldsymbol{\Phi}^{\boldsymbol{u}(x)}_{m;\zeta} \middle| \mathbf{f} \right\rangle , \qquad \mathbf{w}_\alpha(\zeta) = \left\langle \boldsymbol{\Phi}^{\boldsymbol{u}(x)}_{m;\zeta} \middle| \mathbf{w}_\alpha \right\rangle , \qquad \alpha = 1,2,\ldots , \tag{3.19}$$

and that, for any $n' > n = 1,2,\ldots$,

$$\left| \sum_{\alpha=n}^{n'} z_\alpha \mathbf{w}_\alpha(\zeta) \right| = \left| \left\langle \boldsymbol{\Phi}^{\boldsymbol{u}(x)}_{m;\zeta} \middle| \sum_{\alpha=n}^{n'} z_\alpha \mathbf{w}_\alpha \right\rangle \right| \leq \left\| \boldsymbol{\Phi}^{\boldsymbol{u}(x)}_{m;\zeta} \right\| \left\| \sum_{\alpha=n}^{n'} z_\alpha \mathbf{w}_\alpha \right\| . \tag{3.20}$$

Indeed, according (4.2.5), (4.2.8a) and (4.2.12),

$$\left\| \boldsymbol{\Phi}^{\boldsymbol{u}(x)}_{m;\zeta} \right\|^2 = \tilde{Z}^{-1}_{\ell,m} \int_{V^+} \exp(-2\ell m u^0) d\Omega(u) = (2\pi)^{-3} \hat{Z}^{-1}_{\ell,m} , \tag{3.21}$$

so that the partial sum on the left-hand side of (3.20) is uniformly bounded by a ζ-independent expression,

$$\left| \sum_{\alpha=n}^{n'} z_\alpha \mathbf{w}_\alpha(\zeta) \right| \leq (2\pi)^{-3/2} \hat{Z}^{-1/2}_{\ell,m} \left(\sum_{\alpha=n}^{n'} |z_\alpha|^2 \right)^{1/2} , \tag{3.22}$$

which goes to zero as $n, n' \to \infty$ by virtue of the fact that $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ is an orthonormal basis in $\mathbf{F}_x$, and that the series in (3.18) converges strongly (i.e., in the Hilbert-space norm of $\mathbf{F}_x$) to $\mathbf{f} \in \mathbf{F}_x$. This pointwise and uniform convergence of this same series will ensure in the considerations carried in the next section that no ambiguities or inconsistencies occur in any of the expressions for stress-energy tensors and Lagrangian densities, in which we will be dealing with the values of single-boson wave functions, and of their expansions (3.18), at individual points ζ.

With each fixed choice of orthonormal basis $\{w_1, w_2, \ldots\}$ in the typical fibre $\mathbf{F}$, and with each choice of Poincaré frame $\boldsymbol{u} \in P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, we can associate the following sets of canonical annihilation and creation operators,

$$a_\alpha(x) = \varphi^{(-)}(\mathbf{w}_\alpha) , \qquad a_\alpha{}^*(x) = \varphi^{(+)}(\mathbf{w}_\alpha) , \qquad \mathbf{w}_\alpha = (\boldsymbol{\sigma}^{\boldsymbol{u}}_x)^{-1} w_\alpha \in \mathbf{F}_x , \tag{3.23}$$

acting within the corresponding Fock fibre $\mathcal{F}_x$ in $\mathcal{E}$. These operators satisfy the canonical commutation relations for a (countably) infinite number of degrees of freedom,

$$[a_\alpha(x), a_\beta(x)] = [a_\alpha{}^*(x) , a_\beta{}^*(x)] = 0 , \qquad [a_\alpha(x), a_\beta{}^*(x)] = \delta_{\alpha\beta} , \tag{3.24}$$

which follow from the definitions in (3.23) and the commutation relations in (2.8), where, in the derivation of the last set of relations, the help of (4.2.11)-(4.2.13) can be solicited in deriving the second of the following equalities:

$$\begin{aligned} [a_\alpha(x), a_\beta{}^*(x)] &= i \int \mathbf{w}_\alpha{}^*(x;\zeta)\, \Delta^{(+)}_x(\zeta;\zeta') \mathbf{w}_\beta(x;\zeta') d\Sigma(\zeta) d\Sigma(\zeta') \\ &= \int \mathbf{w}_\alpha{}^*(x;\zeta)\, \mathbf{w}_\beta(x;\zeta) d\Sigma(\zeta) = \delta_{\alpha\beta} . \end{aligned} \tag{3.25}$$

Their existence enables us to express the second-quantized frame elements in (3.8) in the more familiar form of Glauber (1963) coherent states,

$$\Phi_{\mathbf{f}} = \exp\left[-\tfrac{1}{2}\sum_{\alpha=1}^{\infty}|z_\alpha|^2 + \sum_{\alpha=1}^{\infty} z_\alpha a_\alpha^*(x)\right]\Psi_{0;x} \ , \tag{3.26}$$

which provide continuous resolutions of the identity in such a Hilbert space as the fibre $\mathcal{F}_x$ in a sense originally established by Bargmann (1961), and later described as a "continual integral" by Berezin (1966). Thus, in accordance with the procedure of the latter, the functional integral in (3.14) is equal to the following weak operator limit in $\mathcal{F}_x$,

$$\int_{\mathbf{F}_x} \left|\Phi_{\mathbf{f}}\right\rangle \mathbf{df}\,\mathbf{df}^* \left\langle\Phi_{\mathbf{f}}\right| = \underset{n\to\infty}{\text{w-lim}}\, \pi^{-n}\int_{\mathbf{R}^{2n}} \left|\Phi_{\hat{z}_n}\right\rangle \mathbf{d}\hat{z}_n\mathbf{d}\hat{z}_n^* \left\langle\Phi_{\hat{z}_n}\right| \ , \tag{3.27a}$$

$$\Phi_{\hat{z}_n} := \exp\left[-\tfrac{1}{2}\sum_{\alpha=1}^{n}|z_\alpha|^2 + \sum_{\alpha=1}^{n} z_\alpha a_\alpha^*(x)\right]\Psi_{0;x} \ ; \tag{3.27b}$$

which simply means that for arbitrary vectors $\Psi, \Psi' \in \mathcal{F}_x$ one has

$$\int_{\mathbf{F}_x} \left\langle\Psi\middle|\Phi_{\mathbf{f}}\right\rangle \mathbf{df}\,\mathbf{df}^* \left\langle\Phi_{\mathbf{f}}\middle|\Psi'\right\rangle = \lim_{n\to\infty} \pi^{-n}\int_{\mathbf{R}^{2n}} \left\langle\Psi\middle|\Phi_{\hat{z}_n}\right\rangle \left\langle\Phi_{\hat{z}_n}\middle|\Psi'\right\rangle \mathbf{d}\hat{z}_n\mathbf{d}\hat{z}_n^* \ , \tag{3.28}$$

where the integration over $\mathbf{R}^{2n}$ is performed with respect to the Lebesgue measure

$$\mathbf{d}\hat{z}_n\mathbf{d}\hat{z}_n^* = \prod_{i=1}^{n} d(\text{Re}\, z_i)d(\text{Im}\, z_i) \tag{3.29}$$

This entire procedure not only provides a mathematically rigorous interpretation of the functional integral in (3.14), but, on account of theorems proven in Sec. 1.2 of (Berezin, 1966), the resulting functional integral is actually independent of any particular choice of orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ in $\mathbf{F}_x$ used in this definition. Thus, by making in (3.27) that choice of basis in each fibre $\mathbf{F}_x$ dependent on a choice of basis within the typical fibre $\mathbf{F}$, we have not actually introduced a gauge-dependent aspect in the final outcome of the computations that will be carried out with such functional integrals in the next section. As we shall see in the next chapter, however, the treatment of fermions will require the introduction of anticommuting supernumbers associated with the choice of basis in the typical fibre, and the extension of that technique to quantum gauge fields which create and annihilate bosons of zero mass will make the introduction of the map in (3.17) mandatory.

5.4. Propagators for Parallel Transport in Fock Bundles

The second quantized frames in (3.7) enable the introduction of a propagator for parallel transport within the Fock bundle $\mathcal{E}$ which is analogous to propagator in (4.6.1) for the parallel transport within the Klein-Gordon quantum bundle $\mathbf{E}$:

$$K_\gamma(\mathbf{f}'';\mathbf{f}') = \left\langle\Phi_{\mathbf{f}''}\middle|\tau_\gamma(x'',x')\Phi_{\mathbf{f}'}\right\rangle \ , \qquad \mathbf{f}' \in \mathbf{F}_{x'} \ , \qquad \mathbf{f}'' \in \mathbf{F}_{x''} \ . \tag{4.1}$$

This *second-quantized boson frame propagator* is, however, a Poincaré gauge independent entity, since neither the above operator for parallel transport, given by (3.1), nor the second-quantized frame elements, given by (3.8), depend on the choice of section $\boldsymbol{s}$ of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$. Clearly, this is a consequence of the fact that, with the Klein-Gordon quantum bundle $\mathbf{E}$ already constructed in the previous chapter, all these geometric objects can be

now defined over $\mathbf{E}$ without any direct recourse to the bundle coordinates in $\mathbf{E}$.

The second-quantized boson frame propagator plays vis-à-vis parallel transport in $\mathcal{E}$ a role totally analogous to that played by the quantum frame propagator in (4.6.1) vis-à-vis parallel transport in $\mathbf{E}$. Thus, after a local quantum state $\Psi_x \in \mathbf{F}_x$ within the Fock bundle $\mathcal{E}$ is parallel transported from x to x' along a smooth curve γ, this second-quantized boson frame propagator mediates the computation of the coordinate wave functions

$$\Psi^{\parallel}_{x'}(\mathbf{f}') := \left\langle \Phi_{\mathbf{f}'} \middle| \Psi^{\parallel}_{x'} \right\rangle , \qquad \Psi^{\parallel}_{x'} = \tau_\gamma(x',x)\Psi_x , \tag{4.2}$$

of the parallel transport of that state, without any direct recourse to the quantum connection within $\mathcal{E}$:

$$\Psi^{\parallel}_{x'}(\mathbf{f}') = \int K_\gamma(\mathbf{f}';\mathbf{f})\,\Psi_x(\mathbf{f})\,\mathbf{df}\,\mathbf{df}^* , \qquad \Psi_x(\mathbf{f}) := \left\langle \Phi_{\mathbf{f}} \middle| \Psi_x \right\rangle . \tag{4.3}$$

We can now proceed as in Sec. 4.6, and insert $\tau_\gamma(x'',x') = \tau_\gamma(x'',x)\,\tau_\gamma(x,x')$ in the right-hand side of the inner product in (4.1) for a $\gamma = \{x(\tau) |\, a \le \tau \le b\}$ with $x' = x(a)$ and $x'' = x(b)$. Upon using (3.14) at $x \in \gamma$ we obtain the following counterpart of (4.6.3):

$$K_\gamma(\mathbf{f}'';\mathbf{f}') = K^*_\gamma(\mathbf{f}';\mathbf{f}'') = \int_{\mathbf{F}_x} K_\gamma(\mathbf{f}'';\mathbf{f})\,K_\gamma(\mathbf{f};\mathbf{f}')\,\mathbf{df}\,\mathbf{df}^* . \tag{4.4}$$

This leads, by iteration, to the following counterpart of (4.6.4):

$$K_\gamma(\mathbf{f}'';\mathbf{f}') = \lim_{\varepsilon \to +0} \int K_\gamma(\mathbf{f}_N;\mathbf{f}_{N-1}) \prod_{n=N-1}^{1} K_\gamma(\mathbf{f}_n;\mathbf{f}_{n-1})\,\mathbf{df}_n\mathbf{df}_n^* , \quad \mathbf{f}' = \mathbf{f}_0 , \ \mathbf{f}'' = \mathbf{f}_N \tag{4.5}$$

In turn, the above limit will prepare the ground for the formulation of a quantum-geometric mode of free-fall propagation of quantum fields.

In view of the fact that all the local states in the Fock bundle $\mathcal{E}$ are superpositions of multi-boson states whose propagation is determined, in the absence of mutual interactions, by the propagation of single-boson states, from a purely geometric point of view the idea of free-fall quantum-geometric propagation in $\mathcal{E}$ emerges from the considerations in Sec. 4.6. We intend to show that, in addition, a type of action integrals for the propagators of parallel transport is also determined by the quantum geometry of these bundles, and that it is related to the quantum-geometric propagation of the proper energy of *local* QFT states in $\mathcal{E}$.

The first step towards that goal consists of establishing a general relationship between the purely geometric formulation of the propagator for parallel transport in (4.1), and the Lagrangian densities used by the customary formulations of quantum field theories in Minkowski space. To understand the motivation behind the derivation of this relationship, one should recall that, first of all, those Lagrangian densities are derived by variational arguments applied with respect to global rather than local Lorentz frames, and, second, that they are *not* derived at the quantum level, but rather by means of Noether's theorem applied to *classical* fields (cf. Bogolubov and Shirkov, 1959; Nakanishi and Ojima, 1990). Thus, what these variational arguments actually yield are *classical* field equations, and *classical* energy-momentum and angular momentum tensors. Consequently, their relevance to conventional QFT manifests itself only in a purely formal manner, rather than as a result of a rigorous chain of arguments based on *quantum* variational principles. For example, in

attempting to use quantum variational principles for a free scalar field one has to give up the well-defined smeared quantum field in (1.4), and use the ill-defined quantum field in (1.3), thus making any such derivation at best heuristic (cf. Schweber, 1961, p. 188) – as opposed to the mathematically sound derivation of Noether's theorem for classical fields.

In view of all this, and in keeping with the thesis (cf. Sec. 4.8) that conservation laws and geometrically formulated physical principles are primary, whereas action principles based on variational arguments are at best secondary, we shall derive an action-type integral for the propagator for parallel transport in (4.1) by relating the infinitesimal generators of spacetime translations and Lorentz transformations in the quantum connection (3.2) to operator-valued stress-energy tensors acting in the Fock fibres of $\mathcal{E}$. To achieve that, we first note that those generators emerge in each fibre $\mathcal{F}_x$ from the representation in (3.4), which is tied in with a choice of section $\boldsymbol{s}$ of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$. In any such Poincaré gauge, the four infinitesimal generators of spacetime translations to which that representation gives rise can be expressed in the form

$$\boldsymbol{P}_{j;\boldsymbol{s}(x)} = i\int \varphi^{(+)}(x;\boldsymbol{\zeta})\,\partial_j \varphi^{(-)}(x;\boldsymbol{\zeta})\,d\Sigma(\boldsymbol{\zeta})\ , \qquad \partial_j = \partial/\partial q^j\ , \quad j = 0,1,2,3\ . \tag{4.6}$$

Indeed, in the single-boson Fock subfibre $\mathcal{F}_{1;x}$ these operators are given by (4.4.10), so that within the n-boson subfibre $\mathcal{F}_{n;x}$ they equal the sum of those operators acting on the local gauge coordinates q^j of each boson in a given local n-boson state. By using (2.4) and (2.5) it can be then easily verified that the matrix elements of this sum of operators in between any two such local n-boson states equals that of the operator in (4.6).

To arrive at an expression of the infinitesimal generators in (4.6) which contains the quantum frame field in (2.6), rather than just boson creation and annihilation operators, we shall take advantage of the fact that the inner product in each single-boson Fock subfibre $\mathcal{F}_{1;x}$ can be expressed in the alternative form (4.2.4) in the case of rest mass $m = 1$, so that in the generic case of any $m > 0$ we have, in accordance with (3.7.19) and (3.21),

$$\langle \boldsymbol{\Psi}_1 | \boldsymbol{\Psi}_2 \rangle = i\hat{Z}_{\ell,m}\int \Psi_1^*(\boldsymbol{\zeta})\,\overleftrightarrow{\partial}_k\,\Psi_2(\boldsymbol{\zeta})\,d\sigma^k(\boldsymbol{q})\,d\Omega(\boldsymbol{v})\ , \qquad \hat{Z}_{\ell,m} = m\,\hat{Z}_{\ell m}\ . \tag{4.7}$$

Since a corresponding expression can be then established for the inner product within each n-boson subfibre $\mathcal{F}_{n;x}$, (4.6) can be recast in the form

$$\boldsymbol{P}_{j;\boldsymbol{s}(x)} = -\tfrac{1}{2}\hat{Z}_{\ell,m}\int :\varphi^{(+)}(x;\boldsymbol{\zeta})\,\overleftrightarrow{\partial}_k\,\partial_j\,\varphi^{(-)}(x;\boldsymbol{\zeta}):\, d\sigma^k(\boldsymbol{q})\,d\Omega(\boldsymbol{v}) \tag{4.8}$$

The advantage of using the form (4.7) of the inner product is that

$$\int \Psi_1(\boldsymbol{\zeta})\,\overleftrightarrow{\partial}_k\,\Psi_2(\boldsymbol{\zeta})\,d\sigma^k(\boldsymbol{q})\,d\Omega(\boldsymbol{v}) = 0\ , \qquad \boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2 \in \mathcal{F}_{1;x}\ , \tag{4.9}$$

as can be established by making the transition to the momentum representation by means of the inverse of the transformation in (3.7.11) for the case in (4.1.9). This implies that

$$\int \varphi^{(\pm)}(x;\boldsymbol{\zeta})\,\overleftrightarrow{\partial}_k\,\partial_j\,\varphi^{(\pm)}(x;\boldsymbol{\zeta})\,d\sigma^k(\boldsymbol{q})\,d\Omega(\boldsymbol{v}) = 0\ , \tag{4.10}$$

as can be easily verified by applying the above operator-valued integrals to any n-boson state, and using (2.4) and (2.5) in the two respective cases of integrands with creation and

annihilation operators. Hence, by using the following counterpart,

$$:\varphi^{(-)}(x;\zeta)\varphi^{(+)}(x;\zeta):\equiv:\varphi^{(+)}(x;\zeta)\varphi^{(-)}(x;\zeta):\equiv\varphi^{(+)}(x;\zeta)\varphi^{(-)}(x;\zeta)\ , \tag{4.11}$$

of the normal ordering of quantum fields in (1.9), we obtain that

$$\boldsymbol{P}_{j;\boldsymbol{s}(x)}=-\tfrac{1}{2}\hat{Z}_{\ell,m}\int:\varphi(x;\zeta)\overleftrightarrow{\partial}_k\partial_j\varphi(x;\zeta):\,d\sigma^k(\boldsymbol{q})d\Omega(\boldsymbol{v}) \tag{4.12}$$

In view of the crucial significance of this very simple derivation, which on the surface appears to be totally analogous to a similar derivations in conventional QFT (cf. Schweber, 1961, Sec. 7d), it should be emphasized that the latter derivation is merely formal, since it deals with the ill-defined quantum field in (1.3), rather than its smeared version in (1.4); whereas, the present derivation is mathematically rigorous, due to the fact that the quantum frame field in (2.6) is a *bona fide* smooth operator-valued function of $\boldsymbol{\zeta}$.

As a consequence of this fact, and of the additional fact that this field satisfies the (geometrically) local Klein-Gordon equation in (2.7), we deduce in a mathematically totally legitimate manner that, if we choose for the 3-surface σ for the $\boldsymbol{q}$-integration in (4.12) any of the hyperplanes corresponding to a constant value of q^0 in the local Poincaré frame $\boldsymbol{s}(x)$, such as the hyperplane with $q^0=0$, then, for $j=0$,

$$\boldsymbol{P}_{0;\boldsymbol{s}(x)}=\tfrac{1}{2}\hat{Z}_{\ell,m}\int_{q^0=0}:m^2\varphi^2(x;\zeta)+\sum\nolimits_{j=0}^{3}\varphi^2_{,j}(x;\zeta):\,d\mathbf{q}\,d\Omega(\boldsymbol{v})\ ,\quad \varphi_{,j}:=\partial\varphi/\partial q^j\ ; \tag{4.13a}$$

moreover, in the case of $j=1,2,3$, by integrating by parts along σ we obtain the following:

$$\boldsymbol{P}_{a;\boldsymbol{s}(x)}=\hat{Z}_{\ell,m}\int_{q^0=0}:\varphi_{,a}(x;\zeta)\varphi_{,0}(x;\zeta):\,d\mathbf{q}\,d\Omega(\boldsymbol{v})\ ,\qquad a=1,2,3\ . \tag{4.13b}$$

In view of the fact that the entire present framework is Poincaré gauge invariant, all four equations in (4.13) can be expressed in the following manifestly covariant form:

$$\boldsymbol{P}_{j;\boldsymbol{s}(x)}=\int:T_{jk}[\varphi(x;\zeta)]:\,d\sigma^k(\boldsymbol{q})d\Omega(\boldsymbol{v}) \tag{4.14}$$

We note that the above integrand is a *bona fide* operator-valued function, given by the normally-ordered values of the following *bona fide* operator-valued stress-energy tensor:

$$T_{jk}[\varphi]=\hat{Z}_{\ell,m}\left(\varphi_{,j}\varphi_{,k}+\tfrac{1}{2}\eta_{jk}\left(m^2\varphi^2-\eta^{il}\varphi_{,i}\varphi_{,l}\right)\right) \tag{4.15}$$

Outwardly, this stress-energy tensor has the same form as the energy-momentum tensor in (1.7), but the existence of the normalization constant appearing in (4.15) could not possibly have been inferred from any classically-based arguments; rather, it represents a purely quantum feature, which together with the normal ordering in (4.14), has emerged from the fundamental fact that, as infinitesimal generators of spacetime translations, the four operators in (4.6) are related to a *quantum* 4-momentum, whose timelike component is related to local energy, and is bounded from below by its zero expectation value in the local quantum vacuum state $\boldsymbol{\Psi}_{0;x}$, and whose remaining values are related to the total boson energies contained in various local quantum states. According to (4.2.6b), this boson quantum state *normalization* constant is fixed by those energies, but it diverges in the

sharp-point limit $\ell \to 0$. Hence, the recovery in the special relativistic regime of the corresponding conventional expressions in (1.6) require an *infinite* renormalization, to which no legitimate physical justification can be assigned even by invoking customary "self-interaction" arguments, since in that regime all the quantum fields are *free* fields.

Thus, we see already at this early stage that the renormalizations required in conventional QFT have their roots in the quantum field localization problem. The smearing required in (1.4) provides a first indication that such is the case, but since it is merely a *mathematical* solution to a *physical* problem, it provides no hints as to the true physical origins of those renormalization constants. On the other hand, the normalization constant $\tilde{Z}_{\ell,m}$ in (3.8.4b) has earlier revealed itself as a local quantum state normalization constant, whose values are unambiguously fixed by the reproducibility property (4.4) of the propagators for parallel transport; whereas, the renormalization constant $\hat{Z}_{\ell,m}$ in (4.7) has now revealed itself as being physically fixed by the spacing of quantum free energy-momentum values.

A derivation analogous to that of (4.14) can be applied to the infinitesimal generators of the Lorentz transformations in (3.4) with respect to the axes of the local Lorentz frame in $\boldsymbol{s}(x)$, which within the single-boson Fock subfibre $\mathcal{F}_{1;x}$ are given by (4.4.12). On account of the fact that the Q-operators in (4.4.12) are defined in (4.4.10) by means of the eight real variables contained in the complex-valued 4-tuple ζ, which determine the components of ζ with respect to the Poincaré frame $\boldsymbol{s}(x)$ in accordance with (4.2.1), such a derivation provides the following expressions for the angular momentum operators:

$$\boldsymbol{M}^{ij}_{\boldsymbol{s}(x)} = \int\!:\!Q^i_{\boldsymbol{s}(x)}T^{jk}[\varphi(x;\zeta)] - Q^j_{\boldsymbol{s}(x)}T^{ik}[\varphi(x;\zeta)]\!:\! d\sigma_k(\boldsymbol{q})\, d\Omega(\boldsymbol{v}) \ . \tag{4.16}$$

The normal ordering in (4.14) and (4.16), which from a classical point of view appears artificial and unjustified, reveals itself as totally natural after the propagator for parallel transport is expressed in the form (4.5), and the "infinitesimal" propagators on its right-hand side are expressed as follows in terms of the infinitesimal generators in (4.14) and (4.16), and the values assumed by the connection coefficients along the path γ:

$$K_\gamma(\mathbf{f}_n;\mathbf{f}_{n-1}) = \Big\langle \boldsymbol{\Phi}_{\mathbf{f}_n(x_n)} \Big| \Big(\mathbf{1}_{x_n} - i\delta x^j_n \boldsymbol{P}_{j;\boldsymbol{s}(x_n)} + \tfrac{i}{2}\tilde{\boldsymbol{\omega}}_{kl}(\delta x_n)\boldsymbol{M}^{kl}_{\boldsymbol{s}(x_n)} \Big) \boldsymbol{\Phi}_{\mathbf{f}_{n-1}(x_n)} \Big\rangle + O\big((\delta\tau_n)^2\big) \ ,$$

$$\mathbf{f}_n \in \mathbf{F}_{x_n} \ , \qquad \mathbf{f}_{n-1} \in \mathbf{F}_{x_{n-1}} \ , \qquad \delta\tau_n = \tau_n - \tau_{n-1} \ . \tag{4.17}$$

If, for the sake of notational simplicity, we restrict our attention to a cross-section $\boldsymbol{s}$ of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ which is identifiable with a cross-section of the Lorentz frame bundle $L\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, then the "infinitesimal" increments in location and the single-boson modes for the second-quantized frames in (4.17) assume the values

$$\delta x^i_n = \boldsymbol{\theta}^i_\mu(x_n)\dot{x}^\mu(\tau_n)\delta\tau_n \ , \qquad \boldsymbol{\delta x}_n = \boldsymbol{e}_i(x_n)\, \delta x^i_n \ , \qquad x_n = x(\tau_n) \ , \tag{4.18a}$$

$$\mathbf{f}_n(x_n) = \mathbf{f}_n \ , \quad \mathbf{f}_{n-1}(x_n) = \Big((\boldsymbol{\sigma}^{\boldsymbol{s}}_{x_n})^{-1} \circ \boldsymbol{\sigma}^{\boldsymbol{s}}_{x_{n-1}} \Big) \mathbf{f}_{n-1} \ , \quad \boldsymbol{s}(x_n) = \big(\mathbf{0}, \boldsymbol{e}_i(x_n)\big) \ , \tag{4.18b}$$

in terms of the local coordinate charts and of the soldering maps at the various chosen subdivision points along γ. In fact, since the state vectors in (4.18b) belong to single-boson Fock subfibres, by applying to those state vectors the soldering maps in (3.5) we obtain

within the typical fibre $\mathbf{F}$ the following coordinate wave functions in the complex-valued Poincaré gauge variables $\zeta \in \mathbf{R}^4 \times \boldsymbol{V}^+$:

$$f(x_n;\zeta) := \left(\boldsymbol{\sigma}^s_{x_n}\mathbf{f}_n(x_n)\right)(\zeta) = f_n(\zeta) \ , \qquad f(x_{n-1};\zeta) = \left(\boldsymbol{\sigma}^s_{x_n}\mathbf{f}_{n-1}(x_n)\right)(\zeta) \ . \tag{4.19}$$

We can now insert in (4.17) the expressions (4.14) and (4.16), and conclude that

$$\begin{aligned} K_\gamma(\mathbf{f}_n;\mathbf{f}_{n-1}) = &\left[1 - i\,\delta x_n^j\, P_j(f(x_n);f(x_{n-1}))\right]\left\langle \boldsymbol{\Phi}_{\mathbf{f}_n(x_n)} \middle| \boldsymbol{\Phi}_{\mathbf{f}_{n-1}(x_n)} \right\rangle \\ &+ i\tilde{\boldsymbol{\omega}}_{kl}(\delta x_n) M^{kl}(f(x_n);f(x_{n-1}))\left\langle \boldsymbol{\Phi}_{\mathbf{f}_n(x_n)} \middle| \boldsymbol{\Phi}_{\mathbf{f}_{n-1}(x_n)} \right\rangle + O\left((\delta\tau_n)^2\right) , \end{aligned} \tag{4.20}$$

where, due to the normal ordering of the integrands in these operator-valued integrals, as well as to the fact that, according to (3.11), the elements of second-quantized boson frames are eigenvectors of the annihilation operators, we have:

$$P_i(f(x_n);f(x_{n-1})) = \int T_{ik}[\bar{f}(x_n;\zeta) + f(x_{n-1};\zeta)]\, d\sigma^k(q) d\Omega(v) \ , \tag{4.21a}$$

$$M^{jk}(f(x_n);f(x_{n-1})) = \int Q^{[j}T^{k]l}[\bar{f}(x_n;\zeta) + f(x_{n-1};\zeta)] d\sigma_l(q) d\Omega(v) \ , \tag{4.21b}$$

$$Q^{[j}T^{k]l} := \tfrac{1}{2}\left(Q^jT^{kl} - Q^kT^{jl}\right) \ . \tag{4.21c}$$

In the above expressions, the bars are used to denote the complex conjugates of the coordinate wave functions belonging to those eigenvectors, so that if those wave functions were to change smoothly along γ, then a classical energy-momentum and angular momentum field would result in (4.21) in the limit $\varepsilon \to +0$. Of course, when used in (4.5), the choices for those coordinate wave functions actually have to cover all possibilities, so that if we view the adoption of various choices as a random process, the likelihood of such "classical" choices is zero. It is in this sense that quantum parallel transport manifests a "stochastic" nature when viewed from the classical point of view.

Upon inserting (4.20) and (4.21) into (4.5) we actually obtain a kind of quantum "stochastic" functional integral of the form

$$K_\gamma(\mathbf{f}'';\mathbf{f}') = \lim_{\varepsilon\to+0} \int \prod_{n=N}^{1}{}' \mathcal{D}[f(x_n)] \exp[i\,\delta\tau_n\, \boldsymbol{L}_\gamma(f(x_n);f(x_{n-1}))] \ , \tag{4.22a}$$

$$\mathbf{f}(x_0) = \mathbf{f}' , \quad \mathbf{f}(x_N) = \mathbf{f}'' , \quad \mathcal{D}[f(x_n)] = \mathbf{d}f_n \mathbf{d}f_n^* , \quad n = 1,\ldots,N-1 \ , \tag{4.22b}$$

where the primed symbol for the product signifies that the integration over the N-th mode is to be omitted. The "Lagrangian" within the above functional integral assumes its physically most revealing form if one takes advantage of the general relation

$$\left\langle \boldsymbol{\Phi}_{\mathbf{f}} \middle| \boldsymbol{\Phi}_{\mathbf{g}} \right\rangle = \exp\left[-\tfrac{1}{2}\left(\|\mathbf{f}\|^2 + \|\mathbf{g}\|^2\right) + \langle \mathbf{f} | \mathbf{g} \rangle\right] , \qquad \mathbf{f},\mathbf{g} \in \mathbf{F}_x \ , \tag{4.23}$$

which follows from (2.9), (3.8) and (3.9). Indeed, the use of (4.23) in the computation of the inner products in (4.20) casts this "Lagrangian" in the following form:

$$\boldsymbol{L}_\gamma(f(x_n);f(x_{n-1})) = \tfrac{i}{2}\big[\langle f(x_n)|\, p(x_n)\rangle - \langle p(x_n)|\, f(x_{n-1})\rangle\big]$$
$$-X_n^{\,i} P_i(f(x_n);f(x_{n-1})) + \tfrac{1}{2}\tilde{\boldsymbol{\omega}}_{jk}(\boldsymbol{X}_n)\boldsymbol{M}^{jk}(f(x_n);f(x_{n-1}))\ , \qquad (4.24a)$$

$$p(x_n) = \big(f(x_n) - f(x_{n-1})\big)/\delta\tau_n\ , \quad \boldsymbol{X}_n = \dot{x}^\mu(\tau_n)\,\partial_\mu\ , \quad x_n = x(\tau_n)\ . \qquad (4.24b)$$

To recast the functional integral in (4.22) in the form of an action integral involving Lagrangian densities rather than Lagrangians, we use the counterpart of (4.7) for the inner product in the typical fibre $\mathbf{F}$ to express the first of the inner products in (4.24a) in the form

$$\langle f(x_n)|\, p(x_n)\rangle = i\hat{Z}_{\ell,m}\int \bar{f}(x_n;\zeta)\tilde{\partial}_k\, p(x_n;\zeta)\, d\sigma^k(q)\, d\Omega(v), \quad f(x_n), p(x_n) \in \mathbf{F}, \qquad (4.25)$$

and treat the second one in a similar manner. This finally expresses the propagator in (4.1) for the parallel transport of second-quantized frames in the form of the functional integral

$$K_\gamma(\mathbf{f}'';\mathbf{f}') = \int \mathcal{D}f\, \exp\big(iS_\gamma[f,p]\big)\,, \qquad \mathcal{D}f = \prod\nolimits_{b>\tau\geq a} \mathcal{D}\big[f(x(\tau))\big]\ , \qquad (4.26)$$

which incorporates the following action along the smooth path γ,

$$S_\gamma[f,p] = \int_a^b d\tau \int L_k^\gamma[f(x(\tau);\zeta), p(x(\tau);\zeta)]\, d\sigma^k(q)\, d\Omega(v)\ . \qquad (4.27)$$

This action incorporates the following "Lagrangian densities"[5] for each one of the four degrees of freedom determined by the elements $\boldsymbol{e}_k(x)$, $k = 0,1,2,3$, of the local Lorentz frame element within the section $\boldsymbol{s}$ at each $x \in \gamma$:

$$L_k^\gamma[f(x(\tau);\zeta), p(x(\tau);\zeta)] =$$
$$\tfrac{1}{2}\hat{Z}_{\ell,m}\big[\bar{f}_{,k}(x(\tau);\zeta)\, p(x(\tau);\zeta) + f_{,k}(x(\tau-0);\zeta)\,\bar{p}(x(\tau);\zeta)\big]$$
$$-\tfrac{1}{2}\hat{Z}_{\ell,m}\big[\bar{f}(x(\tau);\zeta)\, p_{,k}(x(\tau);\zeta) + f(x(\tau-0);\zeta)\,\bar{p}_{,k}(x(\tau);\zeta)\big]$$
$$-X^i(\tau)T_{ik}[\bar{f}(x(\tau);\zeta) + f(x(\tau-0);\zeta)]$$
$$+\eta_{kl}\,\tilde{\boldsymbol{\omega}}_{ij}(\boldsymbol{X}_n(\tau))\,\boldsymbol{Q}^{[i}\boldsymbol{T}^{j]l}[\bar{f}(x(\tau);\zeta) + f(x(\tau-0);\zeta)]\ . \qquad (4.28)$$

The above expressions for the "Lagrangian densities" determining the parallel transport in the direction $\boldsymbol{e}_k(x)$, $x \in \gamma$, follows from the expressions in (4.24) after taking in (4.22) the limit $\varepsilon = \max(\tau_n - \tau_{n-1}) \to +0$. Due to mathematically well-defined nature of all the quantum fields in (4.28), the action integral in (4.27) contains no ambiguities when it is interpreted in the above described manner. Note should be taken of the fact that, due to the presence in (4.28) of the limits from the left, the densities in (4.28) depend not only on

[5] In conventional QFT in Minkowski space, a "Lagrangian density" is the *seemingly* unique one dictated by a corresponding classical field theory. A formally analogous "Lagrangian density" will be derived in Sec. 5.6. However, contrary to formal appearances, the "Lagrangian density" of any conventional QFT model is neither truly "Lagrangian" in the sense of its classical counterpart (which is derivable from well-defined Lagrangian action principles by means of Noether's theorem), nor is it a mathematically well-defined "density." In fact, the discussion in Sec. 6.4 will reveal a host of ambiguities that remain hidden behind the conventional choice of notation.

$x \in \mathbf{M}$, but on the choice of γ as well. However, we have followed a universally adopted notation in the theory of path integrals, and incorporated within the field designated by $p(x_n)$ contributions from $x_n = x(\tau_n)$ as well as from the boson field values that were parallel transported from the preceding point $x(\tau_{n-1})$ – thus combining those contributions as if they behaved smoothly. On the other hand, the "stochastic" nature of the field "history" along γ has to be kept in mind, so that, prior to taking the limit $\varepsilon \to +0$, those two types of contributions have to be treated as independent quantities while we functionally integrate at each τ_n over all possible boson modes within the typical fibre $\mathbf{F}$.

The notation adopted in (4.27) is meant to underscore some of the *formal* similarities with the action integrals in conventional QFT (cf., e.g., Itzykson and Zuber, 1980). On the other hand, the form of the "Lagrangian densities" in (4.28) might appear, at first sight, cumbersome as well as unfamiliar. However, that is due to the generality allowed in the choice of Poincaré gauge, which greatly surpasses that of all the gauges which, according to (4.1.13), correspond to global choices of Lorentz frame in Minkowski space. Such general gauges allow for moving Poincaré frames which are highly non-inertial, so that the terms in the last line of (4.28) correspond to rotations and accelerations[6] of those frames in relation to the inertial frames which describe free-fall conditions.

5.5. Geometric Localization of Boson Field Exciton Modes

As we mentioned in the last section, the basic geometric aspects of free-fall quantum-geometric propagation within a Fock bundle $\mathcal{E}$ can be inferred from the free-fall quantum-geometric propagation of the local single-boson states that constitute its single-boson subbundle $\mathcal{E}_1$. However, the resulting propagators for quantum fields would display a rather unwieldy appearance if they were formulated in terms of propagators for the multitude of constituent quantum tensorial subframes within various n-boson subbundles. Hence, the main goal in this section is to prepare the ground for expressing this propagation in terms of action-based path integrals that clearly display all the essential physical properties of this type of quantum propagation. This propagation will be then intimately tied in with those geometrodynamic evolutions that give rise to globally hyperbolic as well as space and time oriented base Lorentzian manifolds $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$, and which are therefore mathematically equivalent to foliations of $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$ into reference hypersurfaces Σ_t (cf. Sec. 2.7).

Of paramount importance amongst the physical properties that have to be maintained by such propagation is the conservation of the *local* energy-momentum. It is the retention of this fundamental conservation law that will provide one of the physically most essential distinguishing features between the free-fall propagation of quantum fields in the present quantum geometric framework, and the type of free-fall propagation that emerges from the conventional formulation of QFT in curved spacetime.

Indeed, as we saw in Sec. 1.5, the conventional formulation of free-fall quantum field propagation in a stationary but non-static classical spacetime represented by the Loren-

[6] Corresponding terms manifest themselves when these same considerations are carried out in quantum bundles whose base manifolds are Newton-Cartan spacetimes (cf. Prugovečki, 1992, Chapter 4). From the point of view of ordinary Newtonian mechanics, they correspond to "fictional" forces, such as centrifugal and Coriolis forces, that are clearly in evidence in non-inertial frames – cf. e.g., (Honerkamp and Römer, 1993), Sec. 5.1.

tzian manifold $(\mathbf{M}, \boldsymbol{g}^{L})$ gives rise to *ex nihilo* particle production, which violates this most fundamental of all conservation laws. According to conventional wisdom, this purported free-fall phenomenon can be compared with Rindler particle production "observable" by a *non*inertial observer in Minkowski space, since the force field producing the acceleration of the latter supposedly "liberates the correlated fluctuations in a noncausally related region of [Minkowski] spacetime to become a real particle" (Unruh and Wald, 1984, p. 1055).

After considering this physically paradoxical theoretical prediction, which is a consequence of the purely *formal* application of the canonical quantization procedure to *curvilinear* coordinates in Minkowski space, P.C.W. Davies advanced the provocative thesis that in relativistic quantum theory "particles do not exist." In his words: "These 'particles' [in Minkowski space] carry no energy or momentum, yet they still excite a *static* detector! ... The 'particles' that excite a Rindler (accelerating) detector do not carry energy or momentum either. ... So, in a sense, the energy is not where the particles are, and may have long departed from the space-time region of the detector." (Davies, 1984, pp. 72-73).

On the other hand, in nonrelativistic quantum mechanics it is well-known that the purely formal application of the canonical quantization procedure in curvilinear coordinates is liable to yield results which are inconsistent with those obtained in Cartesian coordinates. However, in that context, it has never been suggested that this feature should be interpreted as giving rise to a previously unknown *physical* phenomenon. Rather, as Dirac has pointed out, in general the canonical quantization procedure "of passing from the classical to the quantum theory is not well-defined, ... [as it] depends on considerations of simplicity." (Dirac, 1950, p. 145.) Consequently, in Sec. 3.1 we advanced the alternative thesis that it is not the choice of *coordinates*, but rather the choice of reference *frames* that is of paramount significance to the quantization procedure, so that all the paradoxes that emerge when canonical quantization is carried out in Rindler coordinates in Minkowski space are merely mathematical artifacts resulting from an extrapolation of a procedure on purely formal grounds, rather than reflections of physical facts. However, even if one accepts all the above arguments based on Rindler coordinates, the question still remains: how can the experiences of a *non*-inertial observer in Minkowski spacetime be identified with those of an *inertial* observer in curved spacetime without violating the most fundamental of all CGR principles, namely the strong equivalence principle?

We have seen in Sec. 4.6 that the quantum-geometric propagation of single bosons abides by the strong equivalence principle, since no bosons are produced in *free fall*, and energy-momentum is conserved along all the paths of that propagation, which are in general constructed from arcs of geodesics. Hence, in the quantum-geometric approach, the physical situation with individual boson propagation is essentially the same as it is in the case of the free-fall motion of test particles in classical general relativity.

Like so many problems in QFT, the main *technical* problem with extending the *physical* idea of quantum-geometric propagation of single bosons to quantum fields is rooted in the notion of quantum locality. Indeed, on one hand, the form of the Lagrangian densities in (4.28) indicates that the parallel transport that gives rise to those densities treats the micro-detectors constituting quantum reference frames in a physically correct manner, namely as *passive* devices, whose state of motion does not influence the physical situation by "liberating" *non-existent* energy. On the other hand, the quantum connection in (3.3) is

based on the energy-momentum operators in (4.6), whose form (4.14) is related not only to the local stress-energy tensor at the point $\boldsymbol{O}_x \in T_x\mathbf{M}$ of contact between the tangent space $T_x\mathbf{M}$ and the manifold $\mathbf{M}$, but also to its values at all points $\boldsymbol{q} \in T_x\mathbf{M}$ on the hypersurface σ over which the integration in (4.14) is performed. In the case where the base Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ is the Minkowski space $(\mathbf{R}^4,\boldsymbol{\eta})$, all these points would be indentifiable with all the points x along an entire reference hypersurface Σ_t.

The key to the solution of this problem can be found within the canonical framework for CGR, which was reviewed in Sec. 2.7. As we saw there, the geometrodynamic evolution of 3-geometries is mathematically equivalent to a foliation (2.7.8) of the Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ into reference hypersurfaces Σ_t, and a central role in that evolution is played by the *local* proper energy density ρ and proper 3-momentum components j_a of the nongravitational sources, which in synchronous coordinates are defined by (2.7.21).

To extend (cf. Prugovečki, 1994) these classical concepts in a coordinate-independent manner to the quantum regime, let us introduce the *quantum-geometric boson field*

$$\boldsymbol{\varphi}(x,\boldsymbol{v}) := \varphi(x;-i\ell\boldsymbol{v}) \ , \qquad \boldsymbol{v} = v^i\boldsymbol{e}_i(x) \in \boldsymbol{V}_x^+ \subset T_x\mathbf{M} \ , \quad x \in \mathbf{M} \ , \tag{5.1}$$

as well as the field $\boldsymbol{n}$ of future-pointing unit vectors $\boldsymbol{n}(x) \in T_x\mathbf{M}$ that are normal to the chosen family of reference hypersurface Σ_t. Then, in any Poincaré gauge corresponding to a particular choice of moving frame $\boldsymbol{s} = \{(\boldsymbol{a}(x),\boldsymbol{e}_i(x))|\, x \in \mathbf{M}\}$, we can define the following *local energy-momentum density operators*,

$$\hat{\boldsymbol{P}}_{j;\boldsymbol{s}}(x) = n^k(x)\int_{V_x^+} :T_{jk}[\boldsymbol{\varphi}(x,\boldsymbol{v})]: \ d\Omega(\boldsymbol{v}) \ , \qquad j = 0,1,2,3 \ , \tag{5.2}$$

where $n^k(x)$, $k = 0,1,2,3$, denote the components of $\boldsymbol{n}(x)$ with respect to the local Lorentz frame $\{\boldsymbol{e}_i(x)\}$, $x \in \Sigma_t$, within the chosen cross-section $\boldsymbol{s}$ of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$. As a quantum counterpart of the local proper energy density ρ of a classical scalar field, we can then introduce the local *proper energy density operators* $\boldsymbol{\rho}$ of the quantum-geometric field $\boldsymbol{\varphi}$, defined as follows:

$$\boldsymbol{\rho}(x) = n^j(x)\hat{\boldsymbol{P}}_{j;\boldsymbol{s}}(x) = \int_{V_x^+} :n^j(x)n^k(x)T_{jk}[\boldsymbol{\varphi}(x,\boldsymbol{v})]: \ d\Omega(\boldsymbol{v}) \ . \tag{5.3}$$

Moreover, if the Poincaré moving frame $\boldsymbol{s}$ is *adapted* to the family of reference hypersurface Σ_t, in the sense that all the timelike components of its vierbeins are orthogonal to those hypersurfaces, so that

$$\boldsymbol{e}_0(x) = \boldsymbol{n}(x) \ , \qquad \boldsymbol{e}_a(x) \in T_x\Sigma_t \subset T_x\mathbf{M} \ , \quad x \in \Sigma_t \ , \quad a = 1,2,3 \ , \tag{5.4}$$

then, in accordance with (2.7.21), we can set

$$\boldsymbol{\rho}(x) = \hat{\boldsymbol{P}}_{0;\boldsymbol{s}}(x) \ , \qquad \boldsymbol{j}_a(x) = \hat{\boldsymbol{P}}_{a;\boldsymbol{s}}(x) \ , \quad a = 1,2,3 \ . \tag{5.5}$$

Clearly, the above operators $\boldsymbol{j}_a$ play the role of 3-momentum density components of the quantum-geometric field $\boldsymbol{\varphi}$ in (5.1).

At this stage, the quantum stress-energy tensor in (5.2) is actually still the one defined in (4.15) in terms of the Klein-Gordon quantum frame field φ in (2.6). However, we shall now demonstrate that the Fock bundle $\mathcal{F}$ in (2.3) can be restricted to a bundle $\hat{\mathcal{F}}$ of local-

ized modes of the quantum field $\boldsymbol{\varphi}$ in which that tensor will assume the form

$$T_{jk}[\boldsymbol{\varphi}] = \hat{Z}_{\ell,m}\left(\boldsymbol{\varphi}_{;j}\boldsymbol{\varphi}_{;k} + \tfrac{1}{2}\eta_{jk}(m^2\boldsymbol{\varphi}^2 - \eta^{il}\boldsymbol{\varphi}_{;i}\boldsymbol{\varphi}_{;l})\right) , \tag{5.6}$$

where the semicolons denote differentiation at each $x \in \mathbf{M}$ with respect to the Riemann normal coordinates at x that are determined, as in (4.2.15), by the axes of the local Lorentz frames $\{\boldsymbol{e}_i(x)\}$. As we shall see later in this section, these partial derivatives can be also interpreted as Riemann-covariant derivatives.

To construct the bundle $\hat{\mathcal{E}}$ containing all the localized field modes of the quantum-geometric boson field $\boldsymbol{\varphi}$, let us first introduce the following *standard exciton-mode fibre*:

$$\hat{\mathbf{F}} = \left\{\hat{f} \mid f \in \mathbf{F}\right\} , \quad \hat{f}(v) := f(-i\ell v) , \quad v \in \boldsymbol{V}^+ . \tag{5.7}$$

It should be noted that although this standard fibre is linear, and although its elements are constructed from those of the typical exciton-state fibre $\mathbf{F}$ in (2.1), $\hat{\mathbf{F}}$ is *not* a subspace of $\mathbf{F}$. Consequently, the following *standard boson field-mode fibre* constructed from $\hat{\mathbf{F}}$,

$$\hat{\mathcal{F}} = \bigoplus_{n=0}^{\infty} \hat{\mathcal{F}}_n \quad , \qquad \hat{\mathcal{F}}_n = \hat{\mathbf{F}} \underset{S}{\otimes} \cdots \underset{S}{\otimes} \hat{\mathbf{F}} , \tag{5.8}$$

is not a subspace of the typical Fock fibre $\mathcal{F}$ introduced in (2.1). As a matter of fact, the analyticity[7] of all the wave functions in $\mathcal{F}_n$ ensures that to each n-exciton field mode $\hat{\Psi}_n \in \hat{\mathcal{F}}_n$ belongs to a unique element $\Psi_n \in \mathcal{F}_n$ which is such that

$$\hat{\Psi}_n(v_1,\ldots,v_n) = \Psi_n(-i\ell v_1,\ldots,-i\ell v_n) , \qquad \hat{\Psi}_n \in \hat{\mathcal{F}}_n , \qquad \Psi_n \in \mathcal{F}_n , \tag{5.9}$$

and that this relationship determines an isomorphism between $\hat{\mathcal{F}}_n$ and $\mathcal{F}_n$, which, in turn, establishes the following isomorphism between $\hat{\mathcal{F}}$ and $\mathcal{F}$:

$$\hat{\Psi} \leftrightarrow \Psi , \qquad \hat{\Psi} \in \hat{\mathcal{F}} , \qquad \Psi \in \mathcal{F} . \tag{5.10a}$$

In particular, this isomorphism gives rise to the following identification between standard exciton modes and the coordinate wave functions of their local states:

$$\hat{f} \leftrightarrow f , \qquad \hat{f} \in \hat{\mathbf{F}} , \qquad f \in \mathbf{F} . \tag{5.10b}$$

To be able to transfer later to a quantum field mode bundle $\hat{\mathcal{E}}$ constructed from $\hat{\mathcal{F}}$ all the key geometric features already established for the Fock bundle $\mathcal{E}$, let us consider the following wave functions within the space of exciton modes $\hat{\mathbf{F}}$,

$$\hat{\boldsymbol{\Phi}}_{mv}(v') = \boldsymbol{\Phi}^{\boldsymbol{u}_0}_{m;-i\ell v}(-i\ell v') , \qquad \hat{\boldsymbol{\Phi}}_{mv} \in \hat{\mathbf{F}} , \qquad \boldsymbol{\Phi}^{\boldsymbol{u}}_{m;-i\ell v} \in \mathbf{F} , \tag{5.11}$$

that are associated with the standard frame $\boldsymbol{u}_0$ in $\mathbf{R}^4$, namely with following the 4-tuple:

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[7] An analytic (i.e., holomorphic) function of several complex variables is analytic at a point if and only if it is analytic in a neighborhood of that point in each one of its complex variables separately – cf., e.g., (Vladimirov, 1966), Sec I.4. Hence, this conclusion follows from the well-known result that an analytic function of a single complex variable is uniquely determined by its values on an infinite set which has a point of accumulation where the function is analytic.

$$\boldsymbol{u}_0 = \{\hat{e}_0 = (1,0,0,0),\ \hat{e}_1 = (0,1,0,0),\ \hat{e}_2 = (0,0,1,0),\ \hat{e}_3 = (0,0,0,1)\}\ . \qquad (5.12)$$

On account of (5.10b), the representation (4.1.11) of the Poincaré group induces in $\hat{\mathbf{F}}$ the following representation of the Lorentz group:

$$U(\Lambda)\ :\ \hat{f}(v) = f(-i\ell v)\ \mapsto\ \hat{f}(\Lambda^{-1}v) = f(-i\ell\Lambda^{-1}v)\ ,\quad \Lambda \in \mathrm{SO}_0(3,1)\ . \qquad (5.13)$$

This fact, taken in conjunction with the fact that the standard exciton-mode wave functions in (5.11) are left invariant by space rotations, but not by velocity boosts, implies that the following exciton-mode frame elements,

$$\hat{\boldsymbol{\Phi}}_{mv} = U(\Lambda_v)\,\hat{\boldsymbol{\Phi}}_{m\hat{e}_0}\ ,\qquad v \in \boldsymbol{V}^+\ , \qquad (5.14)$$

represent generalized coherent states of the Lorentz group $\mathrm{SO}_0(3,1)$ in the sense introduced by Perelomov (1972). Furthermore, the invariance of the measure element $d\Omega(v)$ under Lorentz transformations implies that these Perelomov-type coherent states provide a continuous resolution of the identity within the standard fibre $\hat{\mathbf{F}}$, if the latter is viewed as a Hilbert space that is isomorphic to $\mathbf{F}$ on account of the identification implicit in (5.7):

$$\int_{V^+}\left|\hat{\boldsymbol{\Phi}}_{mv}\right\rangle d\Omega(v)\left\langle\hat{\boldsymbol{\Phi}}_{mv}\right| \ =\ Z_{\ell,m}^{-1}\ \mathbf{1}_{\hat{\mathbf{F}}}\ ,\qquad Z_{\ell,m}^{-1} = \int\left|\left\langle\hat{\Phi}_v\middle|\hat{\Phi}_{\hat{e}_0}\right\rangle\middle/\left\|\hat{\Phi}_{\hat{e}_0}\right\|\right|^2 d\Omega(v). \qquad (5.15)$$

The above value of the renormalization constant $Z_{\ell,m}$ for exciton-mode localization follows from Perelomov's theory of generalized coherent states.[8] On account of (4.2.9) and (5.10), it is seen to be explicitly given by the following integral:

$$Z_{\ell,m}^{-1} = \tfrac{1}{2}\left((2\pi)^3\hat{Z}_{\ell,m}^{1/2}\middle/\ell\tilde{Z}_{\ell,m}\right)^2\int_2^\infty\left|\mathrm{K}_1(\ell m\sqrt{2x})\right|^2\frac{dx}{\sqrt{x-2}} \qquad (5.16)$$

Furthermore, (5.15) implies that the inner product in the original typical fibre $\mathbf{F}$ can be expressed in the alternative form,

$$\left\langle f_1\middle|f_2\right\rangle \ =\ Z_{\ell,m}\int_{V^+}\hat{f}_1^*(v)\hat{f}_2(v)\,d\Omega(v) \ :=\ \left\langle\hat{f}_1\middle|\hat{f}_2\right\rangle\ , \qquad (5.17)$$

which can be taken to represent also the inner product in the standard exciton-mode fibre $\hat{\mathbf{F}}$. Consequently, we have in $\hat{\mathbf{F}}$ the following counterpart of (2.13):

$$\Delta^{(+)}(v';v'') := -i\hat{\boldsymbol{\Phi}}_{mv''}(v') = iZ_{\ell,m}\int_{V^+}\Delta^{(+)}(v';v)\Delta^{(+)}(v;v'')d\Omega(v)\ . \qquad (5.18)$$

Upon using (5.10b) to introduce in $\hat{\mathbf{F}}$ the orthonormal basis corresponding to the one in (3.16), so that

$$\sum_{\alpha=1}^{\infty}\left|\hat{w}_\alpha\right\rangle\left\langle\hat{w}_\alpha\right| \ =\ \mathbf{1}_{\hat{\mathbf{F}}}\ ,\qquad \left\langle\hat{w}_\alpha\middle|\hat{w}_\beta\right\rangle = \delta_{\alpha\beta}\ ,\qquad \alpha,\beta = 1,2,\ldots\ , \qquad (5.19a)$$

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[8] Cf. (Perelomov, 1986), Sec. 2.3, and recall that any element Λ of $\mathrm{SO}_0(3,1)$ can be represented as a product of a boost Λ_v and a 3-space rotation R (cf. Miller, 1972, p. 289). Since $d\Omega(v)$ is left invariant by Λ, the equations (2.3.4) and (2.3.5) in (Perelomov, 1986) yield, modulo a mass renormalization, (5.16) and (5.15), respectively.

$$\hat{f}(v) = \sum_{\alpha=1}^{\infty} \hat{z}_\alpha \hat{w}_\alpha(v) , \qquad \hat{z}_\alpha = \left\langle \hat{w}_\alpha \middle| \hat{f} \right\rangle , \qquad \alpha = 1,2,\ldots , \tag{5.19b}$$

we can deduce that, as was the case in (3.19),

$$\hat{f}(v) = \left\langle \hat{\Phi}_{mv} \middle| \hat{f} \right\rangle , \qquad \hat{w}_\alpha(v) = \left\langle \hat{\Phi}_{mv} \middle| \hat{w}_\alpha \right\rangle , \qquad \alpha = 1,2,\ldots , \tag{5.20}$$

and that the convergence of the series in (5.19b) is not only in the strong topology of $\hat{\mathbf{F}}$ (i.e., in its norm), but that it is also pointwise as well as uniform in $v \in \boldsymbol{V}^+$.

These results can be extended to the standard field-mode fibre $\hat{\mathcal{F}}$ in (5.8), which, on account of (5.11), carries the following representation of the Lorentz group:

$$\boldsymbol{U}(\Lambda) \; : \; \hat{\Psi}_n(v_1,\ldots,v_n) \mapsto \hat{\Psi}_n(\Lambda^{-1}v_1,\ldots,\Lambda^{-1}v_n) , \quad \hat{\Psi}_n \in \hat{\mathcal{F}}_n , \quad n = 1,2,\ldots . \tag{5.21}$$

Consequently, we can construct from this typical fibre the following *field-mode bundle*,

$$\hat{\mathcal{E}} = \boldsymbol{LM}_0(\boldsymbol{g}^{\mathrm{L}}) \times_{\mathbf{G}} \hat{\mathcal{F}} , \qquad \mathbf{G} = \mathrm{SO}_0(3,1) , \tag{5.22}$$

which, by its mode of construction, has the Lorentz group as its structure group. Hence, the above quantum field-mode bundle $\hat{\mathcal{E}}$ is identifiable with the Fock bundle $\mathcal{E}$ in (2.3) only if the structure group of the latter is restricted to the Lorentz group $\mathrm{SO}_0(3,1)$ – as we implicitly did in (4.18b), while deriving (4.20) and (4.22).

On the other hand, we observe that during the derivation of (4.20) and (4.22) we could have worked, in any Poincaré gauge $\boldsymbol{s}$, with the creation and annihilation operators

$$\varphi^{(\pm)}(f) := \boldsymbol{\sigma}_x^{\boldsymbol{s}} \varphi^{(\pm)}(\mathbf{f})(\boldsymbol{\sigma}_x^{\boldsymbol{s}})^{-1} , \quad f = (\boldsymbol{\sigma}_x^{\boldsymbol{s}})^{-1}\mathbf{f} \in \mathbf{F} , \quad \mathbf{f} \in \mathbf{F}_x , \tag{5.23a}$$

$$\varphi^{(\pm)}(x;\zeta) := \boldsymbol{\sigma}_x^{\boldsymbol{s}} \varphi^{(\pm)}(x;\boldsymbol{\zeta})(\boldsymbol{\sigma}_x^{\boldsymbol{s}})^{-1} , \quad \varphi^{(+)}(f) = \int \varphi^{(+)}(x;\zeta) f(\zeta) d\Sigma(\zeta) , \tag{5.23b}$$

which act in the typical fibre $\mathcal{F}$, and therefore possess the following counterparts in $\hat{\mathcal{F}}$,

$$\varphi^{(+)}(\hat{f}) := Z_{\ell,m} \int_{\boldsymbol{V}^+} \varphi^{(+)}(x;-i\ell v) \hat{f}(v) d\Omega(v) = \varphi^{(-)}(\hat{f})^* , \qquad \hat{f} \in \hat{\mathbf{F}} . \tag{5.24}$$

This suggests the introduction in $\hat{\mathcal{F}}$ of the counterparts of the canonical creation and annihilation operators in (3.23) for the orthonormal basis in (5.19), so that

$$\hat{a}_\alpha^* := \varphi^{(+)}(\hat{w}_\alpha) = Z_{\ell,m} \int_{\boldsymbol{V}^+} \varphi^{(+)}(-i\ell v) \hat{w}_\alpha(v) d\Omega(v) , \qquad \hat{w}_\alpha \in \hat{\mathbf{F}} . \tag{5.25}$$

In this manner we arrive at the following quantum field-mode counterparts of the second-quantized boson frame elements in (3.8), and of their corresponding representations as Glauber coherent states,

$$\hat{\boldsymbol{\Phi}}_{\hat{f}} = \exp\left[-\tfrac{1}{2} \left\| \hat{f} \right\|^2 + \varphi^{(+)}(\hat{f}) \right] \Psi_0 = \exp\left[-\tfrac{1}{2} \sum_{\alpha=1}^{\infty} |\hat{z}_\alpha|^2 + \sum_{\alpha=1}^{\infty} \hat{z}_\alpha \, \hat{a}_\alpha^* \right] \Psi_0 , \tag{5.26}$$

constituting *standard quantum field-mode frames*. We can then define within the standard Fock fibre $\hat{\mathcal{F}}$ of multi-exciton modes the continuous resolution of the identity

$$\int_{\hat{\mathbf{F}}} \left|\hat{\Phi}_{\hat{f}}\right\rangle \mathbf{d}\hat{f}\,\mathbf{d}\hat{f}^{*}\left\langle\hat{\Phi}_{\hat{f}}\right| = \mathbf{1}_{\hat{\mathcal{F}}} \ , \tag{5.27}$$

by the same method as in (3.27)-(3.29). Moreover, as was the case in (3.11), we also have

$$\varphi^{(-)}(\hat{f})\hat{\Phi}_{\hat{g}} = \left\langle\hat{f}\middle|\hat{g}\right\rangle\hat{\Phi}_{\hat{g}} \ , \qquad \hat{f},\hat{g}\in\hat{\mathbf{F}} \ . \tag{5.28}$$

In view of the feasibility of systematic extrapolation of key concepts and results in typical Fock fibres to standard quantum field-mode fibres, the soldering map in (3.5) can be transferred to the quantum field-mode bundle in (5.22) in an arbitrary Poincaré moving frame $\boldsymbol{s}$,

$$\hat{\boldsymbol{\sigma}}^{\boldsymbol{s}}_{x} \ : \ \hat{\boldsymbol{\Psi}} \mapsto \bigoplus\nolimits_{n=0}^{\infty} \hat{\Psi}_{n} \ , \qquad \hat{\boldsymbol{\Psi}}\in\hat{\mathcal{F}}_{x} = \bigoplus\nolimits_{n=0}^{\infty} \hat{\mathcal{F}}_{x;n} \ , \tag{5.29}$$

by defining the coordinate wave functions corresponding to the standard n-exciton modes in (5.29) as follows:

$$\hat{\Psi}_{n}(v_{1},\ldots,v_{n}) = \Psi_{n}(-a(x)-i\ell v_{1},\ldots,-a(x)-i\ell v_{n}) \ . \tag{5.30}$$

As a consequence, the quantum field-mode bundle $\hat{\mathcal{E}}$ in (5.22) acquires the Poincaré group $\mathrm{ISO}_{0}(3,1)$ as a structure group, and becomes isomorphic to the Fock bundle $\mathcal{E}$. Hence, the quantum connection in (3.3) can be now transferred from the Fock bundle $\mathcal{E}$ to the present quantum field-mode bundle $\hat{\mathcal{E}}$. Similarly, the operators for parallel transport in $\mathcal{E}$, defined in (3.1), give rise to operators for parallel transport between the corresponding fibres of $\hat{\mathcal{E}}$, whose outcome can be derived from (3.1) by taking advantage of (5.29). In other words, the outcome of the parallel transport along a smooth curve γ between two fibres of $\hat{\mathcal{E}}$ can be computed by performing the parallel transport along γ between the corresponding two fibres of $\mathcal{E}$, and then carrying out the identifications between the respective fibres of $\mathcal{E}$ and $\hat{\mathcal{E}}$ that result from (5.10), (5.29) and (5.30).

By exploiting the isomorphism determined by (5.10a) between the typical fibres $\mathcal{F}$ and $\hat{\mathcal{F}}$, we can also transfer the action of the annihilation operators in (5.23b) from $\mathcal{F}$ to $\hat{\mathcal{F}}$, so that, on account of (5.28),

$$\varphi^{(-)}(x;\zeta)\hat{\Phi}_{\hat{f}} = \varphi^{(-)}(\hat{\Phi}^{\boldsymbol{s}(x)}_{m;\zeta})\Phi_{\hat{f}} = \left\langle\hat{\Phi}^{\boldsymbol{s}(x)}_{m;\zeta}\middle|\hat{f}\right\rangle\Phi_{\hat{f}} = \hat{f}_{q}(v)\hat{\Phi}_{\hat{f}} \ , \tag{5.31a}$$

$$\hat{f}_{q}(v) = f(\zeta) \ , \qquad \zeta = q - i\ell v \ , \qquad \hat{f}\in\hat{\mathbf{F}}, \quad f\in\mathbf{F} \tag{5.31b}$$

Consequently, upon introducing the unitary transforms of the 4-momentum operators in (4.14) that are induced in $\mathcal{F}$ by the soldering map in (3.5),

$$P_{j;\boldsymbol{s}(x)} = \boldsymbol{\sigma}^{\boldsymbol{s}}_{x}\,\boldsymbol{P}_{j;\boldsymbol{s}(x)}(\boldsymbol{\sigma}^{\boldsymbol{s}}_{x})^{-1} = \int\!:T_{jk}[\varphi_{q}(x,v)]:\,d\sigma^{k}(q)\,d\Omega(v) \ , \tag{5.32a}$$

$$\varphi_{q}(x,v) = \varphi(x;q-i\ell v) = \boldsymbol{\sigma}^{\boldsymbol{s}}_{x}\,\varphi(x;\boldsymbol{a}+(q^{k}-iv^{k})\boldsymbol{e}_{k})(\boldsymbol{\sigma}^{\boldsymbol{s}}_{x})^{-1} \ , \tag{5.32b}$$

their action can be transferred to $\hat{\mathcal{F}}$, and from there to the fibres of $\hat{\mathcal{E}}$:

$$(\hat{\boldsymbol{\sigma}}^{\boldsymbol{s}}_{x})^{-1} P_{j;\boldsymbol{s}(x)} \hat{\boldsymbol{\sigma}}^{\boldsymbol{s}}_{x} = \int : T_{jk}[\hat{\boldsymbol{\varphi}}_{q}(x,\upsilon)] : d\sigma^{k}(q) d\Omega(\upsilon) \,, \tag{5.33a}$$

$$T_{jk}[\hat{\boldsymbol{\varphi}}_{q}] = \hat{Z}_{\ell,m}\left(\hat{\boldsymbol{\varphi}}_{q;j}\hat{\boldsymbol{\varphi}}_{q;k} + \tfrac{1}{2}\eta_{jk}(m^{2}\hat{\boldsymbol{\varphi}}_{q}^{2} - \eta^{il}\hat{\boldsymbol{\varphi}}_{q;i}\hat{\boldsymbol{\varphi}}_{q;l})\right) , \tag{5.33b}$$

$$\hat{\boldsymbol{\varphi}}_{q}(x,\upsilon) = (\hat{\boldsymbol{\sigma}}^{\boldsymbol{s}}_{x})^{-1}\hat{\boldsymbol{\varphi}}_{q}(x,\upsilon)\hat{\boldsymbol{\sigma}}^{\boldsymbol{s}}_{x} \,, \qquad \hat{\boldsymbol{\varphi}}_{-\boldsymbol{a}(x)}(x,\upsilon) := \hat{\boldsymbol{\varphi}}(x,\upsilon) \equiv \boldsymbol{\varphi}(x,\upsilon) \,. \tag{5.33c}$$

The semicolons in (5.33b) denote partial derivatives with respect to the Poincaré gauge variables q^{j}, but in the case of the quantum field $\hat{\boldsymbol{\varphi}}$ in (5.33c) they can be also interpreted as derivatives with respect to the Riemann normal coordinates y^{j} at $x \in \mathbf{M}$ that were defined in (4.2.15), and can be therefore related to covariant derivatives.

This can be most easily seen by introducing, for any Poincaré gauge $\boldsymbol{s}$ with $\boldsymbol{a}(x) \equiv \mathbf{0}$, the coherent field-mode sections (cf. Sec. 4.4) whose constituents are the local *quantum field-mode frames* with the following elements:

$$\hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\hat{f};x} = (\hat{\boldsymbol{\sigma}}^{\boldsymbol{s}}_{x})^{-1}\hat{\boldsymbol{\Phi}}_{\hat{f}} \in \hat{\mathcal{F}}_{x} \,, \qquad x \in \mathbf{M} \tag{5.34}$$

These elements are eigenvectors of the annihilation operators in (5.32b), so that we have:

$$\left\langle \hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\hat{f};x} \middle| \hat{\boldsymbol{\varphi}}_{q}(x,\upsilon)\hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\hat{g};x} \right\rangle = (\hat{f}^{*}_{q}(x,\upsilon) + \hat{g}_{q}(x,\upsilon))\left\langle \hat{\boldsymbol{\Phi}}_{\hat{f}} \middle| \hat{\boldsymbol{\Phi}}_{\hat{g}} \right\rangle \,, \tag{5.35}$$

$$\left\langle \hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\hat{f};x} \middle| : T_{jk}[\hat{\boldsymbol{\varphi}}_{q}(x,\upsilon)] : \hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\hat{g};x} \right\rangle = T_{jk}[\hat{f}^{*}_{q}(x,\upsilon) + \hat{g}_{q}(x,\upsilon)]\left\langle \hat{\boldsymbol{\Phi}}_{\hat{f}} \middle| \hat{\boldsymbol{\Phi}}_{\hat{g}} \right\rangle \,. \tag{5.36}$$

We observe now that the expansion in (3.15) for vectors belonging to Fock bundles can be generalized to operators acting within such bundles by regarding those operators as being elements of tensorial bundles.[9] Then, we can write

$$\boldsymbol{A} = A^{\mathbf{f}'\mathbf{f}''}\boldsymbol{\Phi}_{\mathbf{f}'} \otimes \overline{\boldsymbol{\Phi}}_{\mathbf{f}''} \,, \qquad A^{\mathbf{f}'\mathbf{f}''} := \left\langle \boldsymbol{\Phi}_{\mathbf{f}'} \middle| \boldsymbol{A}\,\boldsymbol{\Phi}_{\mathbf{f}''} \right\rangle \,, \tag{5.37}$$

for any operator $\boldsymbol{A}$ acting in a fibre $\mathcal{F}_{x}$. A similar expansion holds for an operator acting in a fibre $\hat{\mathcal{F}}_{x}$, so that, due to (5.35)-(5.36), we have in any Poincaré gauge with $\boldsymbol{a}(x) \equiv 0$,

$$\hat{\boldsymbol{\varphi}}(x,\upsilon) = \hat{\varphi}^{\hat{f}\hat{g}}(x,\upsilon)\hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\hat{f};x} \otimes \overline{\hat{\boldsymbol{\Phi}}}^{\boldsymbol{s}}_{\hat{g};x} \,, \qquad \hat{\varphi}^{\hat{f}\hat{g}}(x,\upsilon) = (\hat{f}^{*}_{0}(x,\upsilon) + \hat{g}_{0}(x,\upsilon))\left\langle \hat{\boldsymbol{\Phi}}_{\hat{f}} \middle| \hat{\boldsymbol{\Phi}}_{\hat{g}} \right\rangle \,, \tag{5.38}$$

$$T_{jk}[\hat{\boldsymbol{\varphi}}] = T^{\hat{f}\hat{g}}_{jk}\hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\hat{f};x} \otimes \overline{\hat{\boldsymbol{\Phi}}}^{\boldsymbol{s}}_{\hat{g};x} \,, \qquad T^{\hat{f}\hat{g}}_{jk} = T_{jk}[\hat{f}^{*}_{0}(x,\upsilon) + \hat{g}_{0}(x,\upsilon)]\left\langle \hat{\boldsymbol{\Phi}}_{\hat{f}} \middle| \hat{\boldsymbol{\Phi}}_{\hat{g}} \right\rangle \,. \tag{5.39}$$

The last factor in (5.38) is obviously x-independent. Hence, in accordance with (4.4.18) and (4.4.19), the external derivatives of the quantum field $\hat{\boldsymbol{\varphi}}$ are completely determined by the following external derivatives of local exciton modes:

$$\boldsymbol{\partial}_{j}\hat{f}^{*}_{0}(x,\upsilon) = -\partial\hat{f}^{*}_{q}(x,\upsilon)/\partial q^{j}\big|_{q=0} \,, \qquad \boldsymbol{\partial}_{j}\hat{g}_{0}(x,\upsilon) = -\partial\hat{g}_{q}(x,\upsilon)/\partial q^{j}\big|_{q=0} \,. \tag{5.40}$$

Clearly, the operators on the left-hand side of the above two equations coincide with

[9] Cf. Sec. 4.5, and in particular the expansions in (4.5.10) and (4.5.12).

those for differentiation with respect to the Riemann normal coordinates at $x \in \mathbf{M}$. Moreover, since any Poincaré gauge with $\boldsymbol{a}(x) \equiv 0$ is equivalent to a Lorentz gauge represented by a cross-section of $L\mathbf{M}_0(\boldsymbol{g}^L)$, and since in any Riemann normal coordinates at $x \in \mathbf{M}$ the Christoffel symbols for the Levi-Civita connection vanish at that point x, we see that after that connection is extended from $L\mathbf{M}$ to $L(T\mathbf{M})$ (cf., Poor, 1981, Sec. 9.31), then these operators coincide with those for Riemann-covariant differentiation. Consequently, this substantiates the earlier made claim as to the meaning of the derivatives denoted by semicolons in (5.6), and completes the proof of that relation.

For the sake of mathematical clarity, we have thus far maintained a sharp distinction between operators acting in a Fock fibre $\mathcal{F}_x$, and those acting within the corresponding quantum field-mode fibre $\hat{\mathcal{F}}_x$. However, we shall hereafter take advantage of the already established bundle isomorphism between $\mathcal{E}$ and $\hat{\mathcal{E}}$ to identify all their respective fibres. Then, whenever necessary, we shall use a circumflex to distinguish, as we did in (5.31b), between local multi-exciton state vectors and the local field modes they determine. For example, on account of this convention we shall have

$$\hat{\boldsymbol{\Phi}}_{\mathbf{f}} = \exp\left[-\tfrac{1}{2}\left\|\hat{\mathbf{f}}\right\|^2 + \boldsymbol{\varphi}^{(+)}(\hat{\mathbf{f}})\right] \boldsymbol{\Psi}_{0;x} \in \hat{\mathcal{F}}_x\,, \qquad \mathbf{f} \in \mathbf{F}_x\,, \qquad \hat{\mathbf{f}} \in \hat{\mathbf{F}}_x\,, \tag{5.41}$$

$$\left\langle \hat{\boldsymbol{\Phi}}_{\mathbf{f}} \middle| \hat{\boldsymbol{\Phi}}_{\mathbf{g}} \right\rangle = \exp\left[-\tfrac{1}{2}\left(\left\|\hat{\mathbf{f}}\right\|^2 + \left\|\hat{\mathbf{g}}\right\|^2\right) + \left\langle \hat{\mathbf{f}} \middle| \hat{\mathbf{g}} \right\rangle\right]\,, \qquad \hat{\mathbf{f}}, \hat{\mathbf{g}} \in \hat{\mathbf{F}}_x\,, \tag{5.42}$$

and the continuous resolution of the identity operator within each quantum field mode fibre $\hat{\mathcal{F}}_x$ assumes the following form:

$$\int_{\mathbf{F}_x} \left|\hat{\boldsymbol{\Phi}}_{\mathbf{f}}\right\rangle \mathcal{D}\mathbf{f} \left\langle \hat{\boldsymbol{\Phi}}_{\mathbf{f}}\right| = \hat{\mathbf{I}}_x\,, \qquad \mathcal{D}\mathbf{f} \equiv \mathbf{d}\hat{\mathbf{f}}\,\mathbf{d}\hat{\mathbf{f}}^*\,, \tag{5.43}$$

With these stipulations in mind, the propagator for the parallel transport of local Fock state vectors defined in (4.1) can be also viewed as a propagator of quantum field modes, and therefore expressed in the alternative form

$$K_\gamma(\mathbf{f}'';\mathbf{f}') = \left\langle \hat{\boldsymbol{\Phi}}_{\mathbf{f}''} \middle| \tau_\gamma(x'',x')\,\hat{\boldsymbol{\Phi}}_{\mathbf{f}'} \right\rangle\,, \qquad \hat{\boldsymbol{\Phi}}_{\mathbf{f}'} \in \hat{\mathcal{F}}_{x'}\,, \qquad \hat{\boldsymbol{\Phi}}_{\mathbf{f}''} \in \hat{\mathcal{F}}_{x''}\,. \tag{5.44}$$

We could then proceed with treating this expression within $\hat{\mathcal{E}}$ as we treated (4.1) within $\mathcal{E}$ during the derivation of (4.17)-(4.24). The only key difference in this treatment emerges from the fact that, in a Poincaré gauge with $\boldsymbol{a}(x) \equiv 0$, the first term between the square brackets in (4.24a) is in case of (5.44) given by the expression,

$$\left\langle \hat{f}(x_n) \middle| \hat{p}(x_n) \right\rangle = Z_{\ell,m} \int_{V^*} \hat{f}_0^*(x_n;v)\,\hat{p}_0(x_n;v)\,d\Omega(v)\,, \tag{5.45}$$

and a similar expression holds for the second term. Hence, in this alternative approach no action-type of functional integral, similar to the one in (4.27), results for the propagator of parallel transport in (5.44), since these expressions contain only the field modes localized at $q = 0$, so that no Lagrangian densities are obtained for $q \neq 0$. However, we shall see in the next section that the propagators for parallel transport in $\hat{\mathcal{E}}$ are closely related to quantum-geometric propagators of quantum field modes which provide mathematically well-defined

general relativistic extrapolations of the only formally defined special relativistic Feynman propagators of field configuration modes in conventional QFT.[10]

5.6. Quantum-Geometric Boson Field Propagation in Free Fall

To define the free-fall propagator for the quantum-geometric field $\boldsymbol{\varphi}$ in (5.1), let us proceed as in Sec. 4.6. Thus, we again adopt a foliation of $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ into synchronous reference hypersurfaces Σ_t labeled by a globally defined parameter t, which represents the mean proper time determined by the free-fall propagation of the origins of local quantum frames along those timelike geodesics that are orthogonal to the hypersurfaces Σ_t. We then denote by $\mathbf{S}_n$, $n = 1,\ldots,N$, the base-segments in $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ whose respective inflow and outflow hypersurfaces $\Sigma_{t_{n-1}}$ and Σ_{t_n} are inserted between two fixed hypersurfaces $\Sigma_{t'} = \Sigma_{t_0}$ and $\Sigma_{t''} = \Sigma_{t_N}$ in such a manner that $t_n - t_{n-1} = (t''-t')/N = \varepsilon$. For any choice of Poincaré moving frame $\boldsymbol{s} = \{(\boldsymbol{a}(x),\boldsymbol{e}_i(x))|\, x \in \mathbf{M}\}$ we introduce the following coherent field-mode sections above each one of the inflow-outflow hypersurfaces corresponding to $n = 1,\ldots,N-1$:

$$\hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\varphi_n(x_n)} \in \hat{\mathcal{F}}_{x_n}\,, \qquad \varphi_n(x_n) = (\boldsymbol{\sigma}^{\boldsymbol{s}}_{x_n})^{-1}\varphi_n \in \mathbf{F}_{x_n}\,, \qquad \varphi_n \in \mathbf{F}\,, \qquad x_n \in \Sigma_{t_n}\,. \tag{6.1}$$

The comparison of the expression in (4.6.4) for the propagators for parallel transport of exciton state vectors with that in (4.6.7) for the boson quantum-geometric propagator suggests the following quantum-geometric *free-fall boson field propagator*,

$$K(\varphi(x');\varphi(x')) = \lim_{\varepsilon\to+0}\prod_{n=N}^{1}{}'\int \mathcal{D}\varphi_n \left\langle e^{i\varepsilon\int d\sigma(x_n)\rho(x_n)}\,\hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\varphi_n(x_n)}\,\middle|\,\tau_\gamma(x_n,x_{n-1})\,\hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\varphi_{n-1}(x_{n-1})}\right\rangle\,, \tag{6.2}$$

between two boson field modes corresponding to some prescribed exciton coordinate modes $\varphi(x')$ and $\varphi(x'') = \varphi_N(x_N)$ at the respective base locations $x' = x_0 \in \Sigma_{t'}$ and $x'' = x_N \in \Sigma_{t''}$. According to the extrapolation to the quantum regime of the strong equivalence principle that was adopted in Sec. 4.6, the parallel transport in (6.2) is to be carried out along the geodesic arcs $\gamma(x_{n-1},x_n)$ connecting the points x_{n-1} on the inflow hypersurface $\Sigma_{t_{n-1}}$ with the points x_n on the outflow hypersurface Σ_{t_n} of the base-segment $\mathbf{S}_n$, $n = 1,\ldots,N$. As in Sec. 5.4, the prime next to the product symbol in (6.2) indicates that there are actually no integrations over $\varphi_N(x_N) = \varphi(x'')$ or over $d\sigma(x_N) = d\sigma(x'')$. Thus, the functional integrations in (6.2) are to be carried out over all the coherent field modes above Σ_{t_n}, $n = 1,\ldots,N-1$, with respect to the functional "measure" with element

$$\mathcal{D}\varphi_n = \prod\nolimits_{x_n\in\Sigma_{t_n}} \mathcal{D}[\,\varphi_n(x_n)]\,, \tag{6.3}$$

whose heuristic meaning is suggested by the functional integration in (4.26). However,

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[10] Cf., e.g., Rivers, 1987, Sec. 4.7. However, as we saw in Sec. 5.1, even *free* conventional quantum fields, such as those in (1.3), actually do not exist as operator-valued fields *at a point* $x = (t,\mathbf{x})$, but only in the "smeared" sense of the operator-valued *distributions* in (1.4). Since the latter are incompatible with the notion of "spatial configuration $\varphi(\mathbf{x})$ at a time t," the conventional approach "remains more a statement of expectations than a reality." (Rivers, 1987, p. 24.) As we shall see in the remainder of this chapter, a mathematically legitimate realization of these "expectations" requires a thorough reevaluation of not only the mathematical, but also of the physical postulates underlying the conventional approach to QFT.

(6.2) also incorporates the Riemannian measure element $d\sigma(x_n)$ determined by the 3-metric $\gamma(x_n)$ in (2.7.9b) on each one of the reference hypersurfaces Σ_{t_n}, so that a more careful formulation of this type of functional integration is required (cf. Prugovečki, 1994).

To assign a mathematically precise meaning to the functional integrals in (6.2) over all the exciton modes above each hypersurface Σ_{t_n}, we shall follow the same method as in Riemannian integration. Thus, let us choose on a reference hypersurface Σ_t any region B that has finite Riemannian measure $\sigma(B)$, and partition it into smaller regions B_ν, $\nu = 1,\ldots,M$, that have Riemannian measure $\sigma(B_\nu)$. Then, let us choose within each B_ν a point y_ν, and set, by definition,

$$\int_{\mathbf{E}(\Sigma_t)} \left| \hat{\boldsymbol{\Phi}}^{\mathbf{s}}_{\varphi(x)} \right\rangle \mathcal{D}\varphi \left\langle e^{i\varepsilon \int d\sigma(x)\rho(x)} \hat{\boldsymbol{\Phi}}^{\mathbf{s}}_{\varphi(x)} \right| =$$

$$\lim_{B\to\Sigma_t} \lim_{M\to\infty} \prod_{\nu=1}^{M} \int_{\mathbf{F}_{y_\nu}} \left| \hat{\boldsymbol{\Phi}}^{\mathbf{s}}_{\varphi(y_\nu)} \right\rangle \mathcal{D}[\varphi(y_\nu)] \left\langle e^{i\varepsilon\sigma(B_\nu)\rho(y_\nu)} \hat{\boldsymbol{\Phi}}^{\mathbf{s}}_{\varphi(y_\nu)} \right| , \tag{6.4}$$

on the subbundle $\hat{\mathcal{E}}(\Sigma_t)$ obtained by restricting $\hat{\mathcal{E}}$ to Σ_t. Clearly, the above limits are to be then computed by inserting the right-hand side of (6.4) between smooth cross-sections of $\hat{\mathcal{E}}(\Sigma_t)$. The smoothness of those sections will then help secure the existence of such limits. This is the case in (6.2), since these cross-sections are the coherent sections in (6.1). The Poincaré gauge invariance of the proper energy density operator in (5.3) and of the parallel transport in (5.44) then insures, in conjunction with the fact that the functional integrations in (6.2) are carried out over all possible coherent exciton modes within the fibres above each Σ_{t_n}, that the free-fall propagator defined by these limits is Poincaré gauge invariant.

If we denote the creation and the annihilation operators for the quantum-geometric field $\boldsymbol{\varphi}$ in (5.1) by

$$\boldsymbol{\varphi}^{(\pm)}(x,\boldsymbol{v}) = \varphi^{(\pm)}(x;-i\ell\boldsymbol{v}) \ , \quad x \in \mathbf{M} \ , \quad \boldsymbol{v} \in \boldsymbol{V}^+_x \ . \tag{6.5}$$

then the action of the latter on the elements of the coherent field-mode sections on the left-hand sides of the inner products in (6.2) is given by

$$\boldsymbol{\varphi}^{(-)}(x_n,\boldsymbol{v})\hat{\boldsymbol{\Phi}}^{\mathbf{s}}_{\varphi_n(x_n)} = \varphi_n(x_n,\upsilon)\hat{\boldsymbol{\Phi}}^{\mathbf{s}}_{\varphi_n(x_n)} \ , \qquad \varphi_n(x_n,\upsilon) = (\hat{\boldsymbol{\sigma}}^{\mathbf{s}}_{x_n}\varphi_n(x_n))(\upsilon) \ . \tag{6.6}$$

Thus, as we shall see later in this section, the free-fall propagator in (6.2) can be interpreted as a transition probability amplitude between the two coherent modes of the quantum field $\boldsymbol{\varphi}$. In very general terms, this conceptualization as a transition probability amplitude is the same as in conventional QFT (cf. Schulman, 1981, Chapter 32; Rivers, 1987, Sec. 4.7), but there is a key difference in its realization: the quantum-geometric propagator in (6.2) is mathematically well-defined, and its physical interpretation is supported by a consistent general relativistic theory of quantum measurement which, as we shall demonstrate later in this section, emerges from the existence of the conserved probability current in (3.7.16); whereas, the corresponding conventional QFT propagator is, even in the special relativistic regime, mathematically ill-defined, and it lacks a similar probabilistic interpretation.

The outcome of the parallel transport in (6.2) can be computed by means of (4.22)-(4.26), in which case the Lagrangian densities for parallel transport will manifest them-

selves explicitly. However, there is an alternative method, which is in the present context preferable since it yields an action integral based on a single Lagrangian.

This alternative method consists of employing (4.6.1)-(4.6.4) to parallel transport the exciton mode $\varphi_{n-1}(x_{n-1})$ that gives rise to the quantum field-mode frame at x_{n-1} and set

$$\varphi_{n-1}(x_n) = \tau_\gamma(x_n, x_{n-1})\,\varphi_{n-1}(x_{n-1}) \in \mathbf{F}_{x_n} , \qquad \varphi_{n-1}(x_{n-1}) \in \mathbf{F}_{x_{n-1}} . \tag{6.7}$$

We can then use (5.26) to reconstruct the parallel transported quantum field-mode frame at x_n that appears on the right-hand side of the inner product in (6.2):

$$\hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\varphi_{n-1}(x_n)} = \tau_\gamma(x_n, x_{n-1})\,\hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\varphi_{n-1}(x_{n-1})} , \qquad \gamma = \gamma(x_{n-1}, x_n) . \tag{6.8}$$

Consequently, we now have

$$\boldsymbol{\varphi}^{(-)}(x_n, \boldsymbol{v})\,\hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\varphi_{n-1}(x_n)} = \varphi_{n-1}(x_n, v)\,\hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\varphi_{n-1}(x_n)} , \qquad x_n \in \Sigma_{t_n} , \quad v \in \boldsymbol{V}^+ , \tag{6.9}$$

and (6.2) can be expressed in the following alternative form:

$$K(\varphi(x'');\varphi(x')) = \lim_{\varepsilon\to+0} \prod_{n=N}^{1}{}' \int_{\mathbf{E}(\Sigma_{t_n})} \mathcal{D}\varphi_n \left\langle \hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\varphi_n(x_n)} \middle| e^{-i\varepsilon \int d\sigma(x_n)\rho(x_n)}\, \hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\varphi_{n-1}(x_n)} \right\rangle . \tag{6.10}$$

By adapting (5.42) to the present situation, we can write:

$$\left\langle \hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\varphi_n(x_n)} \middle| \hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\varphi_{n-1}(x_n)} \right\rangle = \exp\left[-\tfrac{1}{2}\left(\left\|\hat{\varphi}_n(x_n)\right\|^2 + \left\|\hat{\varphi}_{n-1}(x_n)\right\|^2 \right) + \left\langle \hat{\varphi}_n(x_n) \middle| \hat{\varphi}_{n-1}(x_n) \right\rangle \right] . \tag{6.11}$$

Since this type of relation is satisfied by all types of Glauber coherent states, it appears in all coherent state representations of path integrals. Consequently, we can take advantage of customary notation (cf., e.g., Schulman, 1981, p. 248), and express (6.11) in the following physically suggestive form:

$$\left\langle \hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\varphi_n(x_n)} \middle| \hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\varphi_{n-1}(x_n)} \right\rangle = \exp\left[i\varepsilon \tfrac{i}{2} \left(\left\langle \hat{\varphi}_n(x_n) \middle| \dot{\varphi}_n(x_n) \right\rangle - \left\langle \dot{\varphi}_n(x_n) \middle| \hat{\varphi}_{n-1}(x_n) \right\rangle \right) \right] , \tag{6.12a}$$

$$\dot{\varphi}_n(x_n) := \left(\hat{\varphi}_n(x_n) - \hat{\varphi}_{n-1}(x_n) \right) / \delta^4 x_n \in \hat{\mathbf{F}}_{x_n} , \qquad \delta^4 x_n = \varepsilon\,\delta\sigma(x_n) . \tag{6.12b}$$

Let us now work in any Poincaré moving frame $\boldsymbol{s}$ adapted to the family of reference hypersurface Σ_t, so that (5.4) and (5.5) are satisfied. Then, on account of (6.4)-(6.6), the formal matrix elements in (6.10) assume the following form:

$$\exp\left\{ -i\delta^4 x_n \int_{V^+} d\Omega(v)\, T_{00}[\overline{\varphi}_n(x_n, v) + \varphi_{n-1}(x_n, v)] \right\} \left\langle \hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\varphi_n(x_n)} \middle| \hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\varphi_{n-1}(x_n)} \right\rangle . \tag{6.13}$$

Hence, upon using (5.17) in the computation of the inner products in (6.12a), the free-fall propagator in (6.10) can be expressed as a limit of integrals over broken paths,

$$K(\varphi(x'');\varphi(x')) = \lim_{\varepsilon\to+0} \prod_{n=N}^{1}{}' \int_{\mathbf{E}(\Sigma_{t_n})} \mathcal{D}\varphi_n \exp\left[i\varepsilon \int d\sigma(x) \int d\Omega(v)\, \mathcal{L}_0(\varphi_n, \varphi_{n-1}) \right], \tag{6.14}$$

where, using the same notation as in (6.4), we can interpret the above functional integrals in the following Riemannian sense:

$$\lim_{B\to\Sigma_{t_n}}\lim_{M\to\infty}\prod_{\nu=1}^{M}\int_{\mathbf{F}_{y_\nu}}\mathcal{D}[\varphi_n(y_\nu)]\exp[i\varepsilon\,\sigma(B_\nu)\int d\Omega(v)\,\mathcal{L}_0(\varphi_n(y_\nu,v),\varphi_{n-1}(y_\nu,v))]. \quad (6.15)$$

The fact that their integrands

$$\mathcal{L}_0(\varphi_n(x,v),\varphi_{n-1}(x,v)) =$$

$$\tfrac{i}{2}Z_{t,m}[\overline{\varphi}_n(x,v)\,\dot{\varphi}_n(x,v)-\dot{\overline{\varphi}}_n(x,v)\,\varphi_{n-1}(x,v)]-T_{00}[\overline{\varphi}_n(x,v)+\varphi_{n-1}(x,v)]\,, \quad (6.16)$$

are smooth in $x\in\Sigma_{t_n}$ helps secure their existence. We note that these integrands incorporate the normalization constant for exciton mode localization defined in (5.16).

At the formal level, the limit $\varepsilon = t_n - t_{n-1}\to +0$ in (6.14) results in the path integral

$$K(\varphi(x'');\varphi(x')) = \int \mathrm{D}\varphi\exp[iS(\varphi)]\,, \qquad \mathrm{D}\varphi = \prod_{t''>t>t'}\prod_{x\in\Sigma_t}\mathcal{D}\varphi(x)\,, \quad (6.17)$$

based on an "action" which can be expressed in the customary notation (cf., e.g., Rivers, 1987, pp. 58 and 80),

$$S(\varphi) = \int_{t'}^{t''}dt\int_{\Sigma_t}d\sigma(x)\int_{V^+}d\Omega(v)\,\mathcal{L}_0(\varphi_t(x,v),\varphi_{t-0}(x,v))\,, \quad (6.18)$$

in terms of the following "Lagrangian density":

$$\mathcal{L}_0 = \tfrac{i}{2}Z_{t,m}\Big[\overline{\varphi}_t(x,v)\,\dot{\varphi}_t(x,v)-\dot{\overline{\varphi}}_t(x,v)\,\varphi_{t-0}(x,v)\Big]-T_{00}[\overline{\varphi}_t(x,v)+\varphi_{t-0}(x,v)] \quad (6.19)$$

The above notation has been chosen to resemble as much as possible the conventional notation,[11] but not to disguise the fact that the "density" in (6.19), which enters the action integral in (6.18), depends not only on the values of the field mode at the initial point of a given oriented arc in a broken path, but also on its values at the final point of that arc. We observe that, in the apparently similar situation encountered in (4.22)-(4.28), the separation between two sequential points on a curve γ converged to each other when $\varepsilon = \max\,\delta\tau_n \to +0$ in (4.22a), so that one could legitimately refer to (4.28) as a "density." However, when $\varepsilon = t_n - t_{n-1}\to +0$ in (6.10), there are always spacelike arcs whose end-points do not approach each other, since t is not a parameter for an *individual* broken path, but a common parameter for *all* the paths followed during the quantum-geometric propagation. Hence, strictly speaking, the Lagrangian "density" in (6.19) is *not* a true density involving the values of the field at the single base locations $x\in\mathbf{M}$.

This is actually also the case in the conventional theory of path integration even in the nonrelativistic regime. However, in that context, the points at which the evaluations of the

[11] Cf., e.g., Eq. (32.4) on p. 318 of (Schulman, 1981), or Eq. (4.104) on p. 80 of (Rivers, 1987), but take special note of the discussions in Sec. 32.1 of (Schulman, 1981) and in Sec. 6.5 of (Rivers, 1987), which reveal some of the difficulties that remain hidden when using the conventional notation.

Hamiltonian are computed are chosen, whenever necessary, somewhere in between the end points of the segments of broken polygonal paths, so that the outcome of path integration would fit an already well-established equation. For example, such is the case in the treatment of the Schrödinger equation with an external vector potential, where those points are chosen to be the midpoints in between the end points (cf. Schulman, 1981, p. 23). This feature of Feynman path integrals is therefore considered an "ambiguity" in the assignment of a propagator to a given Hamiltonian, and it is deemed to represent a counterpart of the ambiguities in the operator ordering that emerge during the canonical quantization procedure (cf. Sec. 3.1). In the conventional coherent state representation of path integration, these ambiguities are "resolved" by neglecting all contributions from the initial points of broken arcs. Hence, such conventional treatments "*appear* to give a unique propagator for any Hamiltonian – but this cannot be right, since, as many have found to their dismay, path integrals do not settle ambiguities in the ordering of operators." (Schulman, 1981, p. 249.)

This fundamental mathematical fact is ascribed to the "roughness" of the paths of quantum propagation. Comparisons are then drawn with the "roughness" of the paths of Brownian motion.[12] However, the propagation along the latter paths are related to a *bona fide* probability measure, namely the Wiener measure. Hence, it is meaningful to talk of the probability of propagation of a Brownian particle along an arbitrarily small (cylinder) neighborhood of a given stochastic path, so that such "roughness" has no deeper physical or mathematical consequences; whereas, no such measure exists for the former paths, due to the fundamental *physical* fact that the very concept of quantum propagation along an *individual* path contradicts the uncertainty principle.

Due to the far-reaching *physical* implications of this issue, it is important to realize that in conventional QFT "the fields entering the functional integral are very rough, much worse than the Brownian motion paths, with short wavelength problems getting more severe with increasing dimension [of spacetime]." (*Ibid.*, p. 320.) However, the customary notation hides the fact that the singular nature of the conventional "Lagrangian densities" exacerbates the aforementioned mathematical ambiguities, which are "actually a reflection of ultraviolet divergencies in the usual formulation of field theory." (*Ibid.*, p. 320.) The option of choosing special points, such as midpoints, in between the end points of arcs so as to fit an equation of motion, exercised by the nonrelativistic theory of path integration, is not available in conventional QFT in the presence of interactions, since the field equations are not well-defined in that case even in Minkowski space. Indeed, the nonlinear terms present in all such equations are mathematically *meaningless* as they stand since, as we saw in Sec. 5.1, conventional quantum fields do not exist at individual points x in Minkowski space; whereas, when those quantum fields are treated in a mathematically correct manner, namely as operator-valued distributions,[13] then all such nonlinear terms become subject to

[12] Almost all (in Wiener measure) stochastic paths for Brownian motion are continuous, but the subset of smooth paths has zero Wiener measure. This means that there is zero probability for a Brownian particle to follow a smooth path, so that almost all of its paths are "rough" – cf., e.g, (Kac, 1959), or (Arnold, 1974).

[13] Thus, the mathematical *interpretation* ascribed to such terms in constructive QFT (cf. Glimm and Jaffee, 1987) is mathematically rigorous, but leads to physically trivial theories (cf. Fröhlich, 1982, 1991). On the other hand, the attitude in conventional QFT is that the path integral is merely "an algorithm for handling the Feynman perturbation series," so that such integrals "are as rigorous – or non-rigorous – as [that] series, irrespective of whether they can be given proper mathematical definition or not." (Rivers, 1987, p. 81).

arbitrary mathematical interpretation, shared by all attempts to give mathematical meaning to the formal expressions involving the square or higher powers of the δ-"function."

This physically and mathematically undesirable situation is exacerbated by the conventional treatment of QFT in curved spacetime, where difficulties with the definition and uniqueness of the stress-energy tensor occur even in the absence of mutual quantum field interactions (cf. Birrell and Davies, 1982, Sec. 6.6; Fulling, 1989, Chapter 9).

None of these ambiguities occur in the quantum-geometric approach to field propagation in QGR, since the proper energy operator $\boldsymbol{\rho}$ in (6.2) entails no singularities; moreover, despite the lack of a *bona fide* measure over paths that would permit a Lebesgue-type formulation of the path integral in (6.17), that action integral is mathematically well-defined by the Riemann-type approach to path integration that is embodied in the transition from (6.2) to (6.14). Of course, at first sight, a high price seems to have been paid, since one is no longer dealing with *bona fide* densities, such as those in (4.28) or (5.43).[14] However, as we have just seen, the situation is in this respect no different than it is in the conventional framework when the earlier discussed terms are not arbitrarily neglected.

On the other hand, as we explained in Sec. 3.5, the teleological nature of action principles makes them suspect as primary physical principles. In fact, already the analysis of the role they play in the nonrelativistic quantum regime reveals that their significance is in that context basically heuristic at the mathematical as well as the physical level – as seen from the discussion in Sec. 3.5 of the epistemological ideas underlying action principles, and of their role in quantum theory. As a consequence, in conventional QFT in Minkowski space the underlying attitude is "that the original lagrangian is only a bookkeeping device enabling us to define the perturbative amplitudes. It was postulated from some intuitive correspondence between classical and quantum mechanics. As such it may be amended by quantum corrections." (Itzykson and Zuber, 1980, p. 326.)

This means that it is on physical principles with geometric origins (such the geodesic postulate), and on fundamental conservation laws (such as that for energy-momentum), that one's attention should be focused in developing a consistent framework for QGR. Hence, it is to this latter topic that we turn our attention next, as we analyze the physical significance of the notion of free-fall propagation of quantum-geometric fields that is mathematically depicted by (6.2)-(6.19).

The physical interpretation of this propagation manifests itself most clearly if we view any local section of the quantum field-mode bundle $\hat{\mathcal{E}}$, whose domain is in general some region R within the base manifold $\mathbf{M}$,

$$x \mapsto \hat{\Psi}_x \in \hat{\mathcal{F}}_x \,, \qquad x \in R \subset \mathbf{M} \,, \tag{6.20}$$

as representing a *quantum-geometric field distribution* over that region. In that case we can consider (cf. (6.1))

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[14] It is for this reason that the previously advocated approaches (Prugovečki, 1987-1992) to geometro-stochastic quantum field propagation were founded exclusively on such *bona fide* densities as those in (4.28), despite the fact that they required not one but four such densities. The present quantum-geometric approach is closer in spirit to the conventional Lagrangian QFT, but the densities in (4.28) or (5.43) are only implicit in the expressions for parallel transport, so that the resulting "Lagrangian" in (6.16) is not represented by a *bona fide* density.

$$\Psi^{\boldsymbol{s}}_{\varphi(x)} = \left\langle \hat{\boldsymbol{\Phi}}_{\mathbf{f}(x)} \middle| \hat{\boldsymbol{\Psi}}_x \right\rangle , \qquad \varphi(x) = (\hat{\boldsymbol{\sigma}}^{\boldsymbol{s}}_x)^{-1} \hat{\mathbf{f}}(x) \in \hat{\mathbf{F}} , \quad \mathbf{f}(x) \in \mathbf{F}_x , \quad \hat{\boldsymbol{\Psi}}_x \in \hat{\mathcal{F}}_x , \tag{6.21}$$

to be the *excitation probability amplitudes* of that field, which are observable in relation to a quantum moving frame $Q^{\boldsymbol{s}}_{\ell}$ associated with the Poincaré moving frame $\boldsymbol{s}$. At the purely epistemological level, this idea is an outgrowth of the corresponding idea in conventional QFT (cf., e.g., Schulman, 1981, p. 315). However, the fundamentally singular nature of conventional quantum fields, illustrated by the lack of convergence of (1.3) for any $x \in \mathbf{M}$ even in the case where the Lorentzian manifold $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$ is the Minkowski space $M^4 = (\mathbf{R}^4, \boldsymbol{\eta})$, makes this concept mathematically ill-defined in that context at individual points $x \in R$. Hence, in conventional QFT such amplitudes can be assigned only to test functions $f \in \mathcal{S}(\mathbf{R}^4)$ with support in R, rather than to individual points in R. Although this "smearing" is *axiomatically* taken to give rise to algebras of "observables" associated with a spacetime region R in M^4 (Haag and Kastler, 1964), or in a globally hyperbolic curved spacetime (Dimock, 1980), we saw in Sec. 5.1 that an actual *physical* interpretation of the elements of these algebras as true observables has never been realized. Furthermore, if the excitations of the quantum field are assumed to manifest themselves in the form of pointlike quantum particles localized in R, whose 4-momenta are therefore determined by the quantum field energy-momentum "local observables" within R, then this entire idea clearly violates the uncertainty principle when R is a *bounded* region which in principle can be chosen to be arbitrarily small.

These problems are resolved in the quantum geometry framework by the introduction of the fundamental length ℓ, and the development of the consistent theory of localization of single bosons, described in Secs. 3.8 and 4.7. In order to extend that theory to quantum fields, whose excitations manifest themselves in general as superpositions of multi-exciton modes generically described by wave functions of stochastic 4-velocity variables,

$$\hat{\Psi}_x = \bigoplus_{n=0}^{\infty} \hat{\Psi}_{x;n} \;\mapsto\; \bigoplus_{n=0}^{\infty} \hat{\Psi}_{x;n}(v_1, \ldots, v_n) , \tag{6.22}$$

we introduce the following generalization of the probability current in (3.7.16a),

$$\begin{aligned} j^k_{\boldsymbol{s}}(x) &= \sum_{n=0}^{\infty} n \int v^k_1 \left| \hat{\Psi}_{x;n}(v_1, \ldots, v_n) \right|^2 d\Omega(v_1) \cdots d\Omega(v_n) \\ &= \sum_{n=0}^{\infty} \sum_{r=1}^{n} \int v^k_r \left| \hat{\Psi}_{x;n}(v_1, \ldots, v_n) \right|^2 d\Omega(v_1) \cdots d\Omega(v_n) . \end{aligned} \tag{6.23}$$

This current is a natural candidate for the local *quantum field-mode probability current* of a quantum-geometric field distribution (6.20), whose field-mode wave functions are given by (6.21).

Indeed, on account of (5.21), (5.22), (5.29) and (5.30), the current in (6.23) is Poincaré gauge covariant, i.e., it behaves as a 4-vector field under changes of Poincaré moving frames:

$$j^k_{\boldsymbol{s}}(x) \mapsto j^{k'}_{\boldsymbol{s}'}(x) = \Lambda^{k'}{}_{k}(x) j^k_{\boldsymbol{s}}(x) , \qquad \boldsymbol{s}'(x) = \boldsymbol{s}(x) \cdot (b(x), \Lambda(x)) . \tag{6.24}$$

To establish that it can be also locally conserved, let us adopt a Poincaré gauge with $\boldsymbol{a}(x) \equiv 0$, and let us then extend this current to non-zero values of the gauge variable q, so that in

accordance with (5.30)

$$j_s^k(x;q) = \sum_{n=0}^{\infty} n \int v_1^k \left| \Psi_n(-q - i\ell v_1, \ldots, -q - i\ell v_n) \right|^2 d\Omega(v_1) \cdots d\Omega(v_n) \ . \qquad (6.25)$$

Then the method[15] used in proving (3.7.16a) yields the equation

$$\partial_k j_s^k(x;q) = 0 \ , \qquad \partial_k := \partial / \partial q^k \ . \qquad (6.26)$$

In case that the values assumed by the quantum-geometric field distribution (6.20) within an immediate neighborhood of a base location $x \in R$ are the parallel transports along geodesic arcs of its value at x, then, according to (4.4.16)-(4.4.19), the partial derivatives in the components q^k of the internal gauge variable q are equal to the corresponding covariant derivatives determined by the Levi-Civita connection. (This can be most easily seen by using, during the parallel transport along each geodesic arc, a section of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ that is adapted[16] to that arc, so that the connection coefficients in (4.4.16) next to all the infinitesimal Lorentz transformations vanish.) Hence, in that case the equation (6.26) coincides at $q = 0$ with the continuity equation

$$\nabla_k j_s^k(x) = 0 \ , \qquad \nabla_k := \nabla_{e_k} \ , \qquad (6.27)$$

and therefore the quantum mode-field current is *locally* conserved.

In view of (6.2), this is exactly the situation that prevails as quantum-geometric propagation progresses from a base location x_n to all the future located points x_{n+1}, since the quantum-geometric field distribution resulting from the propagation of the quantum field mode at x_n is obtained by the parallel transport in (6.8). Moreover, on account of (5.38) and (5.39), the resulting *local* current conservation of the quantum field probability of excitation entails also the local conservation of energy-momentum during that quantum-geometric propagation.

It has to be stressed, however, that this does not entail a global conservation of either field excitation probability or of energy-momentum. In fact, as we shall see in Sec. 5.8, the concept of *total* probability of quantum field excitation over a reference hypersurface Σ_t cannot be consistently defined in the present quantum field theoretical context. Hence, the probability densities determined by the timelike component of the quantum field-mode probability current in (6.23) describe only *relative* probabilities. This is in keeping with the geometrically local nature of the present framework for QGR, which, in turn, is in keeping with the fundamentally local nature of CGR. It represents a basic geometric feature of quantum-geometric propagation which will enable us to formulate in Sec. 5.8 quantum-geometric field propagation in the presence of interaction between fields in a manner which is conceptually simple and yet mathematically rigorous. That formulation will make it manifest that the local energy-momentum is conserved along all the paths of quantum-geometric propagation also in the presence of mutual interactions between quantum fields.

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[15] Cf. Theorem 4.2 on p. 2266 of (Prugovečki, 1978c), or p. 112 of (Prugovečki, 1984). The entire method relies on Lorentz invariance and reality of the kernel of an integral operator, and that same kernel occurs in (6.25).

[16] Cf. Secs. 2.6 and 4.4. In case that the curve is a timelike geodesic, such a section would be provided by the inertial Lorentz moving frames in (2.6.11). However, from a purely geometric point of view, all the constructions and conclusions remain valid for arbitrary smooth curves.

To arrive at a better understanding of the physical significance of this general feature of quantum-geometric field propagation, let us consider the case where the base Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ is the Minkowski space $M^4 = (\mathbf{R}^4,\boldsymbol{\eta})$. Let us then choose a global Lorentz frame $\mathcal{L}=\{\boldsymbol{e}_i(x_0)|\, i = 0,1,2,3\}$ with its origin at some point $x_0 \in \mathbf{M} \equiv \mathbf{R}^4$, and a Poincaré gauge with $-\boldsymbol{a}(x)$ equal to the 4-vector of each $x \in \mathbf{M}$ in relation to the origin at x_0. This gives rise to the cross-section $\boldsymbol{s}_0(\mathcal{L})$ of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ defined in (4.1.13) for $O = x_0$. Such a Poincaré gauge is adapted to the foliation of M^4 into hypersurfaces Σ_t which are hyperplanes σ_t corresponding to constant Minkowski time values in that frame, so that in it we can set $x_n = (t_n,\mathbf{x}_n)$ in (6.2). Hence, upon adopting the resulting Minkowski coordinates for all x_n, we have $d\sigma(x_n) \equiv d\mathbf{x}_n$ along each one of the reference hypersurfaces Σ_{t_n}.

In this Poincaré gauge the identification of all fibres $T_x\mathbf{M}$ of $T\mathbf{M}$ with its typical fibre $\mathbf{R}^4$ can be carried out in the manner which in classical relativity theory underlies the transition from the general relativistic to the special relativistic regime. In view of (2.11), this identification gives rise to an identification of all the fibres $\mathcal{F}_x$ of the Fock bundle $\mathcal{E}$ with its typical fibre $\mathcal{F}$, regarded as a special relativistic Fock space of wave functions associated with the global Lorentz frame $\mathcal{L}$:

$$\Psi(\mathcal{L}) \equiv \bigoplus_{n=0}^{\infty} \Psi_n(\zeta_1,\ldots,\zeta_n)\,. \tag{6.28}$$

A corresponding identification of the fibres $\hat{\mathcal{F}}_x$ of the quantum field-mode bundle $\hat{\mathcal{E}}$ with its typical fibre $\hat{\mathcal{F}}$ cannot be carried out since, as is evident from (6.22), the field-mode wave functions representing elements of $\hat{\mathcal{F}}$ do not incorporate the translational degrees of freedom that would allow for an extension of the representation of the Lorentz group into one for the Poincaré group. (For this reason, the association of the field-mode bundle $\hat{\mathcal{E}}$ with $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ was carried out in Sec. 5.5 in an indirect manner.) On the other hand, the soldering map in (5.29) assigns to the elements of coherent sections (6.20) of the quantum field-mode bundle $\hat{\mathcal{E}}$ coordinate wave functions which can be identified with the special relativistic wave functions in (6.28) as follows:

$$\hat{\boldsymbol{\sigma}}_x^{\boldsymbol{s}_0} : \hat{\boldsymbol{\Psi}}_x \mapsto \bigoplus_{n=0}^{\infty} \hat{\Psi}_{x;n}(\upsilon_1,\ldots,\upsilon_n) \quad \leftrightarrow \quad \Psi(\mathcal{L}) \in \mathcal{F}\,, \tag{6.29a}$$

$$\hat{\Psi}_{x;n}(\upsilon_1,\ldots,\upsilon_n) = \Psi_n(x - i\ell\upsilon_1,\ldots,x - i\ell\upsilon_n)\,, \qquad x \in R \subset \mathbf{R}^4\,. \tag{6.29b}$$

As an outcome of this identification, the four operators given by

$$\boldsymbol{P}_{j;\boldsymbol{s}_0} = \int_{\mathbf{R}^3} d\mathbf{x}\, \hat{\boldsymbol{P}}_{j;\boldsymbol{s}_0}(t,\mathbf{x}) = \int_{x^0 = t} d\sigma(x) \int_{V^+} :T_{j0}[\boldsymbol{\varphi}(x,\upsilon)]:\, d\Omega(\upsilon)\,, \tag{6.30}$$

assume in the Fock space $\mathcal{F}$ the role of generators of spacetime translations in relation to the global Lorentz frame $\mathcal{L}$. In the Schrödinger picture the elements of the coherent field-mode sections corresponding to $n = 1,\ldots, N-1$ in (6.1) can be then identified, in accordance with (6.29b), with Glauber-type coherent states within the Fock space $\mathcal{F}$:

$$\left\{\hat{\boldsymbol{\Phi}}^{\boldsymbol{s}_0}_{\varphi_n(t_n,\mathbf{x}_n)} \in \hat{\mathcal{F}}_{x_n} \,\middle|\, \mathbf{x}_n \in \mathbf{R}^3\right\} \leftrightarrow \boldsymbol{\Phi}_{f_n} \in \mathcal{F}\,, \qquad x_n = (t_n,\mathbf{x}_n) \in \mathbf{R}^4\,, \tag{6.31a}$$

$$\varphi_n(t_n,\mathbf{x}_n,v_n)=f_n(x_n-i\ell v_n)\ ,\qquad \varphi_n(t_n,\mathbf{x}_n)\in\hat{\mathbf{F}}_{x_n}\ ,\qquad f_n\in\mathbf{F}\ . \tag{6.31b}$$

Since in the Poincaré gauge $\mathbf{s}_0(\mathcal{L})$ all the connection 1-forms in (4.4.20) are identically equal to zero (cf. (4.3.3) and (4.3.4)), this gives rise to the following identifications of the quantum field-mode state vectors in (6.8) with Glauber-type coherent states in $\mathcal{F}$:

$$\left\{\hat{\boldsymbol{\Phi}}^{\mathbf{s}_0}_{\varphi_{n-1}(t_n,\mathbf{x}_n)}\in\hat{\mathcal{F}}_{x_n}\,\middle|\,\mathbf{x}_n\in\mathbf{R}^3\right\}\ \leftrightarrow\ \boldsymbol{\Phi}_{f_{n-1}}\in\mathcal{F}\ ,\qquad x_n=(t_n,\mathbf{x}_n)\in\mathbf{R}^4\ . \tag{6.32}$$

According to (5.43), (6.4) and (6.13), the product in (6.10) is therefore equal to

$$\left\langle\boldsymbol{\Phi}_{f_N(t_N)}\left|\prod_{n=N}^{1}{}'\exp[-i\delta t_n\int_{\mathbf{R}^3}d\mathbf{x}_n\int_{V^+}{:}T_{00}[\boldsymbol{\varphi}(0,\mathbf{x}_n;v)]{:}\ d\Omega(v)]\boldsymbol{\Phi}_{f_0(t_1)}\right.\right\rangle$$

$$=\left\langle\boldsymbol{\Phi}_{f''}\left|\prod_{n=N-1}^{1}\exp[-i(t_{n+1}-t_n)\boldsymbol{P}_{0;\mathbf{s}_0}]\boldsymbol{\Phi}_{f'}\right.\right\rangle\ ,\qquad f'=f_0\ ,\quad f''=f_N\ . \tag{6.33}$$

Hence, the quantum-geometric propagator in (6.2) assumes in $\mathcal{F}$ the form

$$K(f'',t'';f',t')=\left\langle\boldsymbol{\Phi}_{f'',t''}\left|\exp[-i\boldsymbol{P}_{0;\mathbf{s}_0}(t''-t')]\boldsymbol{\Phi}_{f',t'}\right.\right\rangle\ , \tag{6.34}$$

which describes the propagation of a free quantum field within the Fock space $\mathcal{F}$.

Thus, in the special relativistic regime the conservation of *total* energy-momentum during quantum-geometric propagation is ensured by the fact that all such propagation proceeds by parallel transport along polygonal broken paths, and that the quantum connection (3.3) governing that parallel transport has generators of infinitesimal spacetime translations that play the dual role of generators of global spacetime translations in Minkowski space and of total energy-momentum operators. This conclusion was reached in a special type of Poincaré gauge, but since the parallel transport governed by a quantum connection is Poincaré gauge invariant, it remains valid in all Poincaré gauges.

5.7. Quantum Field Interactions and Relativistic Locality

Before extending the concept of free-fall quantum-geometric propagation to interacting quantum fields, it is important to understand the basic reasons why the same procedure cannot be implemented within the conventional framework for QFT. The present section is devoted to the examination of this question.

It should be clear by now that these reasons are intimately tied in with the concept of locality of quantum fields. In conventional QFT this concept has been transferred, without any of the adaptations suggested by Heisenberg's uncertainty principle and by his notion of fundamental length, from the classical to the quantum theory of fields. As a result, the concept of "locality" of quantum fields was simply identified with the local commutativity *postulate*. In turn, this postulate was conventionally identified with a concept of "microcausality," which was simply *assumed* to embody the causal independence of any measurements of quantum fields in spacelike separated regions in Minkowski space. However, as the analysis presented in Secs. 5.1 and 5.2 has revealed, this conjecture is actually un-

substantiated by measurement-theoretical results.

This still leaves open all the key questions concerning the mathematical aspects to which the formulation of quantum field interactions gives rise in conventional QFT. These questions require clarification before the quantum-geometric approach to quantum field interactions is presented as a physically as well as mathematically viable alternative to the conventional approach.

When Feynman first formulated the path-integral approach to quantum mechanics and QED in two seminal papers entitled "Space-Time Approach to Non-Relativistic Quantum Mechanics" (Feynman, 1948) and "Space-Time Approach to Quantum Electrodynamics" (Feynman, 1949), respectively, he explicitly stated his hope of developing a new framework for quantum theory, in which quantum processes could be envisaged as taking place by propagation along paths in spacetime, and would be therefore described by spacetime propagators. Thus, after declaring that "there is always the hope that the new [path-integral] point of view will inspire an idea for the modification of present theories," he went on to say that, in his approach, "the general concept of superposition of probability amplitudes in quantum mechanics . . . can be directly extended to define a probability amplitude for any motion or path (position *vs.* time) in space-time." (Feynman, 1948, p. 368.)

The problem of divergences eventually diverted, however, all the attention to the perturbatively more easily manageable momentum space representation of the propagators which Feynman originally introduced in spacetime. Hence, although Feynman rules can be formulated, side-by-side, for Feynman diagrams with spacetime propagators as well as with momentum space propagators (cf. Schweber, 1961, pp. 471 and 478), with the passage of time the latter approach won over in popularity, so that the former is barely mentioned in contemporary textbooks on QFT (cf., e.g., Itzykson and Zuber, 1980, Sec. 6-1-2). This means that most of Feynman's original appeal (cf. Schweber, 1986b) to an intuitive mode of visualization of elementary particle processes in spacetime became lost, and computational expediency won over physical acumen.

Part of the reason for this loss lies in the preoccupation with the *formal* removal of divergencies during the "renormalization" of the "perturbation" series for the S-matrices of interacting quantum fields. On the other hand, despite the suggestive use of the term "perturbation," which is normally associated with "small departures" from an unperturbed solution of a problem, and therefore ordinarily entails convergence, it has been suspected for a long time that such a series is actually an "asymptotic series" (Dyson, 1952) – *assuming*[17] that an S-matrix actually exists in the first place. Hence, in view of the deep-seated causes of the difficulties experienced by conventional QFT, discussed in Sec. 1.4 as well as earlier in this chapter, it became clear that mere computational techniques could not provide more than a formal cure.

In fact, it was pointed out by Feynman himself that: "Actually, the path integral did not then [namely in 1948] provide, nor has it since provided, a truly satisfactory method of avoiding the divergence difficulties of quantum electrodynamics" – cf. (Feynman and Hibbs, 1965, p. viii). And, as mentioned in Sec. 1.4, in his acceptance speech for his 1965

[17] The definition of an *asymptotic series* requires the existence of an analytic function with respect to which that series is asymptotic. In general, such a series does not actually converge to any particular values – as opposed to a true perturbation series, which converges to the values of the function approximated by its partial sums.

Nobel prize, Feynman stated very clearly that "I think that the renormalization theory is simply a way to sweep the difficulties of the divergences of electrodynamics under the rug." (Feynman, 1966, p. 707.)

As we saw in Sec. 1.4, a decade prior to that event, another of the three co-founders of renormalization theory who was honored with that same prize had already declared that the "appearance of divergences, and contradictions, serves to deny the basic measurement hypothesis. . . . A convergent theory cannot be formulated consistently within the framework of present space-time concepts." (Schwinger, 1958, pp. xv-xvi.)

Thus, by the 1960s the root-causes of the difficulties encountered by conventional QFT had been correctly diagnosed, and the physical limitations of that theory in providing an actual cure – rather than merely alleviating the most conspicuous symptoms – had been properly acknowledged by the two principal co-founders of renormalization theory.

Despite these revealing statements by Feynman and Schwinger, and despite similar statements by Dirac, Pauli, and other leading physicists (cf. Chapter 1), the conventional wisdom that eventually prevailed decreed that these divergencies were mere mathematical difficulties that would be removed in due course by the application of more sophisticated mathematical procedures. Amongst those theoretical and mathematical physicists who were not satisfied with the formal rules of renormalization, but insisted on sound mathematics, this belief sprang from the discovery of Gårding and Wightman (1954) that von Neumann's theorem[18] on the uniqueness, modulo unitary transformations, of *all* the irreducible representations of the canonical commutation relations for a given *finite* number n of degrees of freedom does not remain true when that number becomes infinite – as is the case in QFT, due to the phenomenon of pair creation and annihilation.

To arrive at a general formulation of the concept of irreducible representation of the canonical commutation relations that is valid for a finite as well as for an infinite number of degrees of freedom, let us consider a real pre-Hilbert space $\mathfrak{R}$ (i.e., a real inner product space that is not necessarily complete), whose dimension corresponds to the considered (cardinal) number of degrees of freedom. In that setting, a representation of the canonical commutation relation for that number of degrees of freedom consists of a linear association of Q and P operators to all the elements of $\mathfrak{R}$ in such a manner that

$$[Q(\mathrm{f}),P(\mathrm{g})] = i\langle \mathrm{f}|\mathrm{g}\rangle \ , \quad [Q(\mathrm{f}),Q(\mathrm{g})] = [P(\mathrm{f}),P(\mathrm{g})] = 0 \ , \quad \mathrm{f},\mathrm{g} \in \mathfrak{R} \ , \qquad (7.1)$$

where, in general, the above commutators are defined in a dense Gårding domain within the Hilbert space $\mathcal{H}$ in which they act (cf., e.g., Emch, 1972, p. 234). Corresponding annihilation and creation operators can be then defined by setting

[18] A precise statement of that theorem, and a simplified version of the proof originally given by von Neumann (1931), can be found in (Prugovečki, 1981), pp. 342-347. Note should be taken of the fact that, although it has become the custom to speak in this context of "canonical commutation relations," von Neumann's theorem actually pertains to the corresponding Weyl relations, which form the basis of the Weyl-Heisenberg group. In turn, the representations of this group for three degrees of freedom are embedded in the ray representations in (3.1.10), (3.1.15) and (3.3.5). It is also important to note that, whereas the Weyl relations always imply the canonical commutation relations (*ibid.*, pp. 339-341), it was discovered by Nelson (1959) that the converse is not generally true: there are operators, which of necessity are unbounded and therefore only densely defined, which satisfy the canonical commutation relations on a common dense core, without satisfying the Weyl relations. In particular, those operators might commute on that core, without their spectral measures also commuting, so that such representations have to be deemed physically unacceptable.

$$a(\mathrm{f}) = 2^{-1/2}(Q(\mathrm{f}) + iP(\mathrm{f})) \ , \qquad a^*(\mathrm{f}) = 2^{-1/2}(Q(\mathrm{f}) - iP(\mathrm{f})) \ , \qquad \mathrm{f} \in \mathfrak{R} \ . \tag{7.2}$$

Hence, if an orthonormal basis $\{\mathrm{w}_1, \mathrm{w}_2, ...\}$ is introduced in $\mathfrak{R}$, then, by using the usual Einstein summation convention, we can write

$$a(\mathrm{f}) = q^\alpha a_\alpha \ , \qquad a_\alpha := a(\mathrm{w}_\alpha) \ , \qquad \mathrm{f} = q^\alpha \mathrm{w}_\alpha \in \mathfrak{R} \ . \tag{7.3}$$

The following representation of the canonical commutation relations is thereby obtained:

$$[a_\alpha, a^*_\beta] = \delta_{\alpha\beta} \ , \qquad [a_\alpha, a_\beta] = [a^*_\alpha, a^*_\beta] = 0 \ , \qquad \alpha,\beta = 1,2,... \ . \tag{7.4}$$

The Fock representation is then singled out by the requirement that there is a Fock "vacuum vector" Ψ_0 in $\mathcal{H}$, defined by the requirement that

$$a(\mathrm{f})\Psi_0 = q^\alpha a_\alpha \Psi_0 = \mathbf{0} \ , \qquad a(\mathrm{f}) = 2^{-1/2}(Q(\mathrm{f}) + iP(\mathrm{f})) \ , \qquad \mathrm{f} \in \mathfrak{R} \ . \tag{7.5}$$

In the case of a finite number N of degrees of freedom, the existence of such a state is automatically ensured by von Neumann's theorem. Indeed, according to that theorem, all irreducible representations of (7.2) are then unitarily equivalent to the Schrödinger representation, in which the ground state of the nonrelativistic harmonic oscillator in N dimensions can play that role. The fundamental discovery of Gårding and Wightman (1954) was that this is no longer the case when the number of degrees of freedom becomes (countably or uncountably) infinite.

This *mathematical* discovery led some theoretical physicists to the conclusion that "the Main Problem of quantum field theory turned out to be to kill it or cure it: either to show that idealizations involved in the fundamental notions of the theory (relativistic invariance, quantum mechanics, local fields, etc.) are incompatible in some physical sense, or to recast the theory in such a form that it provides a practical language for the description of elementary particle dynamics." (Streater and Wightman, 1964, p. 1.)

The offered "cure" consisted of a reformulation of the key concepts in conventional quantum field theory within an axiomatic framework (Wightman, 1956; Wightman and Gårding, 1964). However, although that framework certainly represented a systematic effort to impart much-needed mathematical rigor to conventional QFT, it did not in any way enrich or improve its *physical* foundations in the manner suggested by Feynman or by Schwinger. Nevertheless, many mathematical physicists became convinced that, despite Schwinger's (1958) well-set emphasis on the need for new physical principles in quantum field theory, the solution to the problem of divergencies could be found exclusively by applying new mathematical techniques to the old physical problem of divergencies.

That hope eventually proved illusory for many fundamental reasons – all centered around the concept of locality in relativistic quantum theory. Thus, after almost a full decade devoted to perfecting the axiomatic method (Streater and Wightman, 1964), and after an additional two decades of concerted effort (Jaffee, 1967; Glimm and Jaffee, 1987) to produce, within the confines of a constructive QFT framework that emerged from the Wightman axioms, at least a single nontrivial and yet mathematically rigorous formulated model, it was discovered (Fröhlich, 1982, 1991) that in more than three spacetime dimensions this method of handling quantum field theoretical interactions led to *physically* trivial

models, i.e., models with an identity S-matrix, which therefore describe no interactions.[19] In particular, the two leading initiators of the constructive QFT programme, which embodied the effort of a large group of researchers, eventually arrived at the following physically undesirable conclusion:

"At present the arguments favoring triviality [of the ϕ^4 quantum field theory] seem to be stronger, but a definitive answer seems to be out of reach of available methods. These arguments apply equally to the four-dimensional Yukawa and electrodynamic (QED) interactions. If these interactions are all trivial, it would mean that a short distance cutoff resulting from the quark interactions is essential to a theory of protons, photons, mesons and electrons as elementary particles. Since it is known experimentally that protons and mesons are not elementary particles, but composites formed from quarks and gauge fields ('gluons'), such a short distance cutoff, set at the proton radius, for example, would not violate [the] experimental facts." (Glimm and Jaffee, 1987, p. 120.)

However, a short distance cutoff *would* violate some of Wightman's key axioms, and most of all the "local" commutativity and anticommutativity postulates. In addition, such a cutoff cannot be justified in the case of gauge particles and leptons, in general, and photons and electrons, in particular, on the same grounds that can be used for "protons and mesons," or for other hadrons, since the former are at the present stage in our understanding of particle physics as "elementary" as the quarks. Finally, it is certainly very difficult to see how such a theory as QED, which from the point of view of all the experimental verifications of its numerical predictions has proved to be so extremely successful, could be *mathematically* consistent and yet be *physically* trivial.

Rather, it appears more prudent to consider the possibility suggested by Feynman (1989), namely that QED, as normally treated by physicists, is a *mathematically inconsistent* theory which, however, is substantially *physically correct* within the limits resulting from the neglect of non-electromagnetic interactions. If that were so, then the negative result reached by constructive QFT is not actually a verdict on the physical validity of QED, or of other experimentally well-established conventional QFT models, but rather on the physical acumen of the methods used to force the physical substance of those models into the mathematical mold provided by Wightman's axioms.

This mathematical mold was in part created for the sake of coping with Haag's theorem (cf. Streater and Wightman, 1964, Sec. 4-5). Within the Wightman framework, that well-known theorem prohibits the mathematically rigorous formulation of theories for interacting fields within a *single* Fock space. It is sometimes claimed that this no-go theorem is a consequence of "locality" and "relativistic invariance." However, the proof of Haag's theorem provided by Streit (1969), and elaborated in (Emch, 1972, Chapter 3, Sec. 1d), reveals that this theorem is *not* a reflection either of local (anti)commutativity, or of the relativistic invariance of the quantum theories for interacting fields. Rather, it is merely a

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[19] There was initial success with the ϕ^4 and with the Yukawa model in two spacetime dimensions, obviously due to the "super-renormalizable" nature of these interactions, and to the rather trivial nature of the Poincaré group in two-dimensional Minkowski space. Some of the Wightman axioms were later verified in three-dimensional Minkowski space (cf. Glimm and Jaffe, 1987). However, in Wightman's words, "it has been shown that the methods used so successfully in spacetime dimensions two and three to construct the ϕ^4 theories of a self-interacting scalar field, ϕ, fail for [ϕ^4 in four or more spacetime dimensions]." (Cf. Fröhlich, 1991, p. v.)

reflection of the required uniqueness of a *global* vacuum state vector Ψ_0, and of the imposition of the demand that the generator H of time translations satisfy (Emch, 1972, p. 251)

$$P(\mathbf{f}) = i[H, Q(\mathbf{f})] \ , \qquad \mathbf{f} \in \mathfrak{R} \ , \tag{7.6}$$

in the context of an irreducible representation of the canonical commutation relations in (7.1) for an infinite number of degrees of freedom. Of course, this has the consequence that such a *unique and global* vacuum state Ψ_0 is left invariant by the time evolution of any QFT model obeying those relations. However, that stands in direct contradiction with the phenomenon of perpetual pair creation and annihilation that one witnesses in Nature.

On the other hand, the concept of such a unique and global vacuum occupies center stage in the formulation of Wightman's Axiom *O* (cf. Streater and Wightman, 1964, p. 97) – namely the very first of all Wightman's axioms. Moreover, that concept is essential to the definition of Wightman's functions as *vacuum* expectations values of various products of quantum fields (cf. Bogolubov *et al.*, 1975, Chapter 10). As such, that axiom plays a crucial role in the derivation of *all* the main results of the axiomatic approach to quantum field theory: PCT, the connection between spin and statistics, "and all that." However, it is very telling that such results can be derived (Lücke, 1979) *without* the use of some of Wightman's key axioms, such as that of local commutativity, so that their confirmation by observations has no actual bearing on the physical suitability of that axiomatic framework.

Of course, *if* a unique and global vacuum were *physically* realizable, then the conservation of energy would demand that it be left invariant, since otherwise *ex nihilo* particle production would result. So, in that sense, *if* the possibility of the *physical* existence of a unique global vacuum state were accepted (as was *formally* the custom in the early stages in the formulation of conventional QFT), then, from a purely logical point of view, there is no alternative but to accept that such a vacuum state is left invariant by the quantum evolution of a physical system.

But what would be the actual physical meaning of a *unique and global* vacuum, once we allow for the fact that *any* observation requires the presence of apparatuses? Indeed, any apparatus is, in the ultimate analysis, built of elementary constituents, which have to interact with a "system" described by a quantum field theoretical configuration in order to produce an "observation." Viewed from such a perspective, it is the *postulate* as to the *mathematical* existence of such a unique and global vacuum that emerges as *un*physical. The fact that the uniqueness of such a vacuum cannot be retained in the presence of the symmetry-breaking mechanisms used in the conventional quantum theory of gauge fields only presents additional corroboration of this basic fact.

Of course, in conventional QFT it is the custom to talk about the "bare vacuum" and of the "dressed vacuum," with the latter being regarded as being populated by unspecified but infinite "clouds" of "bare particles." On the other hand, the actual computations performed in perturbation theory actually begin with expressions for asymptotic states, free propagators, etc., formulated in Fock space, and then progress through a chain of computations dictated by Feynman rules, which have no direct bearing to a mathematically rigorous realization of a non-Fock representation of the canonical commutation relations in (7.1). Hence, in conventional QFT the existence of such a representation, and of a corresponding unique and global "dressed vacuum," is merely a conjecture rather than a

proven mathematical fact.

The constructive QFT programme attempted to actually prove that conjecture by implementing "a standard of absolute truth for testing ideas and hypotheses concerning quantum fields," described as a "truth [that] is more easily accessible by a mathematical proof than it would be by a laboratory measurement." (Glimm and Jaffe, 1987, p. 433.) However, before this attitude can be fully accepted, the fundamental question has to be asked as to what constitutes nowadays a standard of *absolute* truth[20] in mathematics, in light of the fact that Gödel's (1931) incompleteness theorem led to the failure of Hilbert's program (cf. Weyl, 1949; Reid, 1986) to establish a "standard of absolute truth" by *proving* the internal consistency of all the major areas of mathematics – cf., e. g., (Kline, 1980, pp. 260-264). As a result, one is confronted with the following uncomfortable conclusion: "There is no rigorous definition of rigor. A proof is accepted if it obtains the endorsement of the leading specialists of the time or employs the principles that are fashionable at the moment. But no standard is universally accepted today." (*Ibid.*, p. 315).

This means that in applying various areas of contemporary mathematics, meeting the standards of mathematical rigor prevailing in those areas is a *necessary* but not a sufficient criterion for insuring internal consistency and scientific objectivity. In particular, the formal rules used in dealing with infinities in conventional QFT violate all the standards as to how to correctly handle infinities, that have become firmly established in mathematics already in the last century, after the extensive experience with calculus provided ample evidence that lax standards could lead to totally erroneous conclusions. Clearly, it is for that reason that Dirac repeatedly urged the use of "sound mathematics" in handling divergencies in QFT. However, beyond satisfying such an absolutely *necessary* criterion for sound mathematical reasoning, it is difficult to see how any choice of mathematical methodology might provide a "test of ideas and hypotheses" that would be "more easily accessible by a mathematical proof than it would be by a laboratory measurement."

In the absence of an objective and universal criterion for *absolute* truth, it appears more prudent to follow those traditional criteria in science that were advocated by Dirac in his paper "The Evolution of the Physicist's Picture of Nature." In this very enlightening and lucid 1963 paper Dirac made the following statements which, in light of the principal developments in elementary particle physics during the 1980s (cf., e.g., Davies and Brown, 1989), proved prophetic: "There are a good many problems left over concerning particles other than those that come into electrodynamics, ... [such as] how to introduce the fundamental length to physics in some natural way, how to explain the ratio of masses of the elementary particles and how to explain their other properties. I believe that separate ideas will be needed to solve these distinct problems and that they will be solved one at a time through successive stages in the future evolution of physics. At this point I find myself in disagreement with most physicists. They are inclined to think one master idea will be discovered that will solve all these problems together." (Dirac, 1963, p. 50.)

In dealing with this subject, Dirac was even more explicit in a 1968 lecture, which he began with the following explanatory statements: "I shall attempt to give you some idea

[20] A later article by Jaffee and Queen (1993) criticizing "theoretical speculation" does not clarify what this "standard of absolute truth" actually is at a foundational level, and appears more concerned with the control over the publication of papers and with the personalities of the leaders of competing schools of mathematical physics.

how a theoretical physicist works ... One can distinguish between two main procedures for a theoretical physicist. One of them is to work from an experimental basis. ... The other procedure is to work from the mathematical basis. One examines and criticizes the existing theory. One tries to pin-point the faults in it and then tries to remove them. The difficulty here is to remove the faults without destroying the great successes of the existing theory." Dirac ended that lecture[21] with the following declaration: "However, the present quantum electrodynamics does not conform to the high standard of mathematical beauty that one would expect for a fundamental physical theory, and leads one to suspect that a drastic alteration of basic ideas is still needed."

With these lessons in mind, the formulation of quantum field interactions presented in the next section is *not* aimed, as in constructive QFT, at an implementation of a mathematical "standard of absolute truth," which key developments in the foundations of mathematics (Wilder, 1967; Kline, 1980), that took place during this century, cannot justify. Rather, it is aimed at formulating in QGR at the *non*-perturbative level a framework for quantum field interactions that is mathematically sound along the lines advocated by Dirac (1939, 1978) – lines which set most emphasis on the idea of mathematical beauty and simplicity. At the level of perturbation theory, the formulation presented in the next section is primarily meant to provide *only* in the special relativistic regime a perturbation series from which the *formal* manipulations that are customary in conventional special relativistic QFT can be recovered in the *formal* sharp-point limit $\ell \to 0$, so that agreement with the experimental data can be maintained in the case of well-established QFT models.

5.8. Quantum-Geometric Propagation of Interacting Boson Fields

The formulation of quantum field-mode bundles in Sec. 5.5, and their use in the formulation of the free-fall propagator in Sec. 5.6, embodies a conceptualization of field propagation which is at the formal level very similar to the conventional one, but which dispenses with quantum field singularities and all the ensuing infinities, and thereby aims at a mathematically sound and physically consistent framework. These twin goals are achieved by basing the physical interpretation of quantum-geometric field propagation on the conserved and Poincaré gauge invariant field-mode current in (6.23), which altogether dispenses with the idea of a single Fock space in QFT, and with the corresponding postulate of a unique and global vacuum state.

Indeed, the quantum field-mode bundle $\hat{\mathcal{E}}$ introduced in Sec.5.5 is *not* a Fock space, but rather an infinite-dimensional manifold which, considered as a set, is a union of Hilbert fibres built in the manner of Fock spaces:

$$\hat{\mathcal{E}} = \bigcup_{x \in \mathbf{M}} \hat{\mathcal{F}}_x \,, \qquad \hat{\mathcal{F}}_x = \bigoplus_{n=0}^{\infty} \hat{\mathcal{F}}_{x;n} \,. \tag{8.1}$$

However, each one of these quantum field-mode fibres possesses its own vacuum state which, in the absence of quantum field interactions, can be physically interpreted as a *local non*-excited quantum field mode. Moreover, as we have seen in Sec. 5.6, due to their lack

[21] This lecture was entitled "Methods in Theoretical Physics," and it was delivered at the International Center for Theoretical Physics at Trieste. Its text was later reprinted in (Salam, 1990, pp. 125-143).

of intrinsic translational degrees of freedom, these fibres *cannot* be identified with each other even in the case where the base Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ is the Minkowski space $M^4 = (\mathbf{R}^4,\boldsymbol{\eta})$. Consequently, the same is true of all the non-excited local quantum field modes representing their respective vacuum states. Thus, as opposed to the Fock bundle $\mathcal{E}$ defined in (2.3), which in the special relativistic regime can be identified in the manner described in Sec. 5.6 with the single Fock space provided by its typical fibre $\mathcal{F}$, the non-uniqueness of vacuum states is intrinsic to the concept of a quantum field-mode bundle $\hat{\mathcal{E}}$.

To better understand the physical significance of this fact, let us consider a Fock space wave function (6.28) which is normalized along a reference hypersurface Σ_t, and which can be therefore related, in accordance with (3.7.14) and (4.2.3), to following (globally over Σ_t normalized) probabilities

$$2^n \int_B d\sigma(q_1)\cdots\int_B d\sigma(q_n)\int\left|\Psi_n(q_1-i\ell v_1,\ldots,q_n-i\ell v_n)\right|^2 v_1^0\cdots v_n^0\, d\Omega(v_1)\cdots d\Omega(v_n) \quad (8.2)$$

of detecting exactly n excitons within each Borel set B in Σ_t. In case that we adopt in the Minkowski space M^4 a foliation that corresponds to fixed instants of the Minkowski time in some global Lorentz frame, (4.2.3) assumes the form

$$d\Sigma(q,v) = 2v_k\, d\sigma^k(q)d\Omega(v) = d\mathbf{q}\, d\mathbf{v}\,, \quad q=(q^0,\mathbf{q})\in\mathbf{R}^4\,, \quad v=(v^0,\mathbf{v})\in V^+, \quad (8.3)$$

and (8.2) can be expressed also in the alternative form

$$\int_{B\times\cdots\times B} d\mathbf{q}_1\cdots d\mathbf{q}_n \int_{\mathbf{R}^{3n}}\left|\Psi_n(q_1-i\ell v_1,\ldots,q_n-i\ell v_n)\right|^2 d\mathbf{v}_1\cdots d\mathbf{v}_n \quad (8.4)$$

For $B=\Sigma_t$ this gives rise to normalized total probability of detection in Σ_t, which remains normalized with the flow of the Minkowski time t during the free propagation depicted by (6.34). This is due to the fact that the generator for time translations in (6.30) is unitary when it is viewed as an operator in the Fock space $\mathcal{F}$. Hence, we can assert that the *total* probability of n-exciton detection is conserved for each $n = 0,1,2,\ldots$ during that free quantum evolution in $\mathcal{F}$.

On the other hand, the quantum field-mode current in (6.23) gives rise to probabilities

$$\int_B j^0_{\bullet}(x)d\sigma(x) = \tfrac{1}{2}\sum_{n=0}^{\infty} n\int_B d\sigma(x)\int\left|\Psi_n(x-i\ell v_1,\ldots,x-i\ell v_n)\right|^2 d\mathbf{v}_1\cdots d\mathbf{v}_n \quad (8.5)$$

which are not normalized along Σ_t by this procedure. Moreover, although the field-mode wave function in (6.29) could be renormalized in such a manner that the integral in (8.5) would equal 1 for $B=\Sigma_t$, that "total probability" over Σ_t would *not* be left invariant by that same propagation. From the mathematical point of view, this is due to the fact that, although the local energy-momentum density operators in (5.2) are self-adjoint within their respective quantum field mode fibres $\hat{\mathcal{F}}_x$, and although they give rise to the operators in (6.30) for infinitesimal time translations, these latter operators are self-adjoint operators in the Fock space $\mathcal{F}$, rather than in each $\hat{\mathcal{F}}_x$. From the physical point of view, this is due to the geometrically local nature of these latter fibres $\hat{\mathcal{F}}_x$, which does not allow for the mutual identification of the multi-exciton modes describing the excitations of a quantum field at

distinct points in Minkowski space M^4. In other words, those excitations have to be viewed as part of the description of a *local* physical reality, rather than as mere mathematical artifacts that can be transported at will around spacetime.

The multi-boson wave functions in (8.2) might appear, at first sight, to contain more information than the field-mode wave functions in (8.5). However, that is not the case due to the existence of a unique analytic continuation of the latter into the former, that provides the basis of the equality of the inner products in (5.17). Hence, the "particle" point of view, embodied in multi-boson wave functions defined globally on M^4, and the "field-mode" point of view, embodied in the coherent superpositions of multi-exciton modes above individual base locations belonging to M^4, emerge as physically complementary, in the sense that they represent different manners of posing informationally equivalent measurement-theoretical questions: what is the *global* number of excitons and their mean stochastic locations along reference hypersurfaces Σ_t *vs* what are the local quantum field values above various mean locations in Σ_t, as determined by the magnitude of the timelike component of its probabilistic field-mode current. Thus, the first point of view is indeed geometrically global; whereas, the second is geometrically local.

The informational equivalence of these two points of view is not retained in a curved Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$, since parallel transport is no longer path-independent. Hence, information gathered locally cannot be extrapolated globally, and the quantum field-mode point of view is no longer informationally equivalent to the quantum particle point of view. In this manner all the root causes of the inconsistencies of conventional QFT in curved spacetime have been eliminated: in the present quantum-geometric approach to QFT there is no *ex nihilo* boson production in free fall, and all the vestiges of a global concept of "particle" were removed from this framework during the transition from the Fock bundle $\mathcal{F}$ to the field-mode bundle $\hat{\mathcal{F}}$, where they were replaced by the geometrically local concept of quantum field exciton-modes *above* each base location $x \in \mathbf{M}$.

Naturally, since a fundamental length ℓ is intrinsic to the quantum geometry framework, this geometric locality does not entail topological locality, and quantum fluctuations around each base location are intrinsic to the formalism and its physical interpretation. This is in evidence in the very formulation of the quantum-geometric propagator in (6.2), since the probability amplitude of a local state prepared at $x' \in \Sigma_t$, manifests itself, in the form of its parallel transports to $x_1 \in \Sigma_{t_1}$, along the infinitesimally future-located hypersurface Σ_{t_1}. Of course, in accordance with (4.2.15)-(4.2.17), these quantum fluctuations are observationally significant only within neighborhoods of x' of the same order of magnitude as ℓ. Subsequent quantum-geometric propagation spreads those probability amplitudes in a manner characteristic of wave-packet evanescence in quantum mechanics, since at each future base location the contributions from all past locations are superimposed, and only the probability amplitude equal to the outcome of all such superpositions enters (6.23), and is therefore measurable. This is simply another manifestation of the impossibility of transmitting information via the type of quantum correlations that occur in the measurement processes used in the formulation of the EPR paradox (Ghirardi *et al.*, 1980; Jordan, 1983), and constitutes the basis of the notion of QGR causality described in Sec. 4.6: quantum-geometric propagation proceeds along all possible future-oriented stochastic paths associated with a geometrodynamic evolution of a gravitational field, but the superposition prin-

ciple is obeyed, and only the net probability amplitudes are observationally accessible. As a consequence, the field-mode point of view emerges as informationally more realistic than the particle point of view, since it does not entail detailed micro-information with regard to local exciton state vectors everywhere along reference hypersurfaces Σ_t – namely the type of information which could never become accessible in a cosmological setting.

This basic fact is of paramount significance even to the special relativistic theory of quantum field interactions. Indeed, whereas in the case of free quantum fields the two aforementioned physically complementary points of view are mathematically equivalent, that is no longer the case in the presence of quantum field interactions, due to the phenomenon of pair creation and annihilation.

At the purely formal level, the introduction of field interactions into the quantum-geometric propagator in (6.17) proceeds as in conventional QFT, namely via an interaction Lagrangian density $\mathcal{L}_{\mathrm{I}}(\boldsymbol{\varphi})$ which is polynomial in the quantum field $\boldsymbol{\varphi}$ – for which the prototype is the well-known ϕ^4-model[22] with interaction Lagrangian density

$$\mathcal{L}_{\mathrm{I}}[\boldsymbol{\varphi}(x,v)] = -(\lambda/4!):\boldsymbol{\varphi}(x,v)^4: \tag{8.6}$$

However, for a satisfactory treatment of the quantum-geometric propagation of a self-interacting quantum field (or of the generic case of several[23] mutually interacting quantum fields), it is necessary to first solve a key mathematical problem.

This problem stems from the fact that the integral defining the proper energy operator

$$\hat{\boldsymbol{\rho}}_{\mathrm{I}}(x) = -\int_{V_x^+} :\mathcal{L}_{\mathrm{I}}[\boldsymbol{\varphi}(x,v)]:\, d\Omega(v)\,, \tag{8.7}$$

which, in view of (5.3), corresponds to the interaction stress-energy tensor

$$T^{\mathrm{I}}_{jk}[\boldsymbol{\varphi}(x,v)] = -\eta_{jk}\mathcal{L}_{\mathrm{I}}[\boldsymbol{\varphi}(x,v)]\,, \tag{8.8}$$

is not well-defined as a Bochner integral within the quantum field-mode fibre $\hat{\mathcal{F}}_x$. The reason for this technical difficulty is that for the existence of the Bochner integral of a vector-valued function it is necessary that the norms of those vectors be integrable over the entire domain of integration (cf. Prugovečki, 1981, p. 479). However, that is not the case when fourth or higher powers of the quantum field $\boldsymbol{\varphi}$ are applied to some of the quantum field-mode state vectors. For example, in the case of the $\boldsymbol{\varphi}^4$ model, the Bochner integral

$$\hat{\boldsymbol{\rho}}_{\mathrm{I}}(x) = (\lambda/4!)\int_{V_x^+} :\boldsymbol{\varphi}(x,v)^4:\, d\Omega(v)\,, \tag{8.9}$$

does not exist as an operator in $\hat{\mathcal{F}}_x$ since, by repeated use of the commutation relations in (2.8), it can be shown that

$$\left\| :\boldsymbol{\varphi}(x,v)^4: \hat{\boldsymbol{\Phi}}^s_{\varphi(x)} \right\| \geq 4\left[\Delta_x^{(+)}(-i\ell v;-i\ell v)\right]^4 = (m^8/2^{10}\pi^{12})\hat{Z}^{-4}_{\ell,m}\,. \tag{8.10}$$

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[22] Cf. (Nash, 1978) and (Das, 1993) for a detailed treatment of this model in the conventional QFT framework.

[23] The extension of the present treatment to interaction of any finite number of fields poses no problems as long as the interaction Lagrangian densities are polynomials, or, more generally, everywhere analytic in those fields.

This purely technical difficulty can be remedied by introducing the Naimark extension $\mathcal{F}_x^{\uparrow}$ of each quantum field-mode fibre $\hat{\mathcal{F}}_x$. Such extensions are in general nonseparable Hilbert spaces in which the Bochner integrals in (8.7) are mathematically well defined. In practice, the construction of $\mathcal{F}_x^{\uparrow}$ does not have to be carried out explicitly since, as we shall see shortly, only the projection

$$\tilde{\rho}_1(x) = -\int_{V_x^+} \mathbf{P}_{\hat{\mathcal{F}}_x} : \mathcal{L}_1[\varphi(x,v)] : \mathbf{P}_{\hat{\mathcal{F}}_x} \, d\Omega(v) \tag{8.11}$$

onto the original quantum field-mode fibre $\hat{\mathcal{F}}_x$ is required for the computation of the free-fall propagator for interacting quantum-geometric fields, and that projection is well-defined within the field-mode fibre $\hat{\mathcal{F}}_x$. However, in order to exhibit the relation of its extension to the Hilbert spaces carrying non-Fock representations of the canonical commutation relations that were discussed in the preceding section, we shall outline that construction, which is based on the following existence theorem by Naimark:[24]

Let $\mathcal{B}$ be a family of subsets of a set Ω, that incorporates the empty set $\emptyset$, as well as Ω itself, and let $\mathcal{B}$ be closed under the taking of finite numbers of intersections or unions. Assume that a nonnegative self-adjoint operator $\boldsymbol{E}(B) \leq \mathbf{1}$ in a Hilbert space $\mathcal{H}$ is assigned to each $B \in \mathcal{B}$ in such a manner that $\boldsymbol{E}(\emptyset) = \mathbf{0}$, $\boldsymbol{E}(\Omega) = \mathbf{1}$, and $\boldsymbol{E}(B_1 \cup B_2) = \boldsymbol{E}(B_1) + \boldsymbol{E}(B_2)$ whenever $B_1 \cap B_2 = \emptyset$. Then there is a spectral measure $\boldsymbol{E}^{\uparrow}(B)$ on the Boolean σ-algebra generated by $\mathcal{B}$, which consists of projectors acting in a Hilbert space $\mathcal{H}^{\uparrow}$ that is an extension of $\mathcal{H}$, and is such that $\boldsymbol{E}(B) = \mathbf{P}\boldsymbol{E}^{\uparrow}(B)$ for all $B \in \mathcal{B}$, where $\mathbf{P}$ is the projector of $\mathcal{H}^{\uparrow}$ onto $\mathcal{H}$; moreover, $\mathcal{H}^{\uparrow}$ is spanned by all vectors of the form $\boldsymbol{E}^{\uparrow}(B)f$, with $B \in \mathcal{B}$ and $f \in \mathcal{H}$.

To be able to exploit this theorem, and at the same time bring all underlying relationships with non-Fock representations of the canonical commutation relations to the fore, let us take advantage of the orthonormal basis in (5.19a) to associate with the typical single-boson fibre $\mathbf{F}$ the real Hilbert space

$$\mathfrak{R} = \left\{ q^{\alpha} w_{\alpha} \middle| \; q^{\alpha} = \sqrt{2}\, Re\langle w_{\alpha} | f \rangle \in \mathbf{R}^1, \; f \in \mathbf{F}, \; \alpha = 1,2,\ldots \right\}. \tag{8.12}$$

We then obtain within the typical Fock fibre $\mathcal{F}$ the types of operators in (7.1) and (7.2) for a realization of the canonical commutation relations in (7.4):

$$Q(\mathrm{f}) = 2^{-1/2}[a^*(\mathrm{f}) + a(\mathrm{f})], \quad P(\mathrm{f}) = 2^{-1/2} i[a^*(\mathrm{f}) - a(\mathrm{f})], \quad \mathrm{f} \in \mathfrak{R}, \tag{8.13a}$$

$$a(\mathrm{f}) := q^{\alpha} a_{\alpha}, \quad a^*(\mathrm{f}) := q^{\alpha} a_{\alpha}^*, \quad a_{\alpha} := \varphi^{(-)}(w_{\alpha}). \tag{8.13b}$$

Corresponding realizations of the canonical commutation relations are obtained within each Fock fibre $\mathcal{F}_x$ by using, for each given Poincaré gauge $\boldsymbol{s}$, the soldering map in (3.5):

$$a_{\alpha}(x) = (\sigma_x^{\boldsymbol{s}})^{-1} a_{\alpha} \sigma_x^{\boldsymbol{s}}, \quad a_{\alpha}^*(x) = (\sigma_x^{\boldsymbol{s}})^{-1} a_{\alpha}^* \sigma_x^{\boldsymbol{s}}, \quad \mathbf{w}_{\alpha} = (\sigma_x^{\boldsymbol{s}})^{-1} w_{\alpha} \in \mathcal{F}_{x;1}. \tag{8.14}$$

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[24] Cf. Sz.-Nagy (1960), p. 29. In this separately published appendix to the well-known textbook "Functional Analysis" by F. Riesz and B. Sz.-Nagy (1990) the Germanic spelling "Neumark" is used by Sz.-Nagy in referring to the Russian mathematician M. A. Naimark. All the basic definitions and results concerning spectral measures and Boolean σ-algebras, essential to the understanding of this theorem, can be also found in (Prugovečki,1981).

In accordance with (5.23)-(5.25), we can then expand the quantum field $\boldsymbol{\varphi}$ as follows:

$$\boldsymbol{\varphi}(x,v) = \sum_{\alpha=1}^{\infty} \left(\hat{\mathbf{w}}_\alpha(x,v)\, a_\alpha^*(x) + \hat{\mathbf{w}}_\alpha^*(x,v)\, a_\alpha(x) \right). \tag{8.15}$$

The annihilation and creation operators in (8.14) coincide for $\boldsymbol{u} = \boldsymbol{s}(x)$ with those in (3.23). Consequently, we can introduce the real n-tuples

$$\hat{q}_n = (q^1,\ldots,q^n), \quad \hat{p}_n = (p_1,\ldots,p_n), \quad q^\alpha = (z_\alpha + z_\alpha^*)/\sqrt{2}, \quad p_\alpha = (z_\alpha - z_\alpha^*)/\sqrt{2}, \tag{8.16}$$

and use (3.27) to define in each quantum field-mode fibre $\hat{\mathcal{F}}_x$ the following POV measures (cf. Sec. 3.2):

$$\boldsymbol{E}_{\hat{\mathcal{F}}_x}(B_n) = (2\pi)^{-n} \int_{B_n} \left| \hat{\boldsymbol{\Phi}}_{\hat{z}_n} \right\rangle \mathbf{d}\hat{q}_n \mathbf{d}\hat{p}_n \left\langle \hat{\boldsymbol{\Phi}}_{\hat{z}_n} \right|, \qquad B_n \subset \mathbf{R}^{2n}. \tag{8.17}$$

Let Ω denote in the present context the family of all sequences $(q^1,p_1,q^2,p_2,\ldots)$ with $q^\alpha, p_\alpha \in \mathbf{R}^1$. A family $\mathcal{B}$ satisfying the conditions of Naimark's theorem can be obtained by taking finite unions and intersections of those sets in Ω whose elements are sequences whose first $2n$ elements belong, for some positive integer value of n, to one of the Borel sets B_n in (8.17). Then resulting Naimark extension $\mathcal{F}_x^\uparrow$ of each quantum field-mode fibre $\hat{\mathcal{F}}_x$ contains a spectral measure $\boldsymbol{E}_x^\uparrow(B)$ for which

$$\boldsymbol{E}_{\hat{\mathcal{F}}_x}(B_n) = \mathbf{P}_{\hat{\mathcal{F}}_x} \boldsymbol{E}_x^\uparrow(B_n), \qquad B_n \subset \mathbf{R}^{2n}. \tag{8.18}$$

Consequently, the orthogonal projector of $\mathcal{F}_x^\uparrow$ onto $\hat{\mathcal{F}}_x$ is equal to the following weak limit within the Naimark extension $\mathcal{F}_x^\uparrow$:

$$\mathbf{P}_{\hat{\mathcal{F}}_x} = \mathrm{w}\text{-}\lim_{n\to\infty} \boldsymbol{E}_{\hat{\mathcal{F}}_x}(\mathbf{R}^{2n}) = \int_{\mathbf{F}_x} \left| \hat{\boldsymbol{\Phi}}_{\mathbf{f}} \right\rangle \mathcal{D}\mathbf{f} \left\langle \hat{\boldsymbol{\Phi}}_{\mathbf{f}} \right|. \tag{8.19}$$

On the other hand, as a non-separable Hilbert space that can accommodate non-Fock representations of the canonical commutation relations (7.4), the Naimark extension $\mathcal{F}_x^\uparrow$ can also accommodate the proper energy operator in (8.7). By projecting in (8.11) that operator back into the quantum field-mode fibre $\hat{\mathcal{F}}_x$ we implement in the present relativistic regime a counterpart of the projection procedure carried out in (3.5.17) in the nonrelativistic regime.

Indeed, for the generic case of a nonrelativistic resolution generator ξ, the Naimark extension $\mathcal{H}^\uparrow$ of the Hilbert space $\mathcal{H} = \mathbf{P}_\xi L^2(\mathbf{R}^6)$ described in Sec. 3.3 equals $L^2(\mathbf{R}^6)$ (Ali and Prugovečki, 1986). Hence, the projector $\mathbf{P}_\xi$ represents a mathematical counterpart of the one in (8.19). For an external field represented by a potential $V(\mathbf{q},t)$ one could attempt to define the interaction Hamiltonian H_{I} in $\mathcal{H}$ as in conventional nonrelativistic quantum mechanics, namely by multiplying with $V(\mathbf{q},t)$ the wave functions $\psi(\mathbf{q},\mathbf{p})$ belonging to $\mathcal{H}$. However, although the operator resulting from this multiplication is perfectly well-defined in $\mathcal{H}^\uparrow = L^2(\mathbf{R}^6)$, in general it does not leave $\mathcal{H} = \mathbf{P}_\xi L^2(\mathbf{R}^6)$ invariant. On the other hand, it is the Hilbert space $\mathcal{H} = \mathbf{P}_\xi L^2(\mathbf{R}^6)$, rather than its Naimark extension space $\mathcal{H}^\uparrow = L^2(\mathbf{R}^6)$, that carries an irreducible representation of the canonical commutation relations in (3.1.2) for three degrees of freedom. Therefore, in conformity with basic quantum principles, it is

the operator $H_{I,\xi} = \mathbf{P}_\xi H_I \mathbf{P}_\xi$ that has to be adopted as the interaction Hamiltonian. For a generic resolution generator ξ, the interaction Hamiltonian $H_{I,\xi}$ corresponds to a generalization of the "smeared" potential in (3.5.17b) – cf. (Prugovečki, 1984, p. 44).

The spectral analysis embodied in (3.3.13)-(3.3.16) shows that $L^2(\mathbf{R}^6)$ carries an infinity of irreducible ray representations of the Galilei group, and therefore also of the canonical commutation relations for three degrees of freedom. In the case of zero spin these latter representations are irreducible and, on account of (3.3.14), unitarily equivalent to the one in the Hilbert space $\mathcal{H}_\ell = \mathbf{P}_{\xi(\ell)} L^2(\mathbf{R}^6)$, in which the interaction Hamiltonian $H_{I,\xi(\ell)}$ determined by (3.5.17) acts. In this nonrelativistic regime the sharp-point limits discussed in Secs. 3.4 and 3.5 exist, and the transition to the configuration representation, effected as in (3.3.14), reveals that $H_{I,\xi(\ell)} = \mathbf{P}_{\xi(\ell)} H_I \mathbf{P}_{\xi(\ell)}$ is the only interaction Hamiltonian which corresponds to the "smearing" in (3.5.17b) with the confidence function in (3.4.12a); therefore, $H_{I,\xi(\ell)}$ is the only interaction Hamiltonian in $L^2(\mathbf{R}^6)$ which transcends, in the sharp-point limit in (3.4.13), into the correct local interaction Hamiltonian that acts upon nonrelativistic quantum particles that are pointlike – cf. (Prugovečki, 1984), Chapter 1, Eqs. (8.22) and (8.23).

A similar mathematically rigorous sharp-point limit does not exist in the present QFT context, since the three renormalization constants in (4.1.9c), (4.2.5) and (5.16), that enter the Lagrangian density in (6.14), diverge in the limit $\ell \to 0$. However, later in this section we shall indicate how contact with formal perturbation theory in Minkowski space can be effected. Albeit formal, such comparisons provide support for the idea of projecting in (8.11) the interaction proper energy operator (8.7) back into $\hat{\mathcal{F}}_x$. Moreover, studies of interacting conventional quantum fields with cutoffs indicate (cf., e.g., Prugovečki, 1972; Prugovečki and Manoukian, 1972) that the effects of quantum field interactions can be viewed as giving rise to a vacuum renormalization in a given Fock space, corresponding to the fact that the ground state of the Hamiltonian incorporating such an interaction is no longer the original Fock vacuum. Of course, such cutoffs destroy relativistic invariance, but as we saw in the last section, Haag's theorem results from the postulation of a unique global vacuum and the imposition of (7.6), rather than from relativistic covariance or local commutativity. In a framework with an infinity of *local* vacuum states, such as the present one, that theorem no longer plays any role.

It might be thought that the argument based on nonrelativistic quantum theory in $\mathcal{H}_\ell^\uparrow = L^2(\mathbf{R}^6)$ is irrelevant to quantum field theory, since it does not involve self-interactions that give rise to virtual quanta. However, such self-interactions could be introduced in the above considerations by using the same techniques as in the BCS model – cf. (Emch, 1972, Sec. 1.f). Such considerations actually indicate that the *local* Fock vacuum plays in quantum-geometric field theory a role similar to that played by the proper state vector ξ in the nonrelativistic regime: the latter generates the elements of quantum frames which mark the various values around which *finite* fluctuations in spatial localization take place; whereas, the former generates the elements of coherent field mode sections around which the *finite* fluctuations occur in the number of excitons of a given quantum field mode above a mean base location $x \in \mathbf{M}$.

In the present quantum-geometric approach to QFT, the virtual quanta arising from

self-interactions manifest themselves when we add to the free-fall proper energy operator $\rho(x)$ in (5.5) the interaction proper energy operator (8.11). The local vacuum state $\Psi_{0;x}$ is then no longer the ground state of the resulting proper energy operator

$$\tilde{\rho}(x) = \rho(x) + \tilde{\rho}_{\mathrm{I}}(x) = \int_{V_x^+} \mathbf{P}_{\hat{f}_x} : T_{00}[\varphi(x,v)] + T_{00}^{\mathrm{I}}[\varphi(x,v)] : \mathbf{P}_{\hat{f}_x}\, d\Omega(v) \,. \tag{8.20}$$

In view of the fact that the expectation value of the local energy operator (8.19) is zero for the local vacuum state $\Psi_{0;x}$, whereas the expectation value

$$\left\langle \hat{\Phi}^{g}_{\varphi;x} \middle| \tilde{\rho}(x)\, \hat{\Phi}^{g}_{\varphi;x} \right\rangle = \int_{V^+} \tilde{T}_{00}[\overline{\varphi}(x,v) + \varphi(x,v)] d\Omega(v) \,, \qquad \tilde{T}_{00} = T_{00} + T_{00}^{\mathrm{I}} \,, \tag{8.21}$$

is positive whenever the above component of the classical stress-energy tensor is positive, the *local* ground energy

$$E_0(x) = \inf \{ spectrum\ \tilde{\rho}(x) \} \in \mathbf{R}^1 \tag{8.22}$$

is in general negative. Hence, the pure creation terms within the timelike component of the interaction stress-energy tensor in (8.8) can be viewed as giving rise, when acting upon the local vacuum state $\Psi_{0;x}$, to local quantum field excitations consisting of "virtual" exciton modes, which cause fluctuations around the local vacuum expectation value, that remains zero. Since the mathematical cause of this phenomenon lies in the fact that the local vacuum state $\Psi_{0;x}$ is not the ground state of the proper energy operator in (8.20), it might appear that the adoption of the local ground state eigenvector (if it exists) as the "renormalized" local vacuum should be implemented – as it is done in constructive QFT (Glimm and Jaffee, 1970). This, however, would represent a re-assignment of the zero point of the local field energy. Such a re-assignment is not acceptable in a theory involving gravity, since "it is not really true that the zero point of energy can be assigned completely arbitrarily. Energy is equivalent to mass, and mass has a gravitational effect." (Feynman and Hibbs, 1965, p. 245.) Of course, it might be argued that in a framework in which gravity is treated as an external field this does not matter. However, since the present considerations represent one of the many steps leading to the theory of quantum gravity in Chapter 8, such an attitude is not physically admissible.

Let us therefore substitute in place of $\rho(x_n)$ in (6.10) the expression in (8.20) as it stands, in order to arrive at the quantum-geometric propagator

$$\mathcal{K}(\varphi(x''); \varphi(x')) = \lim_{\varepsilon \to +0} \prod_{n=N}^{1}{}' \int_{\mathrm{E}(\Sigma_{t_n})} \mathcal{D}\varphi_n \left\langle \hat{\Phi}^{g}_{\varphi_n(x_n)} \middle| e^{-i\varepsilon \int d\sigma(x_n) \tilde{\rho}(x_n)}\, \hat{\Phi}^{g}_{\varphi_{n-1}(x_n)} \right\rangle \tag{8.23}$$

for a self-interacting quantum field. The matrix element that now replaces the one on the right-hand side of (6.13) is given by

$$\exp[-i\,\delta^4 x_n \int_{V^+} d\Omega(v)\, \tilde{T}_{00}[\overline{\varphi}_n(x_n,v) + \varphi_{n-1}(x_n,v)]] \left\langle \hat{\Phi}^{g}_{\varphi_n;x_n} \middle| \hat{\Phi}^{g}_{\varphi_{n-1};x_n} \right\rangle \,, \tag{8.24a}$$

$$\tilde{T}_{00}[\overline{\varphi}_n + \varphi_{n-1}] = T_{00}[\overline{\varphi}_n + \varphi_{n-1}] + T_{00}^{\mathrm{I}}[\overline{\varphi}_n + \varphi_{n-1}] \,. \tag{8.24b}$$

This matrix element is mathematically well defined, since all powers of exciton-mode wave functions are integrable over V^+ due to the fact those wave functions are smooth and

square-integrable, and that according to (3.19)-(3.22) they are also bounded, so that

$$\left|\overline{\varphi}_n(x_n,\upsilon)+\varphi_{n-1}(x_n,\upsilon)\right|^k \leq \text{const.}\left|\overline{\varphi}_n(x_n,\upsilon)+\varphi_{n-1}(x_n,\upsilon)\right|^2 , \qquad k=2,3,\ldots . \tag{8.25}$$

Its use in (8.23) leads to the modification of the action (6.18) by the addition of an interaction term, so that

$$\mathcal{K}(\varphi(x'');\varphi(x')) = \int \mathrm{D}\varphi \exp[i\tilde{S}(\varphi)] , \qquad \tilde{S}(\varphi) = S(\varphi)+S_{\mathrm{I}}(\varphi) , \tag{8.26}$$

$$S_{\mathrm{I}}(\varphi) = \int_{t'}^{t''} dt \int_{\Sigma_t} d\sigma(x) \int_{V^+} d\Omega(\upsilon)\, \mathcal{L}_{\mathrm{I}}(\overline{\varphi}_t(x,\upsilon)+\varphi_{t-0}(x,\upsilon)) , \tag{8.27}$$

as the interaction "Lagrangian density" in (8.27) is added to the one in (6.19) .

The infinite "clouds" of virtual quanta that "dress" particles in conventional special relativistic QFT result from the adoption of a *global* spacetime point of view. This becomes evident as soon as the Feynman diagram technique is examined in the configuration (cf., e.g., Schweber 1961, Sec. 14b; Das, 1993, Sec. 9.3) rather than the momentum representation. In this S-matrix conceptualization all virtual pair creations and annihilations that accompany the quantum field propagation in Minkowski space are treated *as if* they are incorporated into a single entity, which *concurrently* exists in *all* of spacetime – rather than as an *evolutionary* process, whereby pairs are perpetually *being* created and annihilated, while the process of quantum propagation is taking place. This presupposes the concept of S-matrix as fundamental, rather than as an idealization which is, strictly speaking, *physically* unrealizable in generic curved spacetimes – including those for the "Big Bang" cosmological models. In general, albeit the notion of S-matrix can be retained in asymptotically flat spacetimes, the concept of free asymptotic state on which it relies loses its literal meaning in a realistic physical setting in which there is no shielding from gravitational forces.

On the other hand, in the special relativistic regime all the Schrödinger-picture identifications described in (6.29), (6.31) and (6.32) can be applied to (8.23). By using (5.43), (6.4) and (8.24b) we then see that the right-hand side of (8.23) is equal to

$$\lim_{\varepsilon\to+0} \prod_{n=N}^{1}{}' \int_{\mathbf{F}} \mathcal{D}f_n \left\langle \Phi_{f_n} \middle| \exp[-i\varepsilon \int_{\mathbf{R}^3} d\mathbf{x}_n \int_{V^+} :\tilde{T}_{00}[\varphi]: \delta(\upsilon^2-1)\, d^4\upsilon)] \Phi_{f_{n-1}} \right\rangle . \tag{8.28}$$

Consequently, according to (8.8) and (8.21), the propagator in (8.23) is in the special relativistic regime equal to the following Schrödinger-picture propagator in the Fock space $\mathcal{F}$:

$$\mathcal{K}(f'',t'';f',t') = \left\langle \Phi_{f'',t''} \middle| e^{-iH(t''-t')} \Phi_{f',t'} \right\rangle , \qquad H = H_0 + H_{\mathrm{I}} , \tag{8.29}$$

$$H_0 = \boldsymbol{P}_{0;s_0} , \qquad H_{\mathrm{I}} = \int_{\mathbf{R}^3} d\mathbf{x} \int_{V^+} d\Omega(\upsilon) :\mathcal{L}_{\mathrm{I}}[\varphi(\mathbf{x};\upsilon)]: . \tag{8.30}$$

We arrive at the interaction picture point of view by introducing in (8.29) the freely evolving coherent states

$$\Phi_{f_{\mathrm{in}}(t'),t'} = e^{-iH_0 t'} \Phi_{f_{\mathrm{in}},0} , \qquad \Phi_{f_{\mathrm{out}}(t''),t''} = e^{-iH_0 t''} \Phi_{f_{\mathrm{out}},0} . \tag{8.31}$$

Upon adopting, in accordance with the customary definition,[25]

$$\langle f_{\text{out}}|S|f_{\text{in}}\rangle := \lim_{t''\to+\infty}\lim_{t'\to-\infty}\langle \Phi_{f_{\text{out}},0}|T\exp[-i\int_{t'}^{t''}H_{\text{I}}(t)dt]\Phi_{f_{\text{in}},0}\rangle\ , \tag{8.32}$$

as a representative S-matrix element, we obtain

$$\langle f_{\text{out}}|S|f_{\text{in}}\rangle = \lim_{t''\to+\infty}\lim_{t'\to-\infty}\mathcal{K}(f_{\text{out}}(t''),t'';f_{\text{in}}(t'),t')\ . \tag{8.33}$$

The standard perturbative treatment[25] of (8.32) then yields

$$\langle f_{\text{out}}|S|f_{\text{in}}\rangle = \sum_{n=0}^{\infty}(i^n/n!)\langle f_{\text{out}}|S_n|f_{\text{in}}\rangle\ , \tag{8.34}$$

where in the present case we have

$$S_n = \int_{\mathbf{R}^{4n}} d^4x_1\cdots d^4x_n\int d\Omega(v_1)\cdots d\Omega(v_n)T(:\mathcal{L}_{\text{I}}[\boldsymbol{\varphi}(x_1;v_1)]:\cdots:\mathcal{L}_{\text{I}}[\boldsymbol{\varphi}(x_n;v_n)]:)\ . \tag{8.35}$$

The time-ordering in the above T-products is carried out in the conventional fashion with respect to the timelike components of the 4-vectors $x_1,\ldots,x_n\in M^4$ with respect to the chosen global Lorentz frame $\mathcal{L}$, so that in the case of two fields we have

$$T[\boldsymbol{\varphi}(x,v)\boldsymbol{\varphi}(x',v')] = \theta(x^0-x'^0)\boldsymbol{\varphi}(x,v)\boldsymbol{\varphi}(x',v')+\theta(x'^0-x^0)\boldsymbol{\varphi}(x',v')\boldsymbol{\varphi}(x,v). \tag{8.36}$$

Hence, Wick's theorem can be applied to the matrix elements of the T-products in (8.35) in exactly the same manner as in conventional QFT, thus reducing them to polynomials in the 2-point functions

$$\langle\Psi_0|T[\boldsymbol{\varphi}(x,v)\boldsymbol{\varphi}(x',v')]\Psi_0\rangle = (2\pi)^3 i\tilde{Z}^{-1}_{\ell,m}\Delta_{\text{F}}(z;z')\ , \tag{8.37a}$$

$$z = x-i\ell v\ ,\quad z' = x'-i\ell v'\ ,\quad x,x'\in\mathbf{R}^4\ ,\quad v,v'\in\boldsymbol{V}^+\ , \tag{8.37b}$$

where, in view of (2.8) and the identification of all the fibres of $\mathcal{E}$ with $\mathcal{F}$, we have

$$\Delta_{\text{F}}(z;z') = \theta(x^0-x'^0)\Delta^{(+)}(\bar{z}-z')-\theta(x'^0-x^0)\Delta^{(-)}(z-\bar{z}')\ . \tag{8.38}$$

Furthermore, according to (4.2.8), (4.2.9) and (4.2.13),

$$\Delta^{(+)}(\bar{z}-z') = -(2\pi)^{-3}i\int_{V_m^+}\exp[-i(\bar{z}-z')\cdot k]\,d\Omega_m(k) = -\Delta^{(-)}(\bar{z}'-z)\ , \tag{8.39}$$

where the explicit value of the above integral is provided by (4.2.9).

In view of the above application of Wick's theorem, to every spacetime Feynman diagram[25] can be assigned, side-by-side, a conventional QFT matrix element in which each internal boson line is interpreted as the Feynman 2-point "function" in (3.8.7), and matrix

[25] Cf., e.g., (Schweber, 1961). This textbook presents the derivation of (8.34) in its Secs. 11f and 13a, Wick's theorem in Sec. 13c, and it deals with Feynman diagrams in Chapter 14. Most of this material is standard QFT fare, which can also be found in most other textbooks on conventional QFT. The one major exception is the treatment of spacetime Feynman diagrams in Sec. 14b, which receives scant notice in more recent textbooks.

element of (8.35), in which the same internal boson line is interpreted as a represention of the vacuum expectation value in (8.37). We note that the 2-point "function" in (3.8.7) is actually a distribution (i.e., a generalized function), and that it can be expressed in the form

$$\Delta_F(x-x') = \theta(x^0-x'^0)\Delta^{(+)}(x-x') - \theta(x'^0-x^0)\Delta^{(-)}(x-x') \ , \tag{8.40a}$$

$$\Delta^{(+)}(x-x') = -(2\pi)^{-3} i \int_{V_m^+} \exp[-i(x-x')\cdot k]\, d\Omega_m(k) = -\Delta^{(-)}(x'-x) \ , \tag{8.40b}$$

so that it is equal to the 2-point function in (8.38) when the limit $\ell \to 0$ is taken in the sense of distributions. However, each vertex in a diagram representing a matrix element of (8.35) also entails an integration over a stochastic 4-velocity variable $v \in V^+$. This integration produces a Lorentz-invariant factor at the location $x \in M^4$ of that vertex, such as the following one for the φ^4-model with interaction Lagrangian density (8.6):

$$V_\ell(p) = -(\lambda/4!)\int_{V^+} \exp(-\ell\, v\cdot p)\, d\Omega(v) = -\pi\lambda\, \mathrm{K}_1(\ell\sqrt{p^2})\big/ 12\ell\sqrt{p^2} \ , \tag{8.41a}$$

$$p^2 = p\cdot p > 0 \ , \qquad p = k_1+k_2+k_3+k_4 \ , \qquad k_1,k_2,k_3,k_4 \in V_m^+ \ . \tag{8.41b}$$

In the momentum representation this vertex factor plays the role of a cutoff as long as $\ell > 0$, but in the sharp-point limit $\ell \to 0$ it diverges:

$$-V_\ell(p) \ \sim \ \sqrt{2\pi^3}\lambda\, e^{-\ell\sqrt{p^2}} \big/ 4!\,\ell^3 (p^2)^{3/2} \quad \xrightarrow[\ell \to +0]{} \quad \infty \tag{8.42}$$

Such divergencies have to be absorbed, together with those in (4.2.6), into the renormalizations of the mass and the coupling constant, in order to secure the term by term convergence in the sharp-point limit $\ell \to 0$ of the renormalized matrix elements in (8.34) to their counterparts in the conventional ϕ^4-model.

This illustrates how, at the level of *formal* perturbation theory, agreement can be achieved between S-matrix expansions derived from quantum-geometric propagators in Minkowski space and their conventional counterparts. In the process, it exhibits the mathematical roots of divergencies in conventional QFT. However, this still does not provide a *physical* justification of the *infinite* renormalizations of S-matrix elements between asymptotic states that take an infinite time span to achieve – as is the case in (8.32). Indeed, the association of renormalizations with asymptotic processes requiring infinite spans in *real* time, and therefore depicted by an *infinite* number of Feynman diagrams, reminds one of Feynman's (1966) statement that "renormalization theory is simply a way to sweep the difficulties of the divergences of electrodynamics under the rug." At the foundational level, it raises, even in the special relativistic regime, the earlier posed question as to how physical processes that purportedly give rise to those renormalizations can *concurrently*[26] exist in *all* of spacetime – whereas the observable quantum reality is an *evolutionary* process, during which pairs are *perpetually being* created and annihilated without trace, rather than eternally hovering around, as suggested by formal S-matrix manipulations.

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[26] There is no time scale in relativistic S-matrix theory that would describe an attenuation in the contribution of these processes around the time when an *actual* quantum field interaction takes place. Thus, they all manifest themselves when the interaction is adiabatically "switched on" mathematically rather than physically.

According to the pragmatic attitude that underlies the conventional renormalization of the S-matrix perturbation series, such foundational questions are of no real concern to quantum physicists as long as the right numerical predictions are provided by a given set of computational rules. But, as Heisenberg pointed out in his very last paper,[27] such an attitude affects the type of questions posed during the process of developing physical models intended to describe how Nature functions. Thus, in the second paragraph of that paper, Heisenberg stated: "I believe that certain erroneous developments in particle theory – and I am afraid that such developments do exist – are caused by a misconception by some physicists that it is possible to avoid philosophical arguments altogether. Starting with poor philosophy, they pose the wrong questions. It is only a slight exaggeration to say that good physics has at times been spoiled by poor philosophy. . . . Having witnessed similar mistakes in the development of quantum mechanics fifty years ago, I am in the position to make some suggestions to avoid such errors in the future." (Heisenberg, 1976, p. 32.)

The suggestions which Heisenberg had to offer were drawn from developments which he witnessed first-hand durng the earlier part of this century, and which exemplify how demonstrably erroneous conclusions were reached on the basis of considerations "of purely formal mathematical nature" (*ibid.*, p. 38).

As we saw in Sec. 1.4, this same attitude towards "purely formal" or "artificial" rules was shared by other founders of QFT, and most notably by Dirac. This is illustrated very clearly by the following passage from one of Dirac's 1975 published lectures: "Our present quantum theory is very good provided we do not try to push it too far – we do not try to apply it to particles with very high energies and we do not try to apply it to very small distances. When we try to push it in these directions, we get equations which do not have sensible solutions . . . It is because of these difficulties that I feel that the foundations of quantum mechanics have not yet been correctly established. Working with the present foundations, people have done an awful lot of work in making applications in which they find rules for discarding the infinities. But these rules, even though they may lead to results in agreement with observations, are artificial rules, and I just cannot accept that the present foundations are correct." (Dirac 1978a, p. 20.)

It is in Dirac's spirit of striving to eliminate artificial rules from QFT by appealing to foundational arguments and considerations that we extend in the next chapter the quantum-geometric treatment of spin-0 quantum fields to spin-1/2 Dirac quantum fields in curved spacetime.

[27] This paper represents the English translation of the opening lecture of the German's Physical Society spring 1975 meeting, and was published posthumously. Many of the same points were previously made by Heisenberg in a 1968 talk delivered at ICTP in Trieste, entitled "Theory, Criticism and Philosophy," and reproduced in (Salam, 1990, pp. 85-124). The influences of Einstein and Bohr on these ideas were there explicitly acknowledged.

Chapter 6

Massive Quantum-Geometric Fermion Fields

According to the connection between spin and statistics, fermion fields correspond to half-integral spin values, and obey the Fermi-Dirac statistics. Hence, in their case the role of commutators is taken over by anticommutators. To cope with this problem, Berezin functional integration for fermion fields makes use of Grassmann algebras of anticommuting supernumbers. The quantum geometries required to handle such functional integration consist of superfibres that incorporate these supernumbers. The basic geometric ideas of the preceding chapter can be transferred to fermion fields within such superfibre bundles in a manner that requires some new mathematical techniques, but does not affect the physical substance. These techniques are presented in the context of spin-1/2 fields – which are of primary physical significance – but are equally well applicable to all other fermion fields.

6.1. Fock-Dirac Bundles and Bispinor Quantum Frame Fields

A quantum frame field that generates spin-1/2 charged fermion and antifermion states over a base Lorentzian spacetime manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ can be built (Prugovečki, 1987b; Prugovečki and Warlow, 1989) by following the same basic geometric principles as in the case of the quantum geometries for spin-0 fields, which were studied in the preceding chapter. Thus, a typical Fock-Dirac fibre $\mathcal{F}$ is first constructed from the fermion and the antifermion typical fibres $\mathbf{F}^{(\pm)}$, consisting of the bispinor-valued wave functions in (4.2.18) and (4.2.19), by taking antisymmetric tensor products:

$$\mathcal{F} = \bigoplus_{n,n'=0}^{\infty} \mathcal{F}_{n,n'} , \qquad \mathcal{F}_{n,n'} = (\mathbf{F}^{(+)} \underset{A}{\otimes} \cdots \underset{A}{\otimes} \mathbf{F}^{(+)}) \otimes (\mathbf{F}^{(-)} \underset{A}{\otimes} \cdots \underset{A}{\otimes} \mathbf{F}^{(-)}) . \tag{1.1}$$

These antisymmetric tensor products reflect the presence of Fermi-Dirac statistics.

The two typical fibres $\mathbf{F}^{(\pm)}$ carry irreducible representations of the covering group $\mathrm{ISL}(2,\mathbf{C})$ of the Poincaré group $\mathrm{ISO}_0(3,1)$ induced by the unitary representation,

$$U(a,A) \; : \; \Psi(q - i\ell v) \mapsto \; \Psi'(q - i\ell v) = S(A)\, \Psi(\Lambda^{-1}(q - a - i\ell v)) , \tag{1.2a}$$

$$S(A) = S_0 \begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix} S_0^{-1}, \quad S_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 & \sigma_0 \\ \sigma_0 & -\sigma_0 \end{pmatrix} = S_0^{-1}, \quad A \in \mathrm{SL}(2,\mathbf{C}), \tag{1.2b}$$

that acts within the typical Dirac fibre $\mathbf{F} = \mathbf{F}^{(+)} \oplus \mathbf{F}^{(-)}$. Consequently, by using the definition of $\mathbf{G}$-products in Sec. 4.1, we can construct the Fock-Dirac quantum bundle

$$\mathcal{E} = D\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}}) \times_{\mathbf{G}} \mathcal{F} , \qquad \mathbf{G} = \mathrm{ISL}(2,\mathbf{C}) , \tag{1.3}$$

from the typical Fock-Dirac fibre in (1.1) and the Dirac frame bundle

$$D\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}}) = \bigcup_{\mathbf{u}\in S\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})} \left\{ \boldsymbol{\Phi}^{\mathbf{u}}_{m;\zeta,s} \,\middle|\, q\in\mathbf{R}^4,\ v\in \boldsymbol{V}=\boldsymbol{V}^+\cup\boldsymbol{V}^-,\ s\in\{1,2,3,4\} \right\} . \tag{1.4}$$

This latter bundle is a principal bundle which consists of all the Dirac quantum frames in (4.2.21) that are soldered to the frames $\mathbf{u}$ in the affine spin-frame bundle $S\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$.

The dependence of the quantum frames in (1.4) on the choice of element $\mathbf{u}$ from the affine spin-frame bundle $S\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ manifests itself implicitly through the Dirac matrices

$$\gamma^0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix},\quad \gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix},\quad \gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix},\quad \gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \tag{1.5}$$

which occur in the definition (4.2.21) of their elements, and for which (1.2) gives rise to

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu ,\qquad S(\Lambda) = \pm S(A) ,\qquad \Lambda \leftrightarrow \pm A . \tag{1.6}$$

This particular realization of the Dirac matrices emerges during the transition from the Wigner (1939) type of spinor representations for spin-1/2 and rest mass m, which is directly related to changes of spin frames, to the Dirac type of bispinor representations of ISL(2,**C**) in (1.2a), in which that relationship manifests itself only indirectly in (1.2b), and in the manner in which components of a 4-vector q transform under a change of spin-frames described by some $A\in\mathrm{SL}(2,\mathbf{C})$:

$$A\underline{q}A^* = \Lambda^\mu{}_\nu q^\nu \sigma_\mu = (\Lambda q)^\mu \sigma_\mu ,\quad \underline{q} = q^\mu\sigma_\mu ,\quad \Lambda^\mu{}_\nu = \tfrac{1}{2}\mathrm{Tr}\left(\sigma_\mu A \sigma_\nu A^*\right) . \tag{1.7}$$

However, once these relationships[1] have been established, the principal bundle $D\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ of Dirac quantum frames in (1.4) can be regarded as fundamental in its own right, namely as consisting of quantum frames related to respective Poincaré frames $\boldsymbol{u}\in P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ via the local quantum metric fluctuation amplitudes in (4.2.13b),

$$\boldsymbol{\Phi}^{(\pm)\boldsymbol{u}(x)}_{m;\zeta,s}(\zeta',s') = \left[\left(\tfrac{i}{2} - \tfrac{1}{2m}\gamma^k\partial/\partial q^k\right)\gamma^0\right]_{s's} \Delta^{(\pm)}_x(\zeta';\zeta) , \tag{1.8}$$

rather than via affine spin-frames. The outcome of the change

$$\mathbf{U} \mapsto \mathbf{U}' = \mathbf{U}\cdot(a,A) ,\qquad (a,A)\in \mathrm{ISL}(2,\mathbf{C}) , \tag{1.9}$$

of a local quantum Dirac frame

$$\mathbf{U} = \left\{ \boldsymbol{\Phi}^{(\pm)\boldsymbol{u}}_{m;\zeta,s} \,\middle|\, q\in\mathbf{R}^4,\ v\in\boldsymbol{V}^+,\ s\in\{1,2,3,4\} \right\} \in D\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}}) , \tag{1.10}$$

resulting from the action of elements of ISL(2,**C**) is then given directly by

$$\mathbf{U}' = \left\{ \boldsymbol{\Phi}^{(\pm)\boldsymbol{u}'}_{m;\zeta',s} = U^{-1}(a,A)\boldsymbol{\Phi}^{(\pm)\boldsymbol{u}}_{m;\zeta,s} \,\middle|\, \boldsymbol{u}' = \boldsymbol{u}\cdot(a,\Lambda),\ \zeta' = \Lambda^{-1}(q-a-i\ell v) \right\}. \tag{1.11}$$

Since the quantum fluctuation amplitudes in (4.2.22) reflect the fundamental localiza-

[1] A systematic study of these relationships can be found in (Prugovečki, 1980). A more detailed review of the main aspects and results of this study is provided in Secs. 6.1 and 6.2 of (Prugovečki, 1992).

tion uncertainties displayed by these frames, they can be suitably expressed in the form

$$\mathbf{S}^{(\pm)}_{\mathbf{U}(x)}(\zeta;\zeta') = -\left(\tfrac{1}{2} + \tfrac{i}{2m}\gamma^k \partial/\partial q^k\right) \Delta^{(\pm)}_x(\zeta;\zeta') \ . \tag{1.12}$$

It then follows from (1.6) and (1.8) that

$$\mathbf{S}^{(\pm)}_{\mathbf{U}(x)}(q' - i\ell v'; q'' - i\ell v'') = \mathbf{S}^{(\pm)}_{\mathbf{U}'(x)}(\Lambda^{-1}(q' - a - i\ell v'); \Lambda^{-1}(q'' - a - i\ell v'')) \ , \tag{1.13}$$

under the change of local Dirac quantum frame described by (1.9)-(1.11).

In view of these fundamental features of the Dirac frame bundle $D\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, instead of dealing in the sequel with gauges given by cross-sections of the affine spin-frame bundle $S\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$, we shall work exclusively with the mathematically equivalent concept of *Dirac gauges*, provided by cross-sections $\mathbf{s}$ of the Dirac frame bundle $D\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$. From the physical point of view this is actually more suitable, since the notion of half-integer spin is an intrinsically quantum concept with no counterpart in classical physics.

According to (4.1.3), soldering maps are intrinsically contained in the structure of the Fock-Dirac bundle $\mathcal{E}$, so that at each $x \in \mathbf{M}$ they gives rise to the isomorphisms

$$\boldsymbol{\sigma}^{\mathbf{U}}_x : \ \boldsymbol{\Psi} \mapsto \boldsymbol{\Psi} \in \mathcal{F} \ , \quad \boldsymbol{\Psi} \in \mathcal{F}_x , \qquad \mathbf{U} \in \boldsymbol{\Pi}^{-1}(x) \subset D\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}}) . \tag{1.14}$$

The above map assigns to each state vector $\boldsymbol{\Psi}_{n,n';x}$ within the n-fermion and n'-antifermion subfibre $\mathcal{F}_{n,n';x}$ of the Fock-Dirac fibre $\mathcal{F}_x$ a coordinate wave function,

$$\boldsymbol{\sigma}^{\mathbf{U}}_x : \ \boldsymbol{\Psi}_{n,n';x} \mapsto \Psi_{n,n';x}(\zeta_1, s_1, \ldots, \zeta_n, s_n; \zeta_1', s_1', \ldots, \zeta_{n'}', s_{n'}') \ , \tag{1.15}$$

with respect to spinorial quantum frames. These spinorial quantum frames are built from the Dirac quantum frames in (1.4) in the same manner in which tensorial quantum frames were built in Sec. 4.5 from the quantum frames in (4.2.7):

$$(\boldsymbol{\Phi}^{(+)\boldsymbol{u}}_{m;\zeta_1,s_1} \underset{A}{\otimes} \cdots \underset{A}{\otimes} \boldsymbol{\Phi}^{(+)\boldsymbol{u}}_{m;\zeta_n,s_n}) \otimes (\boldsymbol{\Phi}^{(-)\boldsymbol{u}}_{m;\zeta_1',s_1'} \underset{A}{\otimes} \cdots \underset{A}{\otimes} \boldsymbol{\Phi}^{(-)\boldsymbol{u}}_{m;\zeta_{n'}',s_{n'}'}) \ . \tag{1.16}$$

As in the spin-0 case, the construction of the Fock-Dirac quantum bundle $\mathcal{E}$ in (1.3) results in the assignment of a normalized local vacuum state vector $\boldsymbol{\Psi}_{0;x} \in \mathcal{F}_x$ to each base location $x \in \mathbf{M}$. The vector $\boldsymbol{\Psi}_{0;x}$ spans the one-dimensional vacuum sector $\mathcal{F}_{0;x}$ above that base location.

As a consequence of the alternative form (4.2.20) of the inner product in the fibres of the Dirac quantum bundle, their reproducibility properties (4.2.24) are given by

$$\mathbf{S}^{(\pm)}_{\mathbf{U}}(\zeta';\zeta'') = -i\int \mathbf{S}^{(\pm)}_{\mathbf{U}}(\zeta';\zeta)\gamma^k \mathbf{S}^{(\pm)}_{\mathbf{U}}(\zeta;\zeta'')\, d\sigma_k(q) d\hat{\Omega}(v) \ . \tag{1.17}$$

Hence, in accordance with (4.2.23) and (1.14), the action of the orthogonal projectors

$$\mathbf{P}^{(\pm)}_x = \pm \sum\nolimits_{s=1}^{4} \int d\Sigma(\zeta) \left| \boldsymbol{\Phi}^{(\pm)\boldsymbol{u}}_{m;\zeta,s} \right\rangle \gamma^0_{ss} \left\langle \boldsymbol{\Phi}^{(\pm)\boldsymbol{u}}_{m;\zeta,s} \right| , \qquad \mathbf{F}^{(\pm)}_x = \mathbf{P}^{(\pm)}_x \mathbf{F}_x \ , \tag{1.18}$$

onto fermion and antifermion subspaces in those fibres can be expressed by

$$(\boldsymbol{\sigma}^{\mathbf{U}}_x \mathbf{P}^{(\pm)}_x \Psi_x)(\zeta) = -i\int \mathbf{S}^{(\pm)}_{\mathbf{U}}(\zeta;\zeta')\gamma^k \Psi_x(\zeta') d\sigma_k(q') d\hat{\Omega}(v') \ , \ d\hat{\Omega}(v') = 2\hat{Z}_{\ell m} d\Omega(v') . \tag{1.19}$$

We can now proceed as in Sec. 5.2, and define the fermion creation operators by

$$\left(\psi^{(+)}(\zeta,s)\Psi_{n,n'}\right)_{n+1,n'}(\zeta_1,s_1,\ldots,\zeta_{n+1},s_{n+1};\ldots) = i(n+1)^{-1/2}$$
$$\times\sum_{j=1}^{n+1}(-1)^j S^{(+)}(\zeta_j,s_j;\zeta,s)\,\Psi_{n,n'}(\zeta_1,s_1,\ldots,\hat{\zeta}_j,\hat{s}_j,\ldots,\zeta_{n+1},s_{n+1};\ldots)\,, \quad (1.20)$$

where the circumflexes signify that the variables under them are to be omitted from the above wave function, and the s-variables refer to matrix components, so that

$$S^{(\pm)}(\zeta,s;\zeta',s') := \left[\boldsymbol{S}^{(\pm)}(\zeta;\zeta')\right]_{ss'},\ \ q,q'\in\mathbf{R}^4,\ \ v,v'\in\boldsymbol{V}^+,\ \ s,s'=1,2,3,4\ . \quad (1.21a)$$

The above matrix-valued functions $\boldsymbol{S}^{(\pm)}$ are defined, in accordance with (1.12), by

$$\boldsymbol{S}^{(\pm)}(\zeta;\zeta') = -\left(\tfrac{1}{2}+\tfrac{i}{2m}\gamma^\mu\partial/\partial q^\mu\right)\Delta^{(\pm)}(\zeta;\zeta')\,, \quad (1.21b)$$

in the fundamental frame $\boldsymbol{u}_0$ of $\mathbf{R}^4$ given in (5.5.12).

The spin-1/2 antifermion creation operators are similarly defined by

$$\left(\psi^{(+)}(-\zeta,s)\Psi_{n,n'}\right)_{n,n'+1}(\ldots;\zeta_1,s_1,\ldots,\zeta_{n'+1},s_{n'+1}) = i(n'+1)^{-1/2}$$
$$\times\sum_{j=1}^{n'+1}(-1)^{j+n}S^{(-)}(\zeta_j,s_j;\zeta,s)\,\Psi_{n,n'}(\ldots;\zeta_1,s_1,\ldots,\hat{\zeta}_j,\hat{s}_j,\ldots,\zeta_{n'+1},s_{n'+1})\,. \quad (1.22)$$

These definitions give rise to bounded operators on each n-fermion and n'-antifermion sector $\mathcal{F}_{n,n'}$ of the typical Fock-Dirac fibre $\mathcal{F}$, so that their domains can be immediately extended by linearity to the entire fibre $\mathcal{F}$.

The corresponding fermion and antifermion annihilation operators can be defined by taking adjoints of the above fermion and antifermion creation operators, respectively. The reproducibility properties in (1.17) imply that

$$(\mathbf{P}^{(\pm)}\Psi)(\zeta) = -i\int \boldsymbol{S}^{(\pm)}(\zeta;\zeta')\,\gamma^\mu\,\Psi(\zeta')\,d\sigma_\mu(q')d\hat{\Omega}(v')\,, \quad \mathbf{P}^{(\pm)}\mathbf{F} = \mathbf{F}^{(\pm)}\,, \quad (1.23)$$

within the typical fibre $\mathbf{F}$, so that we obtain:

$$\left(\psi^{(-)}(\zeta,s)\Psi_{n,n'}\right)_{n-1,n'}(\zeta_1,s_1,\ldots,\zeta_{n-1},s_{n-1};\ldots)$$
$$= n^{1/2}\,\Psi_{n,n'}(\zeta,s,\zeta_1,s_1,\ldots,\zeta_{n-1},s_{n-1};\ldots)\,, \quad (1.24)$$

$$\left(\psi^{(-)}(-\zeta,s)\Psi_{n,n'}\right)_{n,n'-1}(\ldots;\zeta_1,s_1,\ldots,\zeta_{n'-1},s_{n-1})$$
$$= n'^{1/2}\,(-1)^n\,\Psi_{n,n'}(\ldots;\zeta,s,\zeta_1,s_1,\ldots,\zeta_{n'-1},s_{n'-1})\,. \quad (1.25)$$

We note that, in compliance with the QFT convention of formally viewing antiparticles as objects propagating backward in time, the above antifermion creation and annihilation operators are defined as functions of the stochastic 4-velocity variable $-v$ that assumes values on the backward 4-velocity hyperboloid $\boldsymbol{V}^-$. At a deeper physical level, the above mode of construction of antifermionic operators ensures local TCP conservation when the bispinor *Dirac quantum frame field*

$$\psi(\zeta) = \psi^{(-)}(\zeta) + \psi^{(+)}(-\zeta) , \qquad \upsilon \in \boldsymbol{V} = \boldsymbol{V}^{+} \cup \boldsymbol{V}^{-}, \tag{1.26}$$

consisting of one-column matrices constructed from the above defined creation and annihilation operators, is transferred to the fibres $\mathcal{F}_x$ of the Fock-Dirac quantum bundle $\mathcal{E}$:

$$\psi_x^{\mathbf{U}}(\zeta) = (\boldsymbol{\sigma}_x^{\mathbf{U}})^{-1} \psi(\zeta) \boldsymbol{\sigma}_x^{\mathbf{U}} , \qquad \mathbf{U} \in \boldsymbol{\Pi}^{-1}(x) \subset D\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}}) \cong S\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}}). \tag{1.27}$$

As opposed to the Klein-Gordon quantum frame field in (5.2.6), which satisfies the commutation relations in (5.2.9), the Dirac quantum frame field in (1.27) satisfies the anti-commutation relations

$$\{\psi_x^{\mathbf{U}}(\zeta,s),\ \psi_x^{\mathbf{U}}(-\zeta',s')\} = -i\boldsymbol{S}_{\mathbf{U}(x)}(\zeta,s;\zeta',s') , \tag{1.28}$$

which follow from those for the above defined fermion creation and annihilation operators:

$$\{\psi^{(-)}(\pm\zeta,s), \psi^{(+)}(\pm\zeta',s')\} = -i\boldsymbol{S}^{(\pm)}(\zeta,s;\zeta',s') , \quad \upsilon,\upsilon' \in \boldsymbol{V}^{+} , \tag{1.29}$$

$$\{\psi^{(-)}(\zeta,s), \psi^{(+)}(-\zeta',s')\} = \{\psi^{(-)}(-\zeta,s), \psi^{(+)}(\zeta',s')\} = 0 , \quad \upsilon,\upsilon' \in \boldsymbol{V}^{+}, \tag{1.30}$$

$$\{\psi^{(+)}(\zeta,s), \psi^{(+)}(\zeta',s')\} = \{\psi^{(-)}(\zeta,s), \psi^{(-)}(\zeta',s')\} = 0 , \quad \upsilon,\upsilon' \in \boldsymbol{V} . \tag{1.31}$$

This means that the method of construction of second-quantized boson frames presented in Sec. 5.3 will have to be modified in the case of these fermion operators by using supernumbers if such key relations as (5.3.13) and (5.3.14) are to be preserved.[2]

6.2. Standard Berezin-Dirac Quantum Superfibres

The first stage in the adaptation of Berezin's (1966) method of functional integration to the fermion frame fields in (1.26) proceeds as in the boson case. Thus, we first adopt within the typical Dirac fibre $\mathbf{F} = \mathbf{F}^{(+)} \oplus \mathbf{F}^{(-)}$ a fixed orthonormal basis $\{\mathrm{w}_1, \mathrm{w}_2, \ldots\}$, so that we can decompose the identity operator $\mathbf{1}_{\mathbf{F}}$ in $\mathbf{F}$ as follows:

$$\sum_{\alpha=1}^{\infty} |\mathrm{w}_\alpha\rangle\langle\mathrm{w}_\alpha| = \mathbf{1}_{\mathbf{F}} , \qquad \langle\mathrm{w}_\alpha|\mathrm{w}_\beta\rangle = \delta_{\alpha\beta} , \qquad \alpha,\beta = 1,2,\ldots . \tag{2.1}$$

Expressed in terms of the version (4.2.20) of the inner product in the typical Dirac fibre $\mathbf{F}$ the orthonormality condition assumes the form

$$\langle\mathrm{w}_\alpha|\mathrm{w}_\beta\rangle = \int_{V^+} d\hat{\Omega}(\upsilon) \int d\sigma_\mu(q) \overline{\mathrm{w}}_\alpha(\zeta) \gamma^\mu \mathrm{w}_\beta(\zeta) = \delta_{\alpha\beta} . \tag{2.2}$$

We shall therefore introduce operator-valued functionals on $\mathbf{F}$ accordingly:

$$\psi^{(+)}(f) = \int_{V^+} d\hat{\Omega}(\upsilon) \int d\sigma_\mu(q) [\psi^{(+)\mathrm{T}}(\zeta) \gamma^\mu f_+(\zeta) + \psi^{(+)\mathrm{T}}(-\zeta) \gamma^\mu f_-(\zeta)] , \tag{2.3a}$$

$$\psi^{(-)}(f) = \int_{V^+} d\hat{\Omega}(\upsilon) \int d\sigma_\mu(q) [\bar{f}_+(\zeta) \gamma^\mu \psi^{(-)}(\zeta) + \bar{f}_-(\zeta) \gamma^\mu \psi^{(-)}(-\zeta)] , \tag{2.3b}$$

[2] Some coherent-state methods for fermions do not make use of supernumbers (Blaizot and Orland, 1980, 1981). However, no coherent state can be an eigenvector of a fermion annihilation operator with a nonzero complex eigenvalue c, as was the case in (5.3.13), since a repeated application of that annihilation operator yields $c^2 = 0$, and that implies $c = 0$ for complex numbers. For this reason in Sec. 6.2 we use the Berezin (1966) method.

$$f = f_+ + f_- \in \mathbf{F} , \qquad f_\pm = \mathbf{P}^{(\pm)} f \in \mathbf{F}^{(\pm)} , \qquad \bar{f}_\pm = f_\pm^* \gamma^0 . \tag{2.3c}$$

By using (1.29) and (1.30) we obtain

$$\{\psi^{(-)}(\mathrm{w}_\alpha), \psi^{(+)}(\mathrm{w}_\beta)\} = -i \int \overline{\mathrm{w}}_\alpha(\zeta) \gamma^\mu [\mathbf{S}^{(+)}(\zeta;\zeta') + \mathbf{S}^{(-)}(\zeta;\zeta')] \times \gamma^\nu \mathrm{w}_\beta(\zeta') d\sigma_\mu(q) d\hat{\Omega}(v) d\sigma_\nu(q') d\hat{\Omega}(v') , \tag{2.4}$$

so that as a consequence of (1.23) and (2.2) we have

$$\{\psi^{(-)}(\mathrm{w}_\alpha), \psi^{(+)}(\mathrm{w}_\beta)\} = \delta_{\alpha\beta} , \qquad \alpha,\beta = 1,2,\ldots . \tag{2.5a}$$

Furthermore, as a direct consequence of (1.31) we also have

$$\{\psi^{(+)}(\mathrm{w}_\alpha), \psi^{(+)}(\mathrm{w}_\beta)\} = \{\psi^{(-)}(\mathrm{w}_\alpha), \psi^{(-)}(\mathrm{w}_\beta)\} = 0 , \tag{2.5b}$$

so that a representation of the canonical anticommutation relations is thus obtained within the Fock-Dirac typical fibre $\mathcal{F}$.

Coherent states of the Glauber type could be now introduced in the Fock-Dirac typical fibre $\mathcal{F}$ in accordance with (5.3.8), but they would not supply a continuous resolution of the identity in $\mathcal{F}$, since this fundamental property of such states emerges only in Hilbert spaces in which canonical commutation rather than anticommutation relations are provided. The basic idea of the Berezin (1966) method of functional integration therefore consists of the modification of such Hilbert spaces into Hilbert superspaces in which anticommuting supernumbers are present, so that each family of anticommutation relations would give rise also to a corresponding family of commutation relations.

The adaptation of Berezin's technique to the Fock-Dirac bundle $\mathcal{E}$ can be achieved by first extending its typical fibre $\mathcal{F}$ into a typical *Berezin-Dirac superfibre* $\mathcal{B}$. Such a typical superfibre is then a Hilbert superspace over an infinite-dimensional Grassmann algebra with involution[3].

In general, a Grassmann algebra with involution is a topological vector space Λ over the field $\mathbf{C}^1$ of complex numbers, within which an associative algebraic operation of multiplication is defined, and in which there is a family of anticommuting generators ϑ_α and $\overline{\vartheta}_\alpha$, which in the sequel we shall take to be countably infinite, so that $\alpha = 1,2,\ldots$, and

$$\{\vartheta_\alpha, \vartheta_\beta\} = \{\vartheta_\alpha, \overline{\vartheta}_\beta\} = \{\overline{\vartheta}_\alpha, \overline{\vartheta}_\beta\} = 0 , \qquad \alpha,\beta = 1,2,3,\ldots . \tag{2.6}$$

Its general element, called in physics literature (DeWitt, 1984a) a *supernumber*, can be written in the form[4]

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[3] All the basic results on Grassmann algebras and Berezin calculus required in this chapter can be found in (Berezin, 1966); whereas mathematically rigorous introductions to superanalysis and supermanifolds can be found in (Berezin, 1987) and (Khrennikov, 1988). An introduction to superanalysis and supermanifolds that might be more easily accessible to readers with a primarily physics background is provided by DeWitt (1984a); whereas, a treatment of Berezin coherent states for fermions is presented at a similar level by Ohnuki and Kashiwa (1978).

[4] Cf. (Berezin, 1966, pp. 60 and 67) and (Khrennikov, 1988, p. 105). It should be noted that a Grassmann algebra does not constitute an algebraic field, since not every nonzero supernumber has a unique inverse. In fact, the supernumber in (2.7a) has a multiplicative inverse if and only if c_0 is non-zero. In physics literature c_0 is called the *body*, and the remaining sum is called the the *soul*, of the supernumber in (2.7a).

$$\vartheta = c_0 + \sum_{m,n=1}^{\infty} \frac{1}{m!n!} \sum_{\alpha_1,\ldots,\beta_n=1}^{\infty} c_{\alpha_1\ldots\alpha_m\beta_1\ldots\beta_n}\, \vartheta_{\alpha_1} \cdots \vartheta_{\alpha_m}\, \overline{\vartheta}_{\beta_1} \cdots \overline{\vartheta}_{\beta_n}\,, \tag{2.7a}$$

$$\|\vartheta\|^2 = |c_0|^2 + \sum_{m,n=1}^{\infty} \sum \left| c_{\alpha_1\ldots\alpha_m\beta_1\ldots\beta_n} \right|^2, \quad \alpha_1 < \alpha_2 < \cdots, \quad \beta_1 < \beta_2 < \cdots, \tag{2.7b}$$

where the above complex coefficients are antisymmetric under the permutation of any pair of indices – regardless of whether it consists of α, or of β, or of α and β indices. We shall adopt in the Grassmann algebra Λ the topology determined by the norm defined by (2.7b) for each $\vartheta \in \Lambda$. The involution operation within the Grassmann algebra Λ then assigns to each such supernumber ϑ a conjugate, which we shall denote by setting a bar above ϑ:

$$\overline{\vartheta} = c_0^* + \sum_{m,n=1}^{\infty} \frac{1}{m!n!} \sum_{\alpha_1,\ldots,\beta_n=1}^{\infty} c^*_{\alpha_1\ldots\alpha_m\beta_1\ldots\beta_n}\, \vartheta_{\beta_n} \cdots \vartheta_{\beta_1}\, \overline{\vartheta}_{\alpha_m} \cdots \overline{\vartheta}_{\alpha_1}\,. \tag{2.8}$$

Any supernumber ϑ can be decomposed into its *even* or *grade-0* component, obtained by setting equal to zero all the coefficients in (2.7a) corresponding to odd values of n, and an *odd* or *grade-1* component, obtained by replacing with zero all the coefficients in (2.7a) corresponding to even values of n, including $n = 0$. This decomposition of each $\vartheta \in \Lambda$ gives rise to the decomposition of Λ into the commutative subalgebra Λ^0 consisting of all even supernumbers, and the vector subspace Λ^1 consisting of all odd supernumbers. The elements of Λ^0 commute not only amongst themselves, but also with all the odd supernumbers. However, Λ^1 does not constitute a subalgebra, since the product of two odd supernumbers is obviously always an even supernumber. This means that the Grassmann algebra $\Lambda = \Lambda^0 \oplus \Lambda^1$ is a $\mathbf{Z}_2$-graded vector space, in which the action of the projection operators $\pi_0 : \Lambda \to \Lambda^0$ and $\pi_1 : \Lambda \to \Lambda^1$ on each supernumber ϑ are obtained by setting equal to zero all the coefficients in (2.7a) next to products consisting of an odd or of an even number of generators, respectively.

By replacing in the fundamental definition of a vector space over an algebraic field, such as the field of complex numbers, the notion of field with that of a ring, one arrives at the concept of algebraic module.[5] Since a Grassmann algebra is a *non*commutative ring, its use in such a definition gives rise to a type of module called in physics literature a *superspace*. Once the concept of superspace is introduced, the definition of a *Hilbert superspace*[6] $\mathcal{H}$ over the Grassmann algebra Λ can be formulated in a manner which is analogous to the one for a Hilbert space over the field $\mathbf{C}^1$ of complex numbers (cf., e.g., Khrennikov, 1988). The inner product in such a Hilbert superspace assumes supernumber values within Λ^0, and the antilinearity and linearity in the first and the second variable, respectively, of an inner product in a complex Hilbert space are extended in conformity with the following rule for that Λ^0-valued inner product:

$$\left\langle \vartheta\Psi \middle| \vartheta'\Psi'\vartheta'' \right\rangle = \overline{\vartheta} \left\langle \Psi\overline{\vartheta}' \middle| \Psi' \right\rangle \vartheta'', \qquad \Psi, \Psi' \in \mathcal{H}\,, \qquad \vartheta, \vartheta', \vartheta'' \in \Lambda\,. \tag{2.9}$$

A particular realization of such a Hilbert superspace, whose construction parallels that of the Hilbert space $L^2(\Sigma_m)$ in Sec. 3.7, is the superspace $L^2(\Sigma_m, \Lambda)$ consisting of all the

[5] Cf., e.g., Chapter 10 of (Dummit and Foote, 1991). This textbook contains also the definitions of all the basic concepts of modern abstract algebra, with that of a module appearing on p. 315.

[6] In physics literature (cf. DeWitt, 1984a, Sec. 5.2) the term *super-Hilbert space* is often used instead.

(Lebesgue-measurable) supernumber-valued functions

$$\Psi(q,v) = c_0(q,v) + \sum_{m,n=1}^{\infty} \frac{1}{m!n!} \sum_{\alpha_1,\ldots,\beta_n=1}^{\infty} c_{\alpha_1\ldots\beta_n}(q,v)\, \vartheta_{\alpha_1} \cdots \overline{\vartheta}_{\beta_n} \,, \tag{2.10}$$

in the variables $q = (q^0,\mathbf{q})$ and $v = (v^0,\mathbf{v})$, with $\mathbf{q},\mathbf{v} \in \mathbf{R}^3$ and $v^0 = (\mathbf{v}^2+1)^{1/2}$, for which the coefficients satisfy (3.7.4) at all $q^0 \in \mathbf{R}^1$, and for which the real-valued norm

$$\|\Psi\|_2 = \left(\int_{q^0=0} \|\Psi(q,v)\|^2 \, d\mathbf{q}\, d\mathbf{v} \right)^{1/2} \tag{2.11}$$

is finite, so that it is equivalent to the Hilbert superspace $L^2(\mathbf{R}^6,\Lambda)$ (cf. Khrennikov, 1988, p. 117). We can then consider the direct sum

$$\mathbf{H} = L^2(\Sigma_m,\Lambda) \oplus L^2(\Sigma_m,\Lambda) \oplus L^2(\Sigma_m,\Lambda) \oplus L^2(\Sigma_m,\Lambda) \tag{2.12}$$

giving rise to a superspace consisting of one-column matrices with four supernumber-valued elements which represent functions that belong to $L^2(\Sigma_m,\Lambda)$. We introduce in this superspace the following even-supernumber-valued inner product,

$$\langle \Psi | \Psi' \rangle = \int_{q^0=0} \sum_{s=1}^{4} \gamma^0_{ss} \left(\pi_0 \overline{\Psi}_s(q,v)\, \pi_0 \Psi'_s(q,v) + \pi_1 \overline{\Psi}_s(q,v)\, \pi_1 \Psi'_s(q,v) \right) d\mathbf{q}\, d\hat{\Omega}(v) \,, \tag{2.13}$$

which is obviously defined in accordance with (4.2.20).

The typical Dirac fibre $\mathbf{F}$ can be embedded most directly into the Hilbert superspace $\mathbf{H}$ by choosing in the representation (2.10) of the supermatrix elements the values

$$c_0(q,v,s) = f(q,v,s) \,, \quad c_{\alpha_1\ldots\beta_n}(q,v,s) \equiv 0 \,, \quad f \in \mathbf{F} \,, \quad s = 1,2,3,4 \tag{2.14}$$

Alternatively, one can assign to all elements in $\mathbf{F}$ and in its complex conjugate $\overline{\mathbf{F}}$ (cf. Sec. 4.5) the following respective Λ^1-valued bispinor wave functions

$$f(\zeta) = \theta^\alpha \mathrm{w}_\alpha(\zeta) \,, \quad \theta^\alpha = \langle \mathrm{w}_\alpha | f \rangle \vartheta_\alpha \,, \tag{2.15a}$$

$$\bar{f}(\zeta) = \overline{\theta}^\alpha \overline{\mathrm{w}}_\alpha(\zeta) \,, \quad \overline{\mathrm{w}}_\alpha(\zeta,s) = \mathrm{w}^*_\alpha(\zeta,s), \tag{2.15b}$$

in which the Einstein summation convention is used. The identification mappings

$$f \mapsto f \in \mathbf{H}, \qquad f^\dagger \mapsto \bar{f} \in \mathbf{H}, \qquad f \in \mathbf{F} \cong \mathbf{F}^\dagger \,, \tag{2.16}$$

which result from (2.15), give rise to the *standard Dirac superfibre* $\mathbf{B}$ and its conjugate superfibre $\overline{\mathbf{B}}$,

$$\mathbf{B} = \{ f |\ f \in \mathbf{F} \} \subset \mathbf{H} \,, \qquad \overline{\mathbf{B}} = \{ \bar{f} |\ f \in \mathbf{F} \} \subset \mathbf{H} \,, \tag{2.17}$$

as linear subsets within the superspace $\mathbf{H}$ defined in (2.12). The fact that $\mathbf{B}$ and $\overline{\mathbf{B}}$ belong to the same superspace implies that the values assumed by supernumber-valued functions representing their respective elements anticommute – a feature that is essential to the later defined notion of Berezin integration.

Since as a Hilbert space the typical fibre $\mathbf{F}$ can be identified with its dual $\mathbf{F}^\dagger$ by means of Riezs' theorem, it might appear simpler to carry out these identifications for $\mathbf{F}^\dagger$ rather

than for $\overline{\mathbf{F}}$. However, Riezs' theorem[7] cannot be applied directly to the standard Dirac superfibre $\mathbf{B}$ since, according to (2.13), the inner product is given within that superfibre by

$$\langle f|f'\rangle = \overline{\theta}^{\alpha}\theta'^{\beta}\langle \mathrm{w}_{\alpha}|\mathrm{w}_{\beta}\rangle = \sum_{\alpha=1}^{\infty}\overline{\theta}^{\alpha}\theta'^{\alpha} , \qquad f,f'\in\mathbf{B}, \tag{2.18}$$

so that it is Λ^0-valued, but it is not complex-valued. Hence, such an identification can proceed only indirectly, namely via the identification of $\mathbf{F}^{\dagger}$ with $\mathbf{F}$.

We can now construct from $\mathbf{H}$ in (2.12) the Hilbert superspace

$$\mathcal{H} = \bigoplus_{n=0}^{\infty}\mathcal{H}_n , \qquad \mathcal{H}_n = \mathbf{H}\otimes\cdots\otimes\mathbf{H} , \tag{2.19}$$

into which the following standard superfibre

$$\mathcal{B} = \bigoplus_{n=0}^{\infty}\mathcal{B}_n \subset \mathcal{H} , \qquad \mathcal{B}_n = \mathbf{B}\underset{S}{\otimes}\cdots\underset{S}{\otimes}\mathbf{B} , \tag{2.20a}$$

can be embedded in accordance with (2.16), together with its conjugate

$$\overline{\mathcal{B}} = \bigoplus_{n=0}^{\infty}\overline{\mathcal{B}}_n \subset \mathcal{H} , \qquad \overline{\mathcal{B}}_n = \overline{\mathbf{B}}\underset{S}{\otimes}\cdots\underset{S}{\otimes}\overline{\mathbf{B}} \tag{2.20b}$$

The following counterparts of the operator-valued functionals in (2.3),

$$\psi^{(+)}(f) = \psi^{(+)}(\mathrm{w}_{\alpha})\theta^{\alpha} , \qquad f\in\mathbf{B} , \tag{2.21a}$$

$$\psi^{(-)}(\bar{f}) = \overline{\theta}^{\alpha}\psi^{(-)}(\mathrm{w}_{\alpha}) , \qquad \bar{f}\in\overline{\mathbf{B}} , \tag{2.21b}$$

are superoperator-valued functionals on the standard Dirac superfibres $\mathbf{B}$ and $\overline{\mathbf{B}}$, respectively, which leave the *standard Berezin-Dirac superfibre* $\mathcal{B}\otimes\overline{\mathcal{B}}$ invariant. It is in terms of these superoperator-valued functionals that we can define, by formal analogy with (5.3.8), the *Berezin-Dirac quantum superframes*

$$\Phi_f = \exp\left(-\tfrac{1}{2}\langle f|f\rangle\right)\exp\left(\psi^{(+)}(f)\right)\Psi_0 , \qquad f\in\mathbf{B} \tag{2.22}$$

The correspondence between state vectors in the typical Fock-Dirac fibre $\mathcal{F}$ and state vectors in the standard Berezin-Dirac superfibre $\mathcal{B}\otimes\overline{\mathcal{B}}$ follows from the mappings

$$\mathrm{w}_{\alpha_1}\underset{A}{\otimes}\cdots\underset{A}{\otimes}\mathrm{w}_{\alpha_n} \leftrightarrow \vartheta_{\alpha_1}\mathrm{w}_{\alpha_1}\underset{S}{\otimes}\cdots\underset{S}{\otimes}\vartheta_{\alpha_n}\mathrm{w}_{\alpha_n}\in\mathcal{B}_n , \qquad \alpha_1,\ldots,\alpha_n = 1,2,\ldots , \tag{2.23}$$

and their conjugates, upon extending the domains of these mappings by linearity and antilinearity, respectively, from $\mathbf{F}$ to $\mathcal{F}$ in accordance with (1.1) and (2.20). These identifications are preserved by the action of fermion and antifermion creation and annihilation operators in (2.5) if we extend that action to the vectors of the superspace $\mathcal{H}$ so that they obey the following anticommutation relations:

---

[7] Cf. Sec. 2.2 of Chapter III in (Prugovečki, 1981). In physics literature the notion of inner product is often based on that of duality. However, note should be taken of the fact that the dual $\mathcal{H}^*$ of a Hilbert space $\mathcal{H}$ is a *topological* rather than an algebraic dual, so that it consists of only those linear functionals which are continuous. Hence, the presence of an inner product is required *prior* to defining duals of Hilbert spaces, as it is only by Riesz's theorem that each element $\langle f|$ of that dual can be identified with a vector $|f\rangle$ in the original space.

$$\{\vartheta_\alpha, \psi^{(\pm)}(\mathrm{w}_\beta)\}\Psi = \{\overline{\vartheta}_\alpha, \psi^{(\pm)}(\mathrm{w}_\beta)\}\Psi = \mathbf{0} \ , \quad \Psi \in \mathcal{H} \ . \tag{2.24}$$

Due to (2.5) and (2.24), this implies that in $\mathcal{H}$we shall have:

$$[\psi^{(-)}(\mathrm{w}_\alpha), \vartheta_\beta \psi^{(+)}(\mathrm{w}_\beta)] = \delta_{\alpha\beta}\vartheta_\beta \ , \qquad \alpha,\beta = 1,2,\ldots \ . \tag{2.25}$$

On account of (2.18) and (2.21), we obtain that

$$\Phi_f = \exp\Big(-\tfrac{1}{2}\sum_{\alpha=1}^{\infty} \overline{\theta}^\alpha \theta^\alpha\Big) \exp\Big(\psi^{(+)}(\mathrm{w}_\alpha)\theta^\alpha\Big)\Psi_0 \ . \tag{2.26}$$

By expanding the above exponentials into power series, we see from (2.24) that

$$\psi^{(-)}(\mathrm{w}_\alpha)\Phi_f = \theta^\alpha \Phi_f \ , \qquad \theta^\alpha = \langle \mathrm{w}_\alpha | f \rangle \vartheta_\alpha \ , \qquad f \in \mathbf{F} \ , \tag{2.27}$$

so that, by using (2.15) and (2.21), we arrive at the following counterpart of (5.3.13):

$$\psi^{(-)}(\pm\zeta, s)\Phi_f = f^{(\pm)}(\zeta, s)\ \Phi_f \ , \quad f^{(\pm)} \leftrightarrow \mathbf{P}^{(\pm)} f \ , \quad f \in \mathbf{B} \tag{2.28}$$

The first of the two main goals motivating the introduction of Berezin-type coherent states has been thus achieved. The second of those goals will be realized in the next section, where we shall show that corresponding Berezin-Dirac quantum frames supply continuous resolutions of the identity within each fibre of the Fock-Dirac quantum bundle $\mathcal{E}$.

6.3. Quantum Berezin-Dirac Superfibre Bundles

The action of the unitary representation (1.2) of ISL(2,**C**) within the typical Dirac fibre **F** can be viewed as action from the right (cf. Sec. 2.3) of the elements of that representation upon the elements of the orthonormal basis $\{\mathrm{w}_1, \mathrm{w}_2, \ldots\}$ introduced in (2.1):

$$U(a,A) : \ \mathrm{w}_\alpha \mapsto \mathrm{w}_\beta U_\alpha^\beta(a,A) \ , \qquad U_\alpha^\beta(a,A) := \langle \mathrm{w}_\beta | U(a,A)\mathrm{w}_\alpha \rangle \ . \tag{3.1}$$

On account of the identifications in (2.16), this gives rise to action from the left within the standard Dirac superfibres **B** and $\overline{\mathbf{B}}$, represented by the following transformations:

$$\mathrm{U}(a,A) \ : \ f \mapsto f' = \theta'^\alpha \mathrm{w}_\alpha \ , \qquad \theta'^\alpha = U_\beta^\alpha(a,A)\theta^\beta , \tag{3.2a}$$

$$\overline{\mathrm{U}}(a,A) \ : \ \bar{f} \mapsto \bar{f}' = \overline{\theta}'^\alpha \overline{\mathrm{w}}_\alpha \ , \qquad \overline{\theta}'^\alpha = \overline{U}_\beta^\alpha(a,A)\overline{\theta}^\beta \tag{3.2b}$$

In turn, the above transformations give rise to corresponding transformations

$$\mathbf{U}(a,A) = \bigoplus_{m,n=0}^{\infty} \left(\mathrm{U}(a,A)^{\otimes m} \otimes \overline{\mathrm{U}}(a,A)^{\otimes n}\right) \tag{3.3}$$

within the standard Berezin-Dirac superfibre $\mathcal{B} \otimes \overline{\mathcal{B}}$, which provide within that superfibre a representation of ISL(2,**C**) by means of superunitary[8] operators. Consequently, we can

[8] As in the case of unitary operators, superunitary operators preserve the supernumber-valued inner product on Hilbert superspaces. They can represent elements of Lie supergroups (cf., e.g., Bruzzo and Cianci, 1984), but in the present context they are homomorphically assigned to the elements of the Lie group ISL(2,**C**).

construct the *Berezin-Dirac superfibre bundle*

$$\mathcal{A} = D\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}}) \times_{\mathbf{G}} \left(\mathcal{B} \otimes \overline{\mathcal{B}}\right) , \qquad \mathbf{G} = \mathrm{ISL}(2,\mathbf{C}) , \tag{3.4}$$

by employing the standard Berezin-Dirac superfibre $\mathcal{B} \otimes \overline{\mathcal{B}}$ in the course of construction of associated bundles by the **G**-bundle method (cf. Sec. 4.1), and considering in the process the elements of the orthonormal basis $\{\mathrm{w}_1, \mathrm{w}_2, ...\}$ as attached to the fundamental frame

$$\mathrm{U}_0 = \left\{ \boldsymbol{\Phi}^{(+)u_0}_{m;\zeta,s} \oplus \boldsymbol{\Phi}^{(-)u_0}_{m;\zeta,s} \,\middle|\, q \in \mathbf{R}^4,\ v \in \boldsymbol{V}^+,\ s \in \{1,2,3,4\} \right\} \tag{3.5}$$

within the typical Dirac fibre **F**. As a consequence, the ensuing soldering map assigns to the superfibre element

$$\boldsymbol{f} = \theta^{\alpha}_{x} \mathbf{w}^{\mathrm{U}(x)}_{\alpha} \in \mathbf{B}_x \subset \mathcal{B}_x , \qquad \mathbf{w}^{\mathrm{U}(x)}_{\alpha} \in \mathbf{F}_x , \tag{3.6}$$

a superfibre coordinate in which the fibre coordinates w_α are frame independent:

$$\boldsymbol{\sigma}^{\mathrm{U}}_{x} : \ \boldsymbol{f} \ \mapsto \ f = \theta^{\alpha}_{x} \mathrm{w}_{\alpha} \in \mathbf{B} , \qquad \theta^{\alpha}_{x} = \left\langle \mathbf{w}^{\mathrm{U}(x)}_{\alpha} \middle| \mathbf{f} \right\rangle \vartheta_{\alpha} = \left\langle \mathrm{w}_{\alpha} \middle| f \right\rangle \vartheta_{\alpha} . \tag{3.7}$$

In the sequel, by a *Dirac gauge* we shall mean a choice of cross-section **s** of the Dirac quantum frame bundle $D\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$. Clearly, a choice **s** of Dirac gauge is equivalent to a choice of cross-section of the affine spin-frame bundle $S\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ mentioned in Secs. 4.2 and 6.1. The transition from a Dirac gauge **s** to another gauge **s'**,

$$\mathbf{s}(x) \ \mapsto \ \mathbf{s}'(x) = \mathbf{s}(x) \cdot (a_x, A_x) , \qquad x \in \mathbf{M} , \tag{3.8}$$

will give rise to the following transformations

$$\boldsymbol{\sigma}^{\mathbf{s}'}_{x} \circ (\boldsymbol{\sigma}^{\mathbf{s}}_{x})^{-1} : \ f^{\mathbf{s}}_{x} \ \mapsto \ f^{\mathbf{s}'}_{x} = \theta'^{\alpha}_{x} \mathrm{w}_{\alpha} , \qquad \theta'^{\alpha}_{x} = U^{\alpha}_{\beta}(a_x, A_x) \theta^{\beta}_{x} , \tag{3.9}$$

of the superfibre coordinates in (3.7). The above Dirac gauge transformations completely determine the corresponding transformations undergone by the *fermion superframe field* creation and annihilation operators

$$\psi^{(+)}_{\mathbf{s}}(x;\boldsymbol{f}) = (\boldsymbol{\sigma}^{\mathbf{s}}_{x})^{-1} \psi^{(+)}(f) \boldsymbol{\sigma}^{\mathbf{s}}_{x} , \qquad \boldsymbol{f} \in \mathbf{B}_x , \tag{3.10a}$$

$$\psi^{(-)}_{\mathbf{s}}(x;\bar{\boldsymbol{f}}) = (\boldsymbol{\sigma}^{\mathbf{s}}_{x})^{-1} \psi^{(-)}(\bar{f}) \boldsymbol{\sigma}^{\mathbf{s}}_{x} , \qquad \bar{\boldsymbol{f}} \in \overline{\mathbf{B}}_x , \tag{3.10b}$$

under such changes of Dirac gauge, due to the fact that, according to (2.21) and (3.7),

$$\boldsymbol{\sigma}^{\mathbf{s}}_{x} \psi^{(+)}_{\mathbf{s}}(x;\boldsymbol{f}) (\boldsymbol{\sigma}^{\mathbf{s}}_{x})^{-1} = \psi^{(+)}(\mathrm{w}_{\alpha}) \theta^{\alpha}_{x} , \quad \boldsymbol{\sigma}^{\mathbf{s}}_{x} \psi^{(-)}_{\mathbf{s}}(x;\bar{\boldsymbol{f}}) (\boldsymbol{\sigma}^{\mathbf{s}}_{x})^{-1} = \bar{\theta}^{\alpha}_{x} \psi^{(-)}(\mathrm{w}_{\alpha}) . \tag{3.11}$$

Moreover, since these operators can be used to define, by formal analogy with (5.3.8), the *fermion quantum superframes*

$$\left| \boldsymbol{\Phi}^{\mathbf{s}}_{f} \right\rangle = \exp\left(-\tfrac{1}{2}\langle \boldsymbol{f} | \boldsymbol{f} \rangle\right) \exp\left(\psi^{(+)}_{\mathbf{s}}(x;\boldsymbol{f})\right) \left| \boldsymbol{\Psi}_{0,x} \right\rangle , \qquad \boldsymbol{f} \in \mathbf{B}_x , \tag{3.12a}$$

$$\left\langle \boldsymbol{\Phi}^{\mathbf{s}}_{f} \right| = \left\langle \boldsymbol{\Psi}_{0,x} \right| \exp\left(\psi^{(-)}_{\mathbf{s}}(x;\bar{\boldsymbol{f}})\right) \exp\left(-\tfrac{1}{2}\langle \boldsymbol{f} | \boldsymbol{f} \rangle\right) , \qquad \bar{\boldsymbol{f}} \in \overline{\mathbf{B}}_x , \tag{3.12b}$$

the transformation properties of these superframes are thereby also determined under any change of Dirac gauge.

We note that (3.12b) does not represent an element of $\overline{\mathcal{B}}_x$, so that the two definitions in (3.12) have to be actually interpreted as representing the operator

$$\left|\Phi_f^{\mathbf{s}}\right\rangle\left\langle\Phi_f^{\mathbf{s}}\right| = \exp(-\langle f|f\rangle)\exp\left(\psi_{\mathbf{s}}^{(+)}(x;f)\right)\left|\Psi_{0,x}\right\rangle\left\langle\Psi_{0,x}\right|\exp\left(\psi_{\mathbf{s}}^{(-)}(x;\bar{f})\right), \quad (3.13)$$

which acts within Berezin-Dirac superfibre $\mathcal{B}_x \otimes \overline{\mathcal{B}}_x$. In order to cope with the functional integration involving this type of operator, Berezin (1966) introduced the "integrals"

$$\int d\vartheta_\alpha = \int d\bar{\vartheta}_\alpha = 0 \ , \qquad \int d\vartheta_\alpha\, \vartheta_\alpha = \int d\bar{\vartheta}_\alpha\, \bar{\vartheta}_\alpha = 1 \ , \qquad \alpha = 1,2,3,\ldots \ , \quad (3.14)$$

together with the following formal anticommutation rules:

$$\{d\vartheta_\alpha, d\vartheta_\beta\} = \{\vartheta_\alpha, d\vartheta_\beta\} = \{\bar{\vartheta}_\alpha, d\vartheta_\beta\} = 0 \ , \qquad \alpha,\beta = 1,2,3,\ldots \quad (3.15)$$

Upon extending these rules by linearity, iteration and involution, one arrives at a concept of Berezin "integral," which is actually a multilinear functional on spaces of functions

$$F(\theta,\bar{\theta}) \ , \quad \theta = \left\{\theta_\alpha = a_\alpha^\beta\, \vartheta_\beta \middle|\ \alpha = 1,\ldots,n\right\} , \quad (3.16)$$

of any finite number of generators of the the Grassmann algebra Λ. One of Berezin's (1966) key results was that, in all those cases where the matrix determined by the above complex coefficients is invertible, so that

$$d\theta_\alpha = b_\alpha^\beta\, d\vartheta_\beta \ , \qquad d\bar{\theta}_\alpha = \bar{b}_\alpha^\beta\, d\bar{\vartheta}_\beta \ , \qquad a_\alpha^\gamma b_\gamma^\beta = \delta_\alpha^\beta \ , \quad (3.17)$$

the following formula holds (cf. Berezin, 1966, p. 55):

$$\int F(\theta,\bar{\theta})\, d\bar{\theta} d\theta = \det\left\|b_\alpha^\beta\right\| \det\left\|\bar{b}_\alpha^\beta\right\| \int F(\theta(\vartheta),\bar{\theta}(\bar{\vartheta}))\, d\bar{\vartheta} d\vartheta \ , \quad (3.18a)$$

$$d\bar{\theta} d\theta := d\bar{\theta}_n d\theta_n \cdots d\bar{\theta}_1 d\theta_1 \ , \qquad d\bar{\vartheta} d\vartheta := d\bar{\vartheta}_n d\vartheta_n \cdots d\bar{\vartheta}_1 d\vartheta_1 \ . \quad (3.18b)$$

This ensures that Berezin integrals are, in general, independent of the choice of basis that is implicit in their definition. In the present context, this mathematical fact ensures that the outcome of functional integrations involving the Berezin-Dirac superframes in (3.12) are independent of the choice of orthonormal basis $\{w_1, w_2, \ldots\}$ in the typical Dirac fibre $\mathbf{F}$.

The central result of Berezin's treatment of functional integration for fermion fields is a theorem which provides a counterpart of the type of continuous resolution of the identity supplied by Glauber coherent states in the case of boson fields. When it is applied to the local state vectors in the fibres of the Fock-Dirac bundle, this theorem yields[9]

$$\langle\Psi|\,\Psi'\rangle = \int_{\overline{\mathbf{B}}_x \times \mathbf{B}_x} \left\langle\Psi\middle|\,\Phi_f^{\mathbf{s}}\right\rangle d\bar{f}\, df \left\langle\Phi_f^{\mathbf{s}}\middle|\,\Psi'\right\rangle , \qquad \Psi, \Psi' \in \mathcal{F}_x \ , \quad (3.19)$$

[9] Cf. (Berezin, 1966, p. 33), but note that according to the convention adopted for Hilbert space inner products on p. 1, such an inner product is linear in the first and antilinear in the second variable. Hence, that convention in (Berezin, 1966) is opposite from the one adopted in physics literature as well as in the present monograph.

with the functional "integral" being defined by the following limit:

$$\int_{\bar{\mathbf{B}}_x \times \mathbf{B}_x} \left\langle \Psi \middle| \Phi_f^{\mathbf{s}} \right\rangle d\bar{f}\, df \left\langle \Phi_f^{\mathbf{s}} \middle| \Psi' \right\rangle = \lim_{n\to\infty} \int \left\langle \Psi \middle| \Phi_f^{\mathbf{s}} \right\rangle d\bar{f}_n\, df_n \left\langle \Phi_f^{\mathbf{s}} \middle| \Psi' \right\rangle , \quad (3.20a)$$

$$f_n = \sum_{\alpha=1}^{n} \theta_x^{\alpha} \mathbf{w}_{\alpha}^{\mathbf{s}(x)}, \quad \bar{f}_n = \sum_{\alpha=1}^{n} \bar{\theta}_x^{\alpha} \bar{\mathbf{w}}_{\alpha}^{\mathbf{s}(x)}, \quad d\bar{f}_n\, df_n = \prod_{\alpha=n}^{1} d\bar{\theta}_x^{\alpha}\, d\theta_x^{\alpha} . \quad (3.20b)$$

In turn, when these definitions and results are combined with (2.24) and (2.25), the values of the functional integrals on the right-hand side of (3.20a) follow unambiguously from (3.7) and (3.14)-(3.18).

The relations (2.24) and (2.25) also imply that, as was the case in (2.28), we have

$$\psi_{\mathbf{s}}^{(-)}(x;\pm\zeta,s) \left| \Phi_f^{\mathbf{s}} \right\rangle = f^{(\pm)}(\zeta,s) \left| \Phi_f^{\mathbf{s}} \right\rangle , \qquad f \in \mathbf{B}_x , \quad (3.21)$$

where, in the adopted Dirac gauge $\mathbf{s}$, the above annihilation operators correspond to the Dirac quantum frame field in (1.27):

$$\psi_{\mathbf{s}}^{(\pm)}(x;\zeta,s) = (\sigma_x^{\mathbf{s}})^{-1} \psi^{(\pm)}(\zeta,s) \sigma_x^{\mathbf{s}} , \qquad v \in \mathbf{V} . \quad (3.22)$$

Moreover, in the course of the computation of the Berezin functional integral in (3.19), these same relations imply that

$$\left\langle \Phi_f^{\mathbf{s}} \right| \psi_{\mathbf{s}}^{(+)}(x;\pm\zeta,s) = \left\langle \Phi_f^{\mathbf{s}} \middle| \bar{f}^{(\pm)}(\zeta,s) \right. , \qquad \bar{f} \in \bar{\mathbf{B}}_x . \quad (3.23)$$

Hence, the formal analogies with the corresponding functional integrals for boson frame fields, treated in Sec. 5.3, extend as far as it is necessary to enable in the next section the formulation of propagators for parallel transport in Fock-Dirac bundles by following the same general lines as in the case the Fock bundles treated in Chapter 5.

6.4. Parallel Transport in Fock-Dirac Bundles

The treatment of connections and parallel transport in the Fock-Dirac bundle $\mathcal{E}$ can be carried out (Prugovečki, 1987; Prugovečki and Warlow, 1989) in the same general manner as in the spin-zero case treated in 5.4, with all the major differences being of a technical nature. At the geometric level, these similarities result from the fact that the operator

$$\tau_\gamma(x'',x') \; : \; \mathcal{F}_{x'} = \bigoplus_{n,n'=0}^{\infty} \mathcal{F}_{n,n';x'} \quad \rightarrow \quad \mathcal{F}_{x''} = \bigoplus_{n,n'=0}^{\infty} \mathcal{F}_{n,n';x''} \quad (4.1)$$

for parallel transport between the Fock-Dirac fibres above the end points x' and x'' of a given piecewise smooth curve γ within the base manifold $\mathbf{M}$ is completely determined by the parallel transport within the principal Dirac bundle $D\mathbf{M}_0(g^{\mathrm{L}})$, or, equivalently, within the principal bundle $S\mathbf{M}_0(g^{\mathrm{L}})$ of affine spin-frames. Consequently, the covariant derivatives within the Fock-Dirac bundle $\mathcal{E}$ can be defined as in (4.4.4),

$$\nabla_X \Psi_x = \mathrm{s}\text{-}\lim_{t\to 0} \frac{1}{t} \left[\tau_\gamma(x,x(t)) \Psi_{x(t)} - \Psi_x \right] , \quad \left\{ \Psi_x \in \mathcal{F}_x \middle| \; x \in \mathbf{M} \right\} \subset \mathcal{E} , \quad (4.2)$$

for any section of the Fock-Dirac bundle $\mathcal{E}$, and in any Dirac gauge.

If such a Dirac gauge is specified as a cross-section $\mathbf{s}$ of $D\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ with elements

$$\mathbf{s}(x) = \left\{ \boldsymbol{\Phi}^{\mathbf{s}(x)}_{m;\zeta,s} = \boldsymbol{\Phi}^{(+)\mathbf{s}(x)}_{m;\zeta,s} \oplus \boldsymbol{\Phi}^{(-)\mathbf{s}(x)}_{m;\zeta,s} \,\middle|\, q \in \mathbf{R}^4,\ v \in \boldsymbol{V}^+,\ s = 1,2,3,4 \right\},\ x \in \mathbf{M}, \quad (4.3)$$

then the gauge transformations (3.8) give rise, on account of the corresponding changes of spinorial quantum frames (1.16), to the unitary operators

$$\boldsymbol{U}_{\mathbf{s}(x)}(a,A) = \bigoplus_{n,n'=0}^{\infty} \left(U^{(+)}_{\mathbf{s}(x)}(a,A)^{\otimes n} \otimes U^{(-)}_{\mathbf{s}(x)}(a,A)^{\otimes n'} \right). \quad (4.4)$$

These operators act within the fibres $\mathcal{F}_x$ of $\mathcal{E}$ in accordance with (1.2), thus giving rise to unitary representations of ISL(2,**C**). The covariant derivative in (4.2) can be expressed in terms of the infinitesimal generators of these representations in the form

$$\nabla_{\boldsymbol{X}} \boldsymbol{\Psi}_x = \left(\partial_{\boldsymbol{X}} - i\tilde{\boldsymbol{\theta}}^i_{\mathbf{s}}(\boldsymbol{X}) \boldsymbol{P}_{i;\mathbf{s}(x)} + \tfrac{i}{2} \tilde{\boldsymbol{\omega}}^{\mathbf{s}}_{jk}(\boldsymbol{X}) \boldsymbol{M}^{jk}_{\mathbf{s}(x)} \right) \boldsymbol{\Psi}_x \,, \quad (4.5)$$

where, on account of (1.7), the connection coefficients are those that correspond to the section $\boldsymbol{s}$ of $P\mathbf{M}_0(\boldsymbol{g}^{\mathrm{L}})$ associated with the Dirac gauge $\mathbf{s}$. The various mathematical technicalities with the domain of definition of this covariant derivative, which arise from the fact that these infinitesimal generators are unbounded operators with the domains that are only dense in $\mathcal{F}_x$, can be handled in the same manner as in Sec. 4.4.

To illustrate this fact, let us first consider the quantum connection in the quantum subbundle $\mathcal{E}_{1,1} = \mathcal{E}_{1,0} \oplus \mathcal{E}_{0,1}$, consisting of single fermion-antifermion local state vectors, so that in (4.2) and (4.5) we have

$$\boldsymbol{\Psi}_x = \overline{\Psi}^{\zeta,s}_x \boldsymbol{\Phi}^{\mathbf{s}(x)}_{m;\zeta,s} \,, \qquad \overline{\Psi}^{\zeta,s}_{(\pm)x} = \pm \sum\nolimits_{s=1}^{4} \gamma^0_{ss'} \left\langle \boldsymbol{\Phi}^{(\pm)\mathbf{s}(x)}_{m;\zeta,s'} \,\middle|\, \boldsymbol{\Psi}_x \right\rangle, \quad (4.6)$$

where, in accordance with (1.18) and the notational convention (4.2.11),

$$\mathbf{P}^{(\pm)}_x \boldsymbol{\Psi}_x = \overline{\Psi}^{\zeta,s}_{(\pm)x} \boldsymbol{\Phi}^{(\pm)\mathbf{s}(x)}_{m;\zeta,s} := \int \overline{\Psi}^{\zeta,s}_{(\pm)x} \boldsymbol{\Phi}^{(\pm)\mathbf{s}(x)}_{m;\zeta,s} d\Sigma(\zeta) \,. \quad (4.7)$$

We can then establish, in the same manner as in (4.4.8a), that

$$\partial_{\boldsymbol{X}} \boldsymbol{\Psi}_x = [X^\mu \partial_\mu (\overline{\Psi}^{\zeta,s}_x)] \boldsymbol{\Phi}^{\mathbf{s}(x)}_{m;\zeta,s} \,. \quad (4.8)$$

However, while the P and the Q operators in (4.4.10) remain the same as in Sec. 4.4, the infinitesimal generators in (4.4.12) have to be replaced by the operators (cf. Prugovečki and Warlow, 1989b)

$$M^{jk}_{\mathbf{s}(x)} = Q^j_{\mathbf{s}(x)} P^k_{\mathbf{s}(x)} - Q^k_{\mathbf{s}(x)} P^j_{\mathbf{s}(x)} + S^{jk} \,, \qquad S^{jk} = \tfrac{i}{4}[\gamma^j, \gamma^k] \,, \quad (4.9)$$

which represent the restriction of the corresponding infinitesimal generators of the representation in (4.4) to the fermion-antifermion subfibre $\mathcal{F}_{1,1;x}$

All the infinitesimal generators of the unitary representation in (4.4) can be related to an operator-valued stress-energy tensor acting within the Fock-Dirac fibre $\mathcal{F}_x$. To show this, we first note that the infinitesimal generators of spacetime translations are given by

$$\boldsymbol{P}_{j;\mathbf{s}} = i\int\left(\overline{\psi}_{\mathbf{s}}^{(+)}(x;\zeta)\,\partial_j\psi_{\mathbf{s}}^{(-)}(x;\zeta) - \overline{\psi}_{\mathbf{s}}^{(+)}(x;-\zeta)\,\partial_j\psi_{\mathbf{s}}^{(-)}(x;-\zeta)\right)d\Sigma(\zeta). \tag{4.10}$$

This can be verified by taking matrix elements of the above operators in between any two states with n fermions and n' antifermions, and using (1.20), (1.22), (1.24) and (1.25) in conjunction with the expression (4.2.19) for the inner product in Dirac fibres.

Upon making the transition to the manifestly covariant form (4.2.20) of the same inner product, we can recast (4.10) into the form

$$\boldsymbol{P}_{j;\mathbf{s}} = i\int_V d\hat{\Omega}(v)\int d\sigma_k(q)\,\overline{\psi}_{\mathbf{s}}^{(+)}(x;\zeta)\,\gamma^k\vec{\partial}_j\psi_{\mathbf{s}}^{(-)}(x;\zeta), \qquad \vec{\partial}_j = \vec{\partial}/\partial q^j. \tag{4.11}$$

By making the transition to the momentum representation it can be established that

$$\int_V d\hat{\Omega}(v)\int d\sigma_k(q)\,\overline{\psi}_{\mathbf{s}}^{(\pm)}(x;\zeta)\,\gamma^k\vec{\partial}_j\psi_{\mathbf{s}}^{(\pm)}(x;\zeta) = 0\,, \tag{4.12}$$

as was the case in (5.4.10). Hence, (4.11) can be expressed in the form

$$\boldsymbol{P}_{j;\mathbf{s}} = \int_{V^+} d\Omega(v)\int d\sigma^k(q)\,{:}T_{jk}[\psi_{\mathbf{s}}(x;\zeta)]{:}\,, \tag{4.13}$$

which incorporates the following normally ordered and renormalized stress-energy tensor:

$$T_{jk}[\psi_{\mathbf{s}}] = i\hat{Z}_{\ell m}\left(\overline{\psi}_{\mathbf{s}}\gamma_k\psi_{\mathbf{s},j} - \overline{\psi}_{\mathbf{s},j}\gamma_k\psi_{\mathbf{s}}\right), \tag{4.14}$$

$$\psi_{\mathbf{s}}(x;\zeta) = \psi_{\mathbf{s}}^{(-)}(x;\zeta) + \psi_{\mathbf{s}}^{(+)}(x;-\zeta)\,, \qquad \psi_{\mathbf{s},j} = \partial\psi_{\mathbf{s}}/\partial q^j. \tag{4.15}$$

We can treat in a similar manner the infinitesimal generators of Lorentz transformations which appear in (4.5), and which are to be therefore taken in relation to the axes of the local Poincaré frame $\boldsymbol{s}(x)$ corresponding to a chosen Dirac gauge $\mathbf{s}$. Thus, we see from (4.9) that in the case of multi-fermion states those generators are given by the operators

$$\boldsymbol{M}_{\mathbf{s}}^{ij} = i\int\overline{\psi}_{\mathbf{s}}^{(+)}(x;\zeta)\left(Q_{\mathbf{s}}^i\partial^j - Q_{\mathbf{s}}^j\partial^i\right)\overline{\psi}_{\mathbf{s}}^{(-)}(x;\zeta)\,d\Sigma(\zeta) + \boldsymbol{S}_{\mathbf{s}}^{ij}\,, \tag{4.16}$$

$$\boldsymbol{S}_{\mathbf{s}}^{ij} = \tfrac{i}{4}\int\left(\overline{\psi}_{\mathbf{s}}^{(+)}(x;\zeta)[\gamma^i,\gamma^j]\,\psi_{\mathbf{s}}^{(-)}(x;\zeta)\right)d\Sigma(\zeta)\,, \tag{4.17}$$

with a corresponding expression holding for multi-antifermion states. By combining these two types of expressions as in the transition from (4.10) to (4.11), we arrive at the following manifestly covariant expression for these generators:

$$\boldsymbol{M}_{\mathbf{s}}^{ij} = \int_{V^+} d\Omega(v)\int d\sigma_k(q)\,{:}M_{\mathbf{s}}^{kij}[\psi_{\mathbf{s}}(x;\zeta)]{:}\,. \tag{4.18}$$

The above form of the infinitesimal generators of Lorentz transformations incorporates the following normally ordered and renormalized angular momentum operator-valued densities:

$$M_{\mathbf{s}}^{kij}[\psi_{\mathbf{s}}] = i\hat{Z}_{\ell m}\left(\tfrac{1}{2}[\overline{\psi}_{\mathbf{s}}(\overleftrightarrow{Q_{\mathbf{s}}^i\partial^j} - \overleftrightarrow{Q_{\mathbf{s}}^j\partial^i})\gamma^k\psi_{\mathbf{s}}] + \tfrac{1}{8}\overline{\psi}_{\mathbf{s}}\{[\gamma^i,\gamma^j],\gamma^k\}\psi_{\mathbf{s}}\right). \tag{4.19}$$

As was the case with the quantum-geometric boson fields treated in the previous chapter, the stress-energy tensor in (4.14) and the angular momentum tensor in (4.19) are

well-defined operator-valued functions – as opposed to their counterparts in conventional QFT, which are singular operator-valued distributions with no well-defined meaning at individual points $x \in \mathbf{M}$. This is essential for the formulation[10] in Sec. 6.6 of a notion of quantum-geometric propagation of fermion fields which is geometrically local, since it is governed at each base location $x \in \mathbf{M}$ by the energy-density operator at that point, rather than by a global Hamiltonian defined only over entire reference hypersurfaces.

6.5. Geometrically Localized Fermion Field Supermodes

The formulation of the parallel transport in the Fock-Dirac bundle $\mathcal{F}$ has led us to the QFT fermionic connection

$$\boldsymbol{\nabla} = \partial - i\tilde{\theta}^i_s \boldsymbol{P}_{i;s} + \tfrac{i}{2}\tilde{\omega}^s_{jk}\boldsymbol{M}^{jk}_s \ , \tag{5.1}$$

whose generators for infinitesimal spacetime translations give rise to the stress-energy tensor in (4.14). The extrapolation to fermion fields of the notion of quantum-geometric propagation formulated in Sec. 5.7 for the case of boson fields requires that this propagation be governed by the local fermionic energy-momentum density operators

$$\hat{\mathbf{P}}_{j;s}(x) = n^k(x)\int_{V_x} :T_{jk}[\boldsymbol{\psi}_s(x,v)]: \, d\Omega(v) \ , \qquad j = 0,1,2,3 \ . \tag{5.2}$$

These operators should act at the point $\boldsymbol{O}_x \in T_x\mathbf{M}$ of contact between the tangent space $T_x\mathbf{M}$ and the manifold $\mathbf{M}$ at various base locations $x \in \mathbf{M}$ along the reference hypersurfaces Σ_t that depict the geometrodynamic evolution of $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$. Consequently, as in the boson case in Sec. 5.5, the *quantum-geometric fermion field*

$$\boldsymbol{\psi}_s(x,v) := \psi^{(+)}_s(x;-i\ell v) + \psi^{(-)}_s(x;i\ell v) \ , \qquad x \in \mathbf{M} \ , \quad v \in \boldsymbol{V}^+ \ , \tag{5.3}$$

occurring in them should act in a fermion superfibre bundle $\hat{\mathcal{E}}$ of localized supermodes for which the tensor in (5.2) can be expressed by means of covariant derivatives:

$$T_{jk}[\boldsymbol{\psi}_s] = i\hat{Z}_{\ell m}\left(\overline{\boldsymbol{\psi}}_s \gamma_k \boldsymbol{\psi}_{s;j} - \overline{\boldsymbol{\psi}}_{s;j}\gamma_k \boldsymbol{\psi}_s\right) . \tag{5.4}$$

To construct this bundle, let us introduce the *standard Dirac exciton fibres*

$$\hat{\mathbf{F}}^{(\pm)} = \left\{ \hat{f}^{(\pm)} \middle| \ f^{(\pm)} \in \mathbf{F}^{(\pm)} \right\} \ , \qquad \hat{f}^{(\pm)}(v) := f^{(\pm)}(-i\ell v) \ , \qquad v \in \boldsymbol{V}^+ \tag{5.5}$$

The analyticity of all the wave functions in $\mathbf{F}^{(\pm)}$ ensures that the above two equalities give rise to the following identification between Dirac exciton modes and fermion or antifermion wave functions, respectively:

$$\hat{f}^{(\pm)} \ \leftrightarrow \ f^{(\pm)} \ , \qquad \hat{f}^{(\pm)} \in \hat{\mathbf{F}}^{(\pm)} \ , \qquad f^{(\pm)} \in \mathbf{F}^{(\pm)} . \tag{5.6}$$

These identifications give rise to the inner products in $\mathbf{F}^{(\pm)}$. When they are applied to the

[10] A formulation of propagators for parallel transport in Fock-Dirac bundles, which is formally analogous to the one provided in Sec. 5.5 for boson Fock bundles, can be found in Sec. 8.4 of (Prugovečki, 1992).

coordinate wave functions of the quantum Dirac frame in (1.10), they yield the wave functions of fermionic exciton-mode frames within $\hat{\mathbf{F}} = \hat{\mathbf{F}}^{(+)} \oplus \hat{\mathbf{F}}^{(-)}$:

$$\hat{\Phi}^{(\pm)u}_{mv,s}(v',s') = \Phi^{(\pm)u}_{m;-i\ell v,s}(-i\ell v',s') \ , \qquad \hat{\Phi}^{(\pm)u}_{mv,s} \in \hat{\mathbf{F}}^{(\pm)} , \quad \Phi^{(\pm)u}_{m;-i\ell v,s} \in \mathbf{F}^{(\pm)} \ . \tag{5.7}$$

As opposed to their boson counterparts in (5.5.11), such a Dirac exciton frame does not consist of Perelomov-type generalized coherent states of the Lorentz group $SO_0(3,1)$. Indeed, spin-1/2 Perelomov-type coherent states are dependent on 3-rotation variables, whereas such variables do not appear in (5.7). Nevertheless, such a frame provides a continuous resolution of the identity within the standard Dirac exciton fibres $\hat{\mathbf{F}}^{(\pm)}$:

$$\sum\nolimits_{s=1}^{4} \gamma^0_{ss} \int_{V^+} \left| \hat{\Phi}^{(\pm)u}_{mv,s} \right\rangle d\Omega(v) \left\langle \hat{\Phi}^{(\pm)u}_{mv,s} \right| = \pm Z^{-1}_{\ell,m} \mathbf{1}_{\hat{\mathbf{F}}^{(\pm)}} \ . \tag{5.8}$$

This can be most easily established by performing in $\mathbf{F}^{(\pm)}$ Foldy-Wouthuysen transformations[11] leading to Wigner-type spinorial representations of fermion and antifermion wave functions, then applying (5.5.15) to each spinorial component of those wave functions, and eventually transferring the outcome to $\hat{\mathbf{F}}^{(\pm)}$ by means of the inverse of that type of transformation.

The identification of the inner product in $\hat{\mathbf{F}}^{(\pm)}$ with that in $\mathbf{F}^{(\pm)}$ leads, in conjunction with (5.8), to the conclusion that

$$\left\langle \hat{f}_1 \middle| \hat{f}_2 \right\rangle = Z_{\ell,m} \int_V \hat{f}^*_1(v) \hat{f}_2(v) \, d\Omega(v) = \left\langle f_1 \middle| f_2 \right\rangle , \qquad \hat{f}_1, \hat{f}_2 \in \hat{\mathbf{F}} \ . \tag{5.9}$$

Hence, upon introducing in $\hat{\mathbf{F}}$ the localized modal counterparts of the functions in (1.21),

$$\boldsymbol{S}^{(\pm)}(\pm v;\pm v') = \boldsymbol{S}^{(\pm)}(\mp i\ell v;\mp i\ell v') \ , \qquad v \in \boldsymbol{V} \ , \tag{5.10}$$

we arrive at the following counterparts of (1.17):

$$\boldsymbol{S}^{(\pm)}(v';v'') = -iZ_{\ell,m} \int_V \boldsymbol{S}^{(\pm)}(v';v) \, \boldsymbol{S}^{(\pm)}(v;v'') d\Omega(v) \ . \tag{5.11}$$

Let us now introduce in $\hat{\mathbf{F}}$ the orthonormal basis corresponding to the one in (2.1):

$$\sum\nolimits_{\alpha=1}^{\infty} \left| \hat{\mathbf{w}}_\alpha \right\rangle \left\langle \hat{\mathbf{w}}_\alpha \right| = \mathbf{1}_{\hat{\mathbf{F}}} \ , \qquad \left\langle \hat{\mathbf{w}}_\alpha \middle| \hat{\mathbf{w}}_\beta \right\rangle = \delta_{\alpha\beta} \ , \qquad \alpha,\beta = 1,2,\ldots \ . \tag{5.12}$$

Then we have, as a counterpart of (2.15),

$$\hat{f}(v) = \theta^\alpha \hat{\mathbf{w}}_\alpha(v) \ , \qquad \hat{\bar{f}}(v) = \bar{\theta}^\alpha \hat{\bar{\mathbf{w}}}_\alpha(v) \ , \tag{5.13}$$

where, as was the case for bosonic exciton modes, the convergence of the above two series in (5.13) is not only in the topology of $\hat{\mathbf{F}}$, but is also pointwise and uniform[12] in $v \in \boldsymbol{V}$. This mediates the introduction of the *standard exciton superfibre*

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[11] Cf. Theorems 3.1 and 3.2 in (Prugovečki, 1980) for the formulation of Foldy-Wouthuysen (1950) transformations of wave functions in relativistic phase space.

[12] The absolute value of supernumber-valued wave functions can be defined by means of the norm in (2.7b), as in the integrand of (2.11).

$$\hat{\mathbf{B}} = \left\{\hat{f} \mid f \in \mathbf{B}\right\} , \quad \hat{f}(v) := f(-i\ell v) , \quad v \in \boldsymbol{V} , \tag{5.14}$$

from which the *standard fermion-mode superfibres*

$$\hat{\mathcal{B}} = \bigoplus_{n=0}^{\infty} \hat{\mathcal{B}}_n , \qquad \hat{\overline{\mathcal{B}}} = \bigoplus_{n=0}^{\infty} \hat{\overline{\mathcal{B}}}_n , \tag{5.15}$$

corresponding to the ones in (2.20), can be constructed.

The analyticity of the wave functions in each $\mathcal{B}_n$ enables their unambiguous identification with a wave function in $\hat{\mathcal{B}}_n$, thus giving rise to the isomorphisms

$$\hat{\Psi} \leftrightarrow \Psi , \quad \hat{\overline{\Psi}} \leftrightarrow \overline{\Psi} , \quad \hat{\Psi} \in \hat{\mathcal{B}} , \quad \Psi \in \mathcal{B} . \tag{5.16}$$

Hence, the same general pattern as in Sec. 5.5 can be followed, whereby the restrictions of (3.2) to $\mathrm{SL}(2,\mathbf{C})$ and to the standard exciton superfibre and its conjugate can be used to construct a fermion field-mode superfibre bundle $\hat{\mathcal{A}}$ identifiable with the Berezin-Dirac superfibre bundle $\mathcal{A}$ in (3.4), after the structure group of the latter is restricted to $\mathrm{SL}(2,\mathbf{C})$. Subsequently, $\mathrm{ISL}(2,\mathbf{C})$ gauge invariance can be established in $\hat{\mathcal{A}}$ by setting

$$\hat{\Psi}_{n,n'}(v_1,s_1,\ldots,v_{n'},s_{n'}) = \Psi_{n,n'}(-a(x)-i\ell v_1,s_1,\ldots,-a(x)-i\ell v_{n'},s_{n'}) \tag{5.17}$$

in an arbitrary Dirac gauge $\mathbf{s}$ associated with a Poincaré moving frame $\boldsymbol{s}$, thus arriving in that gauge at the soldering map

$$\hat{\boldsymbol{\sigma}}_x^{\mathbf{s}} : \hat{\Psi} \mapsto \bigoplus_{n,n'=0}^{\infty} \hat{\Psi}_n , \qquad \hat{\Psi} \in \hat{\mathcal{B}}_x = \bigoplus_{n,n'=0}^{\infty} \hat{\mathcal{B}}_{x;n,n'} , \tag{5.18}$$

and its counterpart in the conjugate fibres within $\hat{\mathcal{A}}$.

By applying this identification to (3.12) and (3.13), we arrive at the operators

$$\left|\hat{\boldsymbol{\Phi}}_f^{\mathbf{s}}\right\rangle\left\langle\hat{\boldsymbol{\Phi}}_f^{\mathbf{s}}\right| = \exp\left(-\left\langle\hat{f}\middle|\hat{f}\right\rangle\right) \exp\left(\psi_{\mathbf{s}}^{(+)}(x;\hat{f})\right)\left|\Psi_{0,x}\right\rangle\left\langle\Psi_{0,x}\right| \psi_{\mathbf{s}}^{(-)}(x;\hat{\bar{f}}) , \tag{5.19}$$

$$\psi_{\mathbf{s}}^{(+)}(x;\hat{f}) = \psi_{\mathbf{s}}^{(+)}(x;\hat{\mathbf{w}}_\alpha^{\mathbf{s}})\,\theta_x^\alpha , \qquad \psi_{\mathbf{s}}^{(-)}(x;\hat{\bar{f}}) = \overline{\theta}_x^\alpha \psi_{\mathbf{s}}^{(-)}(x;\hat{\mathbf{w}}_\alpha^{\mathbf{s}}) , \tag{5.20a}$$

$$\hat{f} = \theta_x^\alpha \hat{\mathbf{w}}_\alpha^{\mathbf{s}(x)} \in \hat{\mathcal{B}}_{x;1,1} , \quad \hat{\bar{f}} = \overline{\theta}_x^\alpha \hat{\overline{\mathbf{w}}}_\alpha^{\mathbf{s}(x)} \in \hat{\overline{\mathcal{B}}}_{x;1,1} , \quad \hat{\mathbf{w}}_\alpha^{\mathbf{s}(x)} = (\hat{\boldsymbol{\sigma}}_x^{\mathbf{s}})^{-1}\hat{\mathbf{w}}_\alpha \in \hat{\mathcal{F}}_{x;1,1} , \tag{5.20b}$$

which act within the fibres of the fermion field-mode superfibre bundle $\hat{\mathcal{A}}$, where, in accordance with (2.3), (2.15), (3.10), (5.9) and (5.17),

$$\psi_{\mathbf{s}}^{(+)}(x;\hat{\mathbf{w}}_\alpha^{\mathbf{s}}) = Z_{\ell,m}\int_V \psi^{(+)\mathrm{T}}(x;-a(x)-i\ell v)\,\hat{\mathbf{w}}_\alpha^{\mathbf{s}}(v)\,d\Omega(v) , \tag{5.21}$$

$$\psi_{\mathbf{s}}^{(-)}(x;\hat{\mathbf{w}}_\alpha^{\mathbf{s}}) = Z_{\ell,m}\int_V \hat{\mathbf{w}}_\alpha^{\mathbf{s}*}(v)\,\psi_{\mathbf{s}}^{(-)}(x;-a(x)-i\ell v)\,d\Omega(v) . \tag{5.22}$$

As in Sec. 6.3, that action can be transferred to modal Fock-Dirac fibres by noting that, due to (2.24) and (2.25),

$$\{\theta_x^\alpha, \psi_{\mathbf{s}}^{(\pm)}(x;\hat{\mathbf{w}}_\beta^{\mathbf{s}})\} = \{\overline{\theta}_x^\alpha, \psi_{\mathbf{s}}^{(\pm)}(x;\hat{\mathbf{w}}_\beta^{\mathbf{s}})\} = 0 , \tag{5.23}$$

$$[\psi_{\mathbf{s}}^{(-)}(x;\hat{\mathbf{w}}_{\alpha}^{\mathbf{s}}),\theta_x^{\beta}\,\psi_{\mathbf{s}}^{(+)}(x;\hat{\mathbf{w}}_{\beta}^{\mathbf{s}})]=\delta_{\alpha\beta}\theta_x^{\beta}\ , \qquad \alpha,\beta=1,2,\ldots\ . \tag{5.24}$$

Hence, in accordance with (3.19), we obtain the resolution

$$\int_{\bar{\mathbf{B}}_x\times\mathbf{B}_x}\left|\hat{\Phi}_f^{\mathbf{s}}\right\rangle\mathcal{D}f\left\langle\hat{\Phi}_f^{\mathbf{s}}\right|=\hat{\mathbf{1}}_x\ , \qquad \mathcal{D}f\equiv d\hat{\bar{f}}d\hat{f}\ , \tag{5.25}$$

for the identity within the Berezin-Dirac as well as the Fock-Dirac fibre over $x\in\mathbf{M}$.

The relations (5.23) and (5.24) also imply that, as in (3.21) and (3.23),

$$\psi_{\mathbf{s}}^{(-)}(x,v,s)\left|\hat{\Phi}_f^{\mathbf{s}}\right\rangle=\hat{f}(v,s)\left|\Phi_f^{\mathbf{s}}\right\rangle, \qquad \hat{f}\in\hat{\mathbf{B}}_x\ ,\ \ \hat{f}=\hat{\sigma}_x^{\mathbf{s}}\hat{f}\in\hat{\mathbf{B}}\ , \tag{5.26a}$$

$$\left\langle\hat{\Phi}_f^{\mathbf{s}}\right|\psi_{\mathbf{s}}^{(+)}(x,v,s)=\left\langle\Phi_f^{\mathbf{s}}\right|\bar{f}(v,s)\ , \qquad \hat{\bar{f}}=\hat{\sigma}_x^{\mathbf{s}}\hat{\bar{f}}\in\hat{\bar{\mathbf{B}}}\ ,\quad v\in\boldsymbol{V}\ . \tag{5.26b}$$

In conjunction with (5.20) they also lead to the conclusion that

$$\left\langle\hat{\Phi}_f^{\mathbf{s}}\middle|\hat{\Phi}_{f'}^{\mathbf{s}}\right\rangle=\exp\left[-\tfrac{1}{2}\left\langle\hat{f}\middle|\hat{f}\right\rangle+\left\langle\hat{f}\middle|\hat{f}'\right\rangle-\tfrac{1}{2}\left\langle\hat{f}'\middle|\hat{f}'\right\rangle\right]\ , \qquad \hat{f},\hat{f}'\in\hat{\mathbf{B}}_x\ , \tag{5.27}$$

where, in accordance with (2.18), the above inner products assume the even-supernumber values

$$\left\langle\hat{f}\middle|\hat{f}'\right\rangle=\theta_x^{\alpha}\theta'^{\beta}_x\left\langle\hat{\mathbf{w}}_{\alpha}^{\mathbf{s}(x)}\middle|\hat{\mathbf{w}}_{\beta}^{\mathbf{s}(x)}\right\rangle=\sum\nolimits_{\alpha=1}^{\infty}\theta_x^{\alpha}\theta'^{\alpha}_x\ , \qquad \hat{f},\hat{f}'\in\hat{\mathbf{B}}_x\ . \tag{5.28}$$

In this manner, the ground is prepared for formulating the quantum-geometric propagation of fermion fields in a geometrodynamically evolving curved spacetime background.

6.6. Quantum-Geometric Fermion Field Propagation

In its principal aspects, the formulation of free-fall propagation of the quantum-geometric fermion field $\boldsymbol{\psi}$ in (5.3) proceeds as in the case of the boson field $\boldsymbol{\varphi}$ in Sec. 5.6. The main differences result between the two cases from the presence of supernumbers in the Berezin integrals for resolutions of the identity in (5.25).

Let us adopt, as in Sec. 5.6, a foliation of $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ into synchronous reference hypersurfaces Σ_t yielding a segmentation of the region between two fixed hypersurfaces $\Sigma_{t'}=\Sigma_{t_0}$ and $\Sigma_{t''}=\Sigma_{t_N}$ into base-segments $\mathbf{S}_n$, $n=1,\ldots,N$, with $t_n-t_{n-1}=(t''-t')/N=\varepsilon$. For any choice of Dirac gauge $\mathbf{s}$ associated with a global Poincaré moving frame $\boldsymbol{s}=\{(\boldsymbol{a}(x),\boldsymbol{e}_i(x))|\,x\in\mathbf{M}\}$, we then introduce coherent field-mode sections of the fermion field above each one of the inflow-outflow hypersurfaces corresponding to $n=1,\ldots,N-1$:

$$\hat{\boldsymbol{\Phi}}_{\psi_n(x_n)}^{\mathbf{s}}\in\hat{\mathcal{B}}_{x_n}\ , \qquad \psi_n(x_n)=(\boldsymbol{\sigma}_{x_n}^{\mathbf{s}})^{-1}\psi_n\in\mathbf{B}_{x_n}\ , \qquad \psi_n\in\mathbf{B}\ , \qquad x_n\in\Sigma_{t_n} \tag{6.1}$$

The quantum-geometric *free-fall fermion field propagator* between two fermion field modes at the respective base locations $x'=x_0\in\Sigma_{t'}$ and $x''=x_N\in\Sigma_{t''}$ is then given as in (5.6.2) by

$$K(\psi(x'');\psi(x'))=\lim_{\varepsilon\to+0}\prod_{n=N}^{1}{}'\int\mathcal{D}\psi_n\left\langle e^{i\varepsilon\int d\sigma(x_n)\rho(x_n)}\,\hat{\boldsymbol{\Phi}}_{\psi_n(x_n)}^{\mathbf{s}}\middle|\,\hat{\boldsymbol{\Phi}}_{\psi_n(x_{n-1})}^{\mathbf{s}}\right\rangle, \tag{6.2a}$$

$$\mathcal{D}\psi_n = \prod_{x_n \in \Sigma_{t_n}} \mathcal{D}[\psi_n(x_n)] \ , \qquad \hat{\boldsymbol{\Phi}}^{\mathbf{s}}_{\psi_n(x_{n-1})} = \tau_\gamma(x_n, x_{n-1})\ \hat{\boldsymbol{\Phi}}^{\mathbf{s}}_{\psi_{n-1}(x_{n-1})} \ , \qquad (6.2b)$$

where the parallel transport in (6.2b) is to be carried out, as in the boson case, along the geodesic arc $\gamma(x_{n-1}, x_n)$ connecting the point x_{n-1} on an inflow hypersurface $\Sigma_{t_{n-1}}$ with the point x_n on an outflow hypersurface Σ_{t_n} of each base-segment $\mathbf{S}_n$, $n = 1,\ldots,N$. According to (5.2) and (5.4), the proper energy density operator in (6.2a) is given by

$$\boldsymbol{\rho}(x_n) = n^k(x_n) \int_{V_{x_n}} : \overline{\boldsymbol{\psi}}(x_n, v)\, \gamma_k\, \boldsymbol{\psi}_{;0}(x_n, v) - \overline{\boldsymbol{\psi}}_{;0}(x_n, v)\, \gamma_k\, \boldsymbol{\psi}(x_n, v) : \, d\Omega(v) \ , \qquad (6.3)$$

where the dependence of the components of the fermion field $\boldsymbol{\psi}$ on $\mathbf{s}$ is implicit. Moreover, the Berezin functional integrals in (6.2a) can be consistently defined, as in (5.6.4), by

$$\int_{\mathcal{I}_{1,1}(\Sigma_t)} \left| \hat{\boldsymbol{\Phi}}^{\mathbf{s}}_{\psi(x)} \right\rangle \mathcal{D}\psi \left\langle e^{i\varepsilon \int d\sigma(x) \boldsymbol{\rho}(x)}\, \hat{\boldsymbol{\Phi}}^{\mathbf{s}}_{\psi(x)} \right| =$$

$$\lim_{B \to \Sigma_t} \lim_{M \to \infty} \prod_{\nu=1}^{M} \int_{\overline{\mathbf{B}}_{y_\nu} \times \mathbf{B}_{y_\nu}} \left| \hat{\boldsymbol{\Phi}}^{\mathbf{s}}_{\psi(y_\nu)} \right\rangle \mathcal{D}[\psi(y_\nu)] \left\langle e^{i\varepsilon \sigma(B_\nu) \boldsymbol{\rho}(y_\nu)}\, \hat{\boldsymbol{\Phi}}^{\mathbf{s}}_{\psi(y_\nu)} \right| \ , \qquad (6.4)$$

since, according to (5.19), (5.28) and (6.3), the "integrands" in (6.4) assume only even supernumber values, which therefore commute with each other. The Dirac gauge invariance of the proper energy density operator in (6.3) and of the parallel transport in (6.2b) then insures that the resulting free-fall fermion field propagator is also Dirac gauge invariant.

On account of (5.26) and (5.27), the considerations in (5.6.9)-(5.6.13) can be easily adapted to the present case of fermion fields. Instead of (5.6.12) we now have

$$\left\langle \hat{\boldsymbol{\Phi}}^{\mathbf{s}}_{\psi_n(x_n)} \middle| \hat{\boldsymbol{\Phi}}^{\mathbf{s}}_{\psi_n(x_{n-1})} \right\rangle = \exp\left[i\varepsilon \tfrac{i}{2} \left(\left\langle \hat{\psi}_n(x_n) \middle| \dot{\psi}_n(x_n) \right\rangle - \left\langle \dot{\psi}_n(x_n) \middle| \hat{\psi}_{n-1}(x_n) \right\rangle \right) \right] , \qquad (6.5)$$

$$\dot{\psi}_n(x_n) := \left(\hat{\psi}_n(x_n) - \hat{\psi}_{n-1}(x_n) \right) \Big/ \delta^4 x_n \ , \quad \hat{\psi}_{n-1}(x_n) = \tau_\gamma(x_n, x_{n-1})\, \hat{\psi}_{n-1}(x_{n-1}), \qquad (6.6)$$

where, according to (5.9) and (5.28), the first of the inner products on the right-hand side of (6.5) can be expressed in the form

$$\left\langle \hat{\psi}_n(x_n) \middle| \dot{\psi}_n(x_n) \right\rangle = \sum_{\alpha=1}^{\infty} \overline{\theta}^{\alpha}_{x_n} \dot{\theta}^{\alpha}_{x_n} = Z_{\ell,m} \int_V \overline{\psi}^{\mathrm{T}}_n(x_n, v)\, \dot{\psi}_n(x_n, v)\, d\Omega(v) \ , \qquad (6.7a)$$

$$\overline{\psi}_n(x_n, v) = \overline{\theta}^{\alpha}_{x_n} \hat{\mathbf{w}}^{\mathbf{s}(x_n)}_{\alpha}(v) \ , \qquad \dot{\psi}_n(x_n, v) = \dot{\theta}^{\alpha}_{x_n} \hat{\mathbf{w}}^{\mathbf{s}(x_n)}_{\alpha}(v) \ , \qquad (6.7b)$$

and similar expressions hold for the second of those two inner products. This leads to the conclusion that

$$K(\psi(x''); \psi(x')) = \lim_{\varepsilon \to +0} \prod_{n=N}^{1}{}' \int_{\mathcal{I}_{1,1}(\Sigma_{t_n})} \mathrm{D}\, \psi_n \exp\left[i\varepsilon \int d\sigma(x) \int d\Omega(v)\, \mathcal{L}_0(\psi_n, \psi_{n-1}) \right] , \qquad (6.8)$$

where in a Dirac gauge $\mathbf{s}$ corresponding to a Poincaré moving frame $\boldsymbol{s}$ adapted to the family of reference hypersurface Σ_t,

$$\mathcal{L}_0(\psi_n(x,v), \psi_{n-1}(x,v)) =$$

$$\tfrac{i}{2} Z_{\ell,m} \left[\overline{\psi}^{\mathrm{T}}_n(x,v)\, \dot{\psi}_n(x,v) - \dot{\overline{\psi}}^{\mathrm{T}}_n(x,v)\, \psi_{n-1}(x,v) \right] - T_{00} \left[\overline{\psi}_n(x,v) + \psi_{n-1}(x,v) \right] . \qquad (6.9)$$

Hence, the formal limit $\varepsilon = t_n - t_{n-1} \to +0$ gives rise to the path integral representation

$$K(\psi(x''); \psi(x')) = \int \mathrm{D}\psi \exp[iS(\psi)] \ , \quad \mathrm{D}\psi = \prod_{t''>t>t'} \prod_{x \in \Sigma_t} \mathcal{D}\psi(x) \ , \quad (6.10)$$

of the free-fall fermion field propagator in (6.2). This path integral is based on the "action"

$$S(\psi) = \int_{t'}^{t''} dt \int_{\Sigma_\tau} d\sigma(x) \int_V d\Omega(v) \mathcal{L}_0(\psi_t(x,v), \psi_{t-0}(x,v)) \ , \quad (6.11)$$

which, in turn, is based on the "Lagrangian density"

$$\mathcal{L}_0 = \tfrac{i}{2} Z_{t,m} \left[\overline{\psi}_t^{\mathrm{T}}(x,v) \dot{\psi}_t(x,v) - \dot{\overline{\psi}}_t^{\mathrm{T}}(x,v) \psi_{t-0}(x,v) \right] - T_{00} [\overline{\psi}_t(x,v) + \psi_{t-0}(x,v)] \ . \quad (6.12)$$

Naturally, all the comments made in Sec. 5.6 with regard to the boson counterparts of these formulae remain in effect in the present fermion case.

For a quantum-geometric fermion field distribution over a region R within $\mathbf{M}$,

$$x \mapsto \hat{\boldsymbol{\Psi}}_x \in \hat{\mathcal{F}}_x \ , \quad \hat{\Psi}_x = \bigoplus_{n,n'=0}^{\infty} \hat{\Psi}_{x;n,n'} \in \hat{\mathcal{F}} \ , \quad x \in R \ , \quad (6.13)$$

the covariant local quantum field-mode probability current is given by

$$j_{\mathrm{s}}^k(x) = \sum_{n,n'=0}^{\infty} \sum_{\rho=1}^{n+n'} \int \sum_{r_\rho, s_\rho=1}^{4} \hat{\overline{\Psi}}_{x;n,n'}(r_\rho) \gamma^k_{r_\rho s_\rho} \hat{\Psi}_{x;n,n'}(s_\rho) d\Omega(v_1) \cdots d\Omega(v_{n+n'}) , \quad (6.14)$$

$$\hat{\Psi}_{x;n,n'}(s_\rho) = \hat{\Psi}_{x;n,n'}(v_1, s_1, \ldots, v_\rho, s_\rho, \ldots, v_{n+n'}, s_{n+n'}) \ , \quad s_\rho = 1,2,3,4 \ . \quad (6.15)$$

As in the case of its bosonic counterpart in (5.6.23), this current is conserved[13] at base locations $x \in R$ in whose neighborhoods the values assumed by the quantum-geometric field distribution (6.13) are the parallel transports along geodesics of the value at x.

The treatment of interactions with other massive boson or fermion quantum-geometric fields presents no new problems, and can be achieved by taking Whitney direct products[14] of the field-mode bundles for all the mutually interacting quantum-geometric fields. The resulting action integral for the quantum-geometric propagators of those interacting fields is then simply obtained by adding the interaction Lagrangian densities to the sum of their free-fall Lagrangian densities. In the case where the base Lorentzian manifold is the Minkowski space the counterparts of the equations (5.6.29)-(5.6.33) and (5.8.26)-(5.5.40) can be arrived at without any basic difficulties.

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[13] Since the counterpart of the Dirac current for single fermion-antifermion states in special relativistic phase space is conserved (Prugovečki, 1980), the same argument as in Sec. 5.6 can be applied to the current in (6.14).

[14] As in the case of Whitney direct sum, the concept of Whitney direct product which is normally defined for real and complex vector bundles with finite-dimensional fibres (cf. Osborn, 1982, p. 170), can be extended in a straightforward manner to Hilbert bundles with infinite-dimensional fibres – cf. (Fell and Doran, 1988, pp. 149-151). In the case of associated bundles that are defined as G-products, such direct Whitney sums and products can also be arrived at by adopting in the construction of those G-products typical fibres which are Hilbert direct sums and tensor products, respectively, of the typical fibres or superfibres of the constituent associated bundles.

Chapter 7

Massless Quantum-Geometric Gauge Fields

The Gupta-Bleuler method of quantization of the electromagnetic field in Minkowski space employs pseudo-Hilbert spaces with indefinite metric, known in mathematics literature as Krein spaces. Quantum-geometric electromagnetic fields are therefore formulated in Gupta-Bleuler bundles whose fibres carry indefinite metrics. The quantization of Yang-Mills fields with generically compact but non-Abelian gauge groups has led to Faddeev-Popov "ghost" fields, whose geometric interpretation has its origins in the theory of connections on principal fibre bundles. This interpretation involves gauge groups of the second kind constituting vertical automorphisms on such bundles, and leads to a Yang-Mills gauge fields which act on generalized Higgs fields that play the role of matter fields defined on these bundles. The geometric formulation of the ensuing BRST symmetries serves as the basis for a quantum-geometric framework for quantized Yang-Mills fields over curved spacetimes.

7.1. Hilbert Space Representations of the Poincaré Group

The theory of induced representations, formulated in its most general form by Mackey (1951-1953) for the case of locally compact topological groups, provides a systematic method for deriving and classifying the irreducible unitary representations of such groups (cf. Barut and Raczka, 1986, Chapters 16-19; Thaller, 1992, Chapter 3). In the case of the restricted Poincaré group $\mathrm{ISO}_0(3,1)$, or, more generally, of the orthochronous[1] Poincaré group $\mathrm{ISO}^{\uparrow}(3,1)$, this method reproduces the results of Wigner (1939), who classified its irreducible unitary representations by pinpointing the stability groups (called by him "little groups") as subgroups of $\mathrm{ISO}_0(3,1)$ or $\mathrm{ISO}^{\uparrow}(3,1)$, and their respective orbits in M^4.

Amongst the six general types of irreducible unitary representations of $\mathrm{ISO}_0(3,1)$ or $\mathrm{ISO}^{\uparrow}(3,1)$ (cf. Barut and Raczka, 1986, p. 521), the only ones which are of indubitable[2] physical significance are those originating from the sets

$$\boldsymbol{V}_m^{\pm} = \left\{ \boldsymbol{k} = k^{\mu}\boldsymbol{e}_{\mu} \in M^4 \;\middle|\; k_0 = \pm\sqrt{\mathbf{k}^2 + m^2} \right\} , \qquad k^{\mu} = \eta^{\mu\nu}k_{\nu} , \tag{1.1a}$$

$$\boldsymbol{e}_{\mu}\cdot\boldsymbol{e}_{\nu} = \eta_{\mu\nu} = \eta^{\mu\nu} , \qquad \mu,\nu = 0,1,2,3 , \qquad \mathbf{k} = k^{a}\boldsymbol{e}_{a} = -\sum\nolimits_{a=1}^{3} k_{a}\boldsymbol{e}_{a} , \tag{1.1b}$$

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[1] The *orthochronous Poincaré group* $\mathrm{ISO}^{\uparrow}(3,1)$ incorporates space reflections, but not time reversals, and therefore it contains the restricted Poincaré group $\mathrm{ISO}_0(3,1)$ as a subgroup. The introduction of $\mathrm{ISO}^{\uparrow}(3,1)$ is necessitated by the helicity formalism (cf., e.g., Scadron, 1979, Sec. 3.D), which will be employed in Sec. 7.2.

[2] Those corresponding to pure imaginary values of m in (1.1) describe tachyons (cf. Recami, 1977, 1987; Fanchi, 1988, 1990), whose behavior in the classical regime violates Einstein causality, and gives rise to physical paradoxes (cf., e.g., Friedman, 1983, p. 162).

which are mathematically identifiable with orbits within the character group of spacetime translations, but whose physical interpretation is, respectively, that of forward and backward mass hyperboloids in the case of $m > 0$, and of forward and backward light cones in the case of $m = 0$. Clearly, for any fixed global Lorentz frame $\{\boldsymbol{e}_\mu \mid \mu = 0,1,2,3\}$ in the Minkowski space M^4, all the points on the orbits for $m > 0$ merge in the limit $m \to +0$ into those on the respective orbits

$$\boldsymbol{V}_0^{\pm} = \left\{ \boldsymbol{k} = k^\mu \boldsymbol{e}_\mu \,\middle|\, k_0 = \pm |\mathbf{k}| \right\} \subset M^4 , \tag{1.2}$$

for $m = 0$ if the values of the 3-vector $\mathbf{k}$ are kept fixed while taking that zero-mass limit. The same is true of the Poincaré-invariant measures on those orbits,

$$d\Omega_m(k) = \delta(k^2 - m^2)\, d^4k , \qquad k^2 = k \cdot k := \eta_{\mu\nu} k^\mu k^\nu , \qquad m \geq 0 , \tag{1.3}$$

if their measure elements are deemed to be directly dependent upon the variable $\mathbf{k}$, so that

$$d\Omega_m(\mathbf{k}) = d\mathbf{k} \,/ 2\sqrt{\mathbf{k}^2 + m^2} \quad \xrightarrow[m \to +0]{} \quad d\Omega_0(\mathbf{k}) = d\mathbf{k} \,/ 2|\mathbf{k}| \tag{1.4}$$

However, the same is not true of the respective stability groups – which, by definition, consist of all the elements of $SO_0(3,1)$ that leave invariant vectors on the respective orbits in (1.1) and (1.2).

Indeed, for $m > 0$, the stability group for any vector on one of the mass hyperboloids in (1.1) is isomorphic to the rotation group[3] $SO(3)$ in $\mathbf{R}^3$. In a given global Lorentz frame $\boldsymbol{u} = \{\boldsymbol{e}_\mu \mid \mu = 0,1,2,3\}$ on whose elements $SO_0(3,1)$ acts from the right (and therefore on whose vector components it acts from the left), such a stability group is given by

$$S_{\hat{\boldsymbol{m}}}(3) = \left\{ \Lambda \,\middle|\, \Lambda \hat{m} = \hat{m} \right\} = \left\{ \Lambda \in \mathrm{SO}_0(3,1) \,\middle|\, \Lambda^\mu{}_0 = \delta^\mu{}_0 \right\} , \qquad \hat{\boldsymbol{m}} := \pm m \boldsymbol{e}_0 \in \boldsymbol{V}_m^{\pm} \tag{1.5}$$

On the other hand, for $m = 0$, the stability group is usually chosen to be the subgroup of $SL(2,\mathbf{C})$ defined by

$$S_{\hat{\boldsymbol{k}}}(2) = \left\{ A \in \mathrm{SL}(2,\mathbf{C}) \,\middle|\, A \underset{\sim}{\hat{k}} A^* = \underset{\sim}{\hat{k}} \right\} , \qquad \hat{\boldsymbol{k}} = \pm \tfrac{1}{2}(\boldsymbol{e}_0 + \boldsymbol{e}_3) \in \boldsymbol{V}_0^{\pm} , \tag{1.6a}$$

so that its elements are given by (cf. Barut and Raczka, 1986, p. 520)

$$A(\theta, z) = \begin{pmatrix} e^{i\theta/2} & z e^{-i\theta/2} \\ 0 & e^{-i\theta/2} \end{pmatrix} \in S_{\hat{\boldsymbol{k}}}(2) , \qquad \theta \in [0, 4\pi) , \quad z \in \mathbf{C}^1 . \tag{1.6b}$$

This group covers twofold the Euclidean group $SE(2)$ in $\mathbf{R}^2$. In turn, $SE(2)$ is equal to the semidirect product $T^2 \wedge O(2)$ of the group T^2 of translations in $\mathbf{R}^2$ with the group $O(2)$ of rotations in $\mathbf{R}^2$, and is usually expressed in the form

$$\mathrm{SE}(2) = \left\{ (z,\theta) \,\middle|\, z \in \mathbf{C}^1, \theta \in [0, 2\pi) \right\} , \qquad (z,\theta)(z',\theta') = (z + e^{i\theta} z', \theta + \theta') . \tag{1.7}$$

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[3] In the case of half-integer spin, the role of $ISO_0(3,1)$ is taken over by its covering group $ISL(2,C)$, and therefore the role of $SO(3)$ is taken over by its covering group $SU(2)$ – cf. (Barut and Raczka, 1986, pp. 518-520).

The Lie algebra $se(2)$ of the Euclidean group SE(2) can be obtained as a contraction[4] of the Lie algebra $so(3)$ of SO(3) by taking a limit $R \to +\infty$. The parameter R used in this contraction can be geometrically envisaged as playing the role of the radius of a 3-sphere on whose points the rotations in SO(3) are deemed to act. Hence, R is not directly related to the parameter m, which for $m > 0$ plays the physical role of the rest mass of quantum particles in the case of 4-momenta k on forward mass hyperboloids, or antiparticles in the case of 4-momenta k on backward mass hyperboloids.

The presence of the stability subgroups of $ISO_0(3,1)$ is very noticeable in Wigner-type representations for spin-1 particles. In the case of massive quantum particles of rest mass $m > 0$ these representations are usually formulated in the Hilbert space

$$\hat{\mathcal{H}}_m = L^2(V_m^+, d\Omega_m) \oplus L^2(V_m^+, d\Omega_m) \oplus L^2(V_m^+, d\Omega_m) \ , \tag{1.8}$$

consisting of wave functions in the momentum representation, and with inner product

$$\left\langle \hat{f} \middle| \hat{f}' \right\rangle = \sum_{s=-1}^{+1} \int_{V_m^+} \hat{f}_s^*(k) \hat{f}_s'(k) \, d\Omega_m(k) \ , \qquad \hat{f}, \hat{f}' \in \hat{\mathcal{H}}_m \ . \tag{1.9}$$

The action of unitary operators belonging to these representations upon the elements of these spaces, which are represented by triples of complex-valued functions arranged into one-column matrices, is given by

$$\hat{U}_m(a,\Lambda) \ : \hat{f}(k) \ \mapsto \ \exp(ia \cdot k) D^1(R_{k,\Lambda}) \hat{f}(\Lambda^{-1}k) \ , \qquad \hat{f} \in \hat{\mathcal{H}}_m \ , \tag{1.10a}$$

$$R_{k,\Lambda} \colon \ \mathbf{k}' = k'^a \, \boldsymbol{e}_a \ \mapsto \ \mathbf{k}'' = k''^a \, \boldsymbol{e}_a \ , \qquad k'' = \Lambda_{k/m}^{-1} \Lambda \, \Lambda_{\Lambda^{-1}k/m} \, k' \in M^4 \tag{1.10b}$$

The operator in (1.10b) is called a Wigner rotation (cf., e.g., Ohnuki, 1976, p. 35), and D^1 in (1.10a) is its representative SO(3) matrix, which acts upon the real and the imaginary values assumed by complex-valued triples in $\mathbf{C}^3$ as it does in $\mathbf{R}^3$.

On the other hand, in the case of massless quantum particles of spin-1 the corresponding Wigner-type representations are usually formulated in the Hilbert space

$$\hat{\mathcal{H}}_0 = L^2(V_0^+, d\Omega_0) \oplus L^2(V_0^+, d\Omega_0) \ . \tag{1.11}$$

This space consists of two-component wave functions, and carries the inner product

$$\left\langle \hat{f} \middle| \hat{f}' \right\rangle = \sum_{r=\pm 1} \int_{V_0^+} \hat{f}_r^*(k) \, \hat{f}_r'(k) \, d\Omega_0(k) \ . \tag{1.12}$$

Hence, the unitary operators belonging to this representation act upon elements of the Hilbert space (1.11) which are represented by pairs of complex-valued functions arranged into one-column matrices, and are given by

$$\hat{U}_0(a,\Lambda) \ : \hat{f}(k) \ \mapsto \ \exp(ia \cdot k) D_{\hat{k}}(\hat{\Lambda}_k^{-1} \Lambda \hat{\Lambda}_{\Lambda^{-1}k}) \hat{f}(\Lambda^{-1}k) \ , \qquad \hat{f} \in \hat{\mathcal{H}}_0 \ , \tag{1.13a}$$

[4] Cf. (Inönü and Wigner, 1953) and (Gilmore, 1974, p. 450). A review of the basic results on such contractions can be found also in (Kim and Noz, 1986). More recent results on these stability groups, which are of relevance to the considerations in this section, can be found in (Vassiliadis, 1989) and in (Kim and Wigner, 1987, 1990).

$$\hat{\Lambda}_k = R(\hat{\mathbf{k}})\Lambda_{\omega(k)\boldsymbol{e}_3} , \quad \boldsymbol{k} = k^\mu \boldsymbol{e}_\mu , \quad \hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}| , \quad \mathbf{k} = k^a \boldsymbol{e}_a , \tag{1.13b}$$

where $R(\hat{\mathbf{k}})$ is the rotation taking $\mathbf{e}_3$ into $\hat{\mathbf{k}}$ around the axis orthogonal to these two 3-vectors, and $\Lambda_{\omega\boldsymbol{e}_3}$ is the Lorentz boost along $\boldsymbol{e}_3$ which maps the vector $\hat{\boldsymbol{k}}$ in (1.6) into the vector $k_0(\boldsymbol{e}_0 + \boldsymbol{e}_3)/2$; whereas, the finite-dimensional representation in (1.13a) is given by

$$D_{\hat{\boldsymbol{k}}} : \ \Lambda(A(\theta,z)) \ \mapsto \ A(2\theta,0) \in S_{\hat{\boldsymbol{k}}}(2) , \quad \theta \in [0,4\pi) , \quad z \in \mathbf{C}^1 , \tag{1.14}$$

in accordance with (1.5) and (1.6).

As we shall see in the next section, the two-component wave functions of the Wigner representation possess a physical interpretation, according to which their two components provide the momentum space probability amplitudes for the clockwise and the counter-clockwise circular polarization of single-photon states represented by those wave functions. However, since the Wigner-type representations in (1.10) and (1.13) for spin-1 massive and massless quantum particles, respectively, have a different number of components, the wave functions for the massive case cannot be directly set in correspondence in the limit $m \to +0$ with those of the massless case. We shall therefore consider next indefinite-metric representations, for which such a smooth transition from the massive to the massless case can be carried out.

7.2. Krein Space Representations of the Poincaré Group

Pseudo-Hilbert spaces with indefinite metric were first introduced in physics by Dirac (1942). Their use in the quantization of the electromagnetic field was later advocated by Gupta (1950) and Bleuler (1950), and in Minkowski space it led to the indefinite-metric Gupta-Bleuler framework for quantum electrodynamics (QED). This framework eventually became more popular than the Dirac-Schwinger framework, which employed a positive-definite inner product in the space of state vectors of the quantum electromagnetic field. Although these two frameworks can be shown to be mathematically interrelated (Dürr and Rudolph, 1969; Ferrari *et al.*, 1974), the greater attraction of the Gupta-Bleuler framework lay in its apparently manifest special-relativistic invariance, and in the formal "locality" displayed by all its electromagnetic quantum field modes. However, as will gradually transpire in subsequent sections, the actual state of affairs is, in both these respects, more involved than is commonly believed to be the case. Hence, from the point of view of the present considerations, the decided advantage of the Gupta-Bleuler framework lies in its treatment of photon wave functions, since it very clearly shows the relationship of these wave functions to the corresponding wave functions of massive quantum particles.

From the mathematical point of view, the structure of the pseudo-Hilbert space of single-photon states employed by the Gupta-Bleuler framework is that of a Krein space.

In general, a *Krein space* (cf., e.g., Bognár, 1974; Azizov and Iokhvidov, 1989) is a real or a complex topological vector space $\mathbf{K}$ which is a Hilbert space with respect to a family of positive-definite inner products, called its *J-inner products*, and on which there is also given a non-degenerate Hermitian inner product specified by an indefinite sesquilinear form; moreover, for each J-inner product in $\mathbf{K}$ there is a *fundamental decomposition* of the

entire Krein space $\mathbf{K}$ into a direct sum of two closed subspaces $\mathbf{K}^+$ and $\mathbf{K}^-$, which is such that on $\mathbf{K}^+$ the indefinite sesquilinear form is positive-definite and equal to that J-inner product; whereas on $\mathbf{K}^-$ it is negative-definite, and its values are equal to the negative values assumed by that J-inner product.[5]

In the momentum representation, the Krein space $\tilde{\mathbf{K}}$ required by the Gupta-Bleuler formalism can be constructed (cf. Bogolubov *et al.*, 1990, p. 420) out of the Hilbert space of all 4-tuples of functions on the forward half of the light-cone $\boldsymbol{V}_0$ in the Minkowski space $(\mathbf{R}^4,\boldsymbol{\eta})$ for which the J-norm corresponding to the J-inner product

$$\left(\tilde{f}\middle|\tilde{f}'\right)_{\tilde{J}} = \sum_{\mu=0}^{3} \int_{V_0^+} \tilde{f}_\mu^*(k)\, \tilde{f}'_\mu(k)\, d\Omega_0(k) \ , \tag{2.1}$$

exists. With this Hilbert space given, the corresponding Krein space $\tilde{\mathbf{K}}$ is obtained by introducing in it the indefinite *Gupta-Bleuler inner product*

$$\left\langle \tilde{f}\middle|\tilde{f}'\right\rangle = -\int_{V_0^+} \eta^{\mu\nu} \tilde{f}_\mu^*(k)\, \tilde{f}'_\nu(k) d\Omega_0(k) \tag{2.2}$$

Indeed, we then obviously have in $\tilde{\mathbf{K}}$ a fundamental decomposition into the two closed subspaces,

$$\tilde{\mathbf{K}}_{\tilde{J}}^+ = \left\{\tilde{f} \in \tilde{\mathbf{K}}\middle|\, \tilde{f}_0 \equiv 0\right\}, \qquad \tilde{\mathbf{K}}_{\tilde{J}}^- = \left\{\tilde{f} \in \tilde{\mathbf{K}}\middle|\, \tilde{f}_a \equiv 0,\ a = 1,2,3\right\} \ , \tag{2.3}$$

on which the Gupta-Bleuler inner product is positive-definite and negative-definite, respectively, and in terms of which we can write

$$\tilde{\mathbf{K}} = \tilde{\mathbf{K}}_{\tilde{J}}^+ \oplus \tilde{\mathbf{K}}_{\tilde{J}}^- , \quad \left(\tilde{f}\middle|\tilde{f}'\right)_{\tilde{J}} = \left\langle \tilde{f}\middle|\tilde{J}\tilde{f}'\right\rangle , \quad \tilde{J} = \mathbf{P}_{\tilde{J}}^+ - \mathbf{P}_{\tilde{J}}^- , \quad \tilde{\mathbf{K}}_{\tilde{J}}^\pm = \mathbf{P}_{\tilde{J}}^\pm \tilde{\mathbf{K}} \ . \tag{2.4}$$

In general, the orthogonal projectors $\mathbf{P}^\pm$ onto the closed subspaces $\mathbf{K}^\pm$ of a fundamental decomposition of a Krein space $\mathbf{K}$ are called *fundamental projections*, and the operator $J = \mathbf{P}^+ - \mathbf{P}^-$ is called the *fundamental symmetry* corresponding to that fundamental decomposition. The norm corresponding to a J-inner product is called a *J-norm*.

The Minkowski space $M^4 \equiv (\mathbf{R}^4,\boldsymbol{\eta})$ provides an example of a 4-dimensional Krein space if its J-inner product is chosen to be the Euclidean inner product corresponding to a given choice of a global Lorentz frame $\boldsymbol{u} = \{\boldsymbol{e}_\mu \mid \mu = 0,1,2,3\}$. Hence, in this case the fundamental projections are the orthogonal projectors onto the subspaces of M^4 spanned by $\{\boldsymbol{e}_0\}$ and $\{\boldsymbol{e}_a \mid a = 1,2,3\}$, respectively. Clearly, the fundamental decomposition for two global Lorentz frames related to each other by a Lorentz boost are distinct, and therefore so are the corresponding J-inner products.

The same is obviously true of the J-inner products in (2.1), and of the fundamental decompositions in (2.4): they are distinct for two global Lorentz frames related to each other by a Lorentz boost. On the other hand, the *majorant topologies* (Bognár, 1974)

[5] Note should be taken of the fact that, as opposed to the terminology of Bognár (1974) adopted in the present monograph, Azizov and Iokhvidov (1989) refer to the indefinite metric determined by this indefinite sesquilinear form as a J-metric. Although this latter reference is more recent, its terminology appears to be less suitable in applications to quantum field theory, so that we shall not employ it in the sequel.

determined by all the corresponding J-norms are equivalent, i.e., the families of all open sets in them coincide. These open sets are generated by the open balls in a given J-norm, with centers at various points in $\tilde{\mathbf{K}}$.

The equivalence of these majorant topologies follows from the fact that the operators

$$\tilde{U}(a,\Lambda) \; : \; \tilde{f}_\mu(k) \; \mapsto \; \tilde{f}'_\mu(k) = \exp(ia\cdot k)\Lambda_\mu{}^\nu \tilde{f}_\nu(\Lambda^{-1}k) \; , \quad (a,\Lambda) \in \mathrm{ISO}^\uparrow(3,1), \tag{2.5}$$

are J-bounded – namely that, as can be seen from (2.1), they are bounded with respect to a given J-norm by virtue of the fact that, for any $\tilde{f} \in \tilde{\mathbf{K}}$,

$$\left\|\tilde{U}(a,\Lambda)\tilde{f}\right\|_J^2 = \sum_{\mu=0}^3 \int_{V_0^+} \left|\Lambda_\mu{}^\nu \tilde{f}_\nu(\Lambda^{-1}k)\right|^2 d\Omega_0(k) \le \sum_{\mu=0}^3 \max_{\nu=0,\ldots,3} \left|\Lambda_\mu{}^\nu\right|^2 \left\|\tilde{f}\right\|_J^2 \tag{2.6}$$

However, the operators in (2.5) are not generally unitary in any given J-inner product. Rather, they are all pseudo-unitary, i.e., they preserve the indefinite Gupta-Bleuler inner product in (2.2):

$$\left\langle \tilde{U}(a,\Lambda)\tilde{f} \middle| \tilde{U}(a,\Lambda)\tilde{f}' \right\rangle = \left\langle \tilde{f} \middle| \tilde{f}' \right\rangle \; , \qquad \tilde{f},\tilde{f}' \in \tilde{\mathbf{K}} \; , \quad (a,\Lambda) \in \mathrm{ISO}^\uparrow(3,1) \tag{2.7}$$

Hence, a J-inner product can play no direct physical role similar to that played by inner products in the Hilbert spaces which carry representations for massive quantum particles. Nevertheless, these J-inner products are mathematically essential since they determine the earlier mentioned majorant topology. In turn, this topology is needed in defining all the continuity properties of the representations in (2.5) of the Poincaré group, as well as in defining other mathematical features which require Hausdorff topologies, since no such separable topologies can be extracted from indefinite inner products.[6]

In addition to (2.7) the operators in (2.5) also satisfy the composition law

$$\tilde{U}(a,\Lambda)\tilde{U}(a',\Lambda') = \tilde{U}(a+\Lambda a', \Lambda\Lambda') \; , \qquad (a,\Lambda),(a',\Lambda') \in \mathrm{ISO}^\uparrow(3,1) \; , \tag{2.8}$$

so that they provide a pseudo-unitary representation of orthochronous Poincaré group $\mathrm{ISO}^\uparrow(3,1)$. However, this representation is not irreducible. Indeed the following *Lorenz space*[7]

$$\tilde{\mathbf{K}}^L = \left\{ \tilde{f} \in \tilde{\mathbf{K}} \,\middle|\, k_\mu \tilde{f}^\mu(k) = 0 \right\} \subset \tilde{\mathbf{K}} \quad , \qquad \tilde{f}^\mu := \eta^{\mu\nu}\tilde{f}_\nu \; , \tag{2.9}$$

is left invariant by all the operators in (2.5), due to the fact that the equation in (2.9), to

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[6] A topological space (cf. Note 1 in Chapter 2) is *Hausdorff* if any two distinct points in it possess disjoint neighborhoods. For any two points in a Krein space $\mathbf{K}$, the J-norm supplies two such neighborhoods in the form of any two open balls with centers at those points and with radii whose sum is smaller than the distance between those points, as determined by the J-norm. However, for an indefinite inner product, such as the Gupta-Bleuler inner product (2.2) or the Minkowski inner product in M^4, there are *null-vectors* (namely vectors whose inner product with themselves is zero) which are *not* equal to the zero vector, and yet whose "distance" from the zero vector, as determined by that indefinite inner product, is equal to zero.

[7] Many readers might know the related gauge condition under the name of "Lorentz gauge" since, due to a widespread historical misconception, in most of contemporary physics literature it is credited to H. A. Lorentz, whose name was confused with that of "Ludwig Lorenz (1829-1891) of Copenhagen, who independently developed an electromagnetic theory of light a few years after the publication of Maxwell's memoirs" (Whittaker, 1951, pp. 267-268). It is to L. Lorenz that this type of gauge condition has to be rightfully attributed, since he was the first to formulate it in 1867 – cf. also (Penrose and Rindler, 1984), p. 321.

which we shall refer as the *Lorenz gauge* condition in $\tilde{\mathbf{K}}$, is obviously left invariant by the transformations in (2.5). Furthermore, the *null subspace* of $\tilde{\mathbf{K}}$,

$$\tilde{\mathbf{K}}^0 = \left\{ \tilde{f} \in \tilde{\mathbf{K}} \,\middle|\, k_\mu \tilde{f}_\nu(k) - k_\nu \tilde{f}_\mu(k) = 0 \,,\ \mu,\nu = 0,1,2,3 \right\} , \tag{2.10}$$

is obviously also left invariant by this pseudo-unitary representation. This null subspace consists of all the null elements of the Krein space $\tilde{\mathbf{K}}$, and turns out to be also a subspace of the Lorenz space in (2.9).

To see that this is indeed the case, let us introduce for each given global Lorentz frame $\boldsymbol{u} = \{\boldsymbol{e}_\mu | \mu = 0,1,2,3\}$ the *linear polarization tetrads* $\{\boldsymbol{\varepsilon}_{(\alpha)}(k) | \alpha = 0,1,2,3\}$, which are attached to all points on the forward light-cone described in (1.2) in such a manner that

$$\boldsymbol{\varepsilon}_{(0)}(k) = \boldsymbol{e}_0 \,,\ \ \boldsymbol{\varepsilon}_{(3)}(k) = -|\mathbf{k}|^{-1} \sum\nolimits_{a=1}^{3} k_a \boldsymbol{e}_a \,,\ \ k_\mu \varepsilon^\mu_{(1)}(k) = k_\mu \varepsilon^\mu_{(2)}(k) = 0 \tag{2.11}$$

According to this construction $\boldsymbol{k} = k_0 (\boldsymbol{\varepsilon}_{(0)}(k) + \boldsymbol{\varepsilon}_{(3)}(k))$, so that we arrive at the conclusion that the two subspaces in (2.9) and (2.10) can be characterized in the following alternative manner, which proves the earlier made statement:

$$\tilde{\mathbf{K}}^L = \left\{ \tilde{f} = \tilde{f}^{(\alpha)}(k) \boldsymbol{\varepsilon}_{(\alpha)}(k) \in \tilde{\mathbf{K}} \,\middle|\, \tilde{f}^{(0)}(k) = \tilde{f}^{(3)}(k) \right\} , \tag{2.12}$$

$$\tilde{\mathbf{K}}^0 = \left\{ \tilde{f} \in \tilde{\mathbf{K}}^L \,\middle|\, \tilde{f}^{(1)}(k) = \tilde{f}^{(2)}(k) = 0 \right\} = \left\{ \tilde{f} \in \tilde{\mathbf{K}}^L \,\middle|\, \left\langle \tilde{f} \middle| \tilde{f} \right\rangle = 0 \right\} . \tag{2.13}$$

We note that the last two relations in (2.11), which physically characterize the two transversal components of a linear polarization tetrad, actually do not determine by themselves those two components unambiguously. We therefore take advantage of the Lorentz transformation introduced in (1.13b) to set

$$\boldsymbol{\varepsilon}_{(r)}(k) = \hat{\Lambda}_k \boldsymbol{e}_r = R(\hat{\mathbf{k}}) \boldsymbol{e}_r \,,\quad \boldsymbol{k} = k^\mu \boldsymbol{e}_\mu \in \boldsymbol{V}_0^+ \,,\quad r = 1,2 \,. \tag{2.14}$$

This determines both those transversal components smoothly and unambiguously on the forward light cone in relation to the chosen global Lorentz frame $\boldsymbol{u}$, except for the set of $\boldsymbol{k}$-values for which $\mathbf{k}$ and $\boldsymbol{e}_3$ point in opposite directions, so that the rotation in (1.13b) is not well-defined. However, the Ω_0-measure of this set is zero, so that within the Krein space $\tilde{\mathbf{K}}$ we can express (2.1) and (2.2) in the respective forms

$$\left\langle \tilde{f} \middle| \tilde{f}' \right\rangle_{\tilde{J}} = \sum\nolimits_{\alpha=0}^{3} \int_{V_0^+} \tilde{f}^{(\alpha)*}(k) \tilde{f}'^{(\alpha)}(k) d\Omega_0(k) \,, \tag{2.15}$$

$$\left\langle \tilde{f} \middle| \tilde{f}' \right\rangle = -\int_{V_0^+} \eta_{\alpha\beta} \tilde{f}^{(\alpha)*}(k) \tilde{f}'^{(\beta)}(k) \, d\Omega_0(k) \,, \tag{2.16}$$

without encountering any ambiguities when the above integrations are performed.

Upon comparing (2.15) with (2.13), we immediately see that the Lorenz space $\tilde{\mathbf{K}}^L$ can be decomposed in the J-dependent direct sum

$$\tilde{\mathbf{K}}^L = \tilde{\mathbf{K}}^0 \underset{J}{\oplus} \tilde{\mathbf{K}}^{\mathrm{T}}_J \,, \qquad \tilde{\mathbf{K}}^{\mathrm{T}}_J = \left\{ \tilde{f}^{(\alpha)}(k) \boldsymbol{\varepsilon}_{(\alpha)}(k) \in \tilde{\mathbf{K}}^L \,\middle|\, \tilde{f}^{(0)}(k) = \tilde{f}^{(3)}(k) = 0 \right\} . \tag{2.17}$$

The second element of this direct sum is therefore also $\boldsymbol{u}$-dependent, but only to the extent that, as can be seen from (2.5), it is not left invariant by those Poincaré transformations in $\mathrm{ISO}^{\uparrow}(3,1)$ for which $\boldsymbol{e}_0 \cdot \Lambda \neq \boldsymbol{e}_0$. On the other hand, the null subspace $\tilde{\mathbf{K}}^0$ constituting the first element of this direct sum is left invariant by all these transformations. As a consequence, the representation in (2.5) induces a representation of the orthochronous Poincaré group $\mathrm{ISO}^{\uparrow}(3,1)$ on the quotient space

$$\tilde{\mathbf{K}}^L/\tilde{\mathbf{K}}^0 = \left\{ [\tilde{f}] \;\middle|\; \tilde{f} \in \tilde{\mathbf{K}}^L, \;\; [\tilde{f}] = [\tilde{f}'] \Leftrightarrow \left\langle \tilde{f} - \tilde{f}' \middle| \tilde{f} - \tilde{f}' \right\rangle = 0 \right\}, \tag{2.18}$$

since both spaces in this quotient are left invariant by this representation.

The elements $[\tilde{f}]$ of this quotient space consist of equivalence classes of vectors $\tilde{f} \in \tilde{\mathbf{K}}^L$ which are equal to each other modulo null vectors from $\tilde{\mathbf{K}}^L$. Consequently, $\tilde{\mathbf{K}}^L/\tilde{\mathbf{K}}^0$ is a *bona fide* Hilbert space with respect to the inner product induced on these equivalence classes by the indefinite inner product in (2.16). In fact, it follows from (2.16) and (2.17) that, in any fixed global Lorentz frame $\boldsymbol{u}$, this quotient space coincides with the second element of the direct sum in (2.17), in which the Gupta-Bleuler inner product in (2.16) is positive-definite, since it assumes there the following form:

$$\left\langle \tilde{f} \middle| \tilde{f}' \right\rangle = \sum_{\alpha=1}^{2} \int_{V_0^+} \tilde{f}^{(\alpha)*}(k) \tilde{f}'^{(\alpha)}(k) d\Omega_0(k) \;, \qquad \tilde{f}, \tilde{f}' \in \tilde{\mathbf{K}}_J^L \,. \tag{2.19}$$

Thus, $\tilde{\mathbf{K}}^L/\tilde{\mathbf{K}}^0$ is identifiable with the set of all pairs of transversal linear polarization probability amplitudes

$$\tilde{\mathbf{K}}^L/\tilde{\mathbf{K}}^0 \cong \left\{ \left(\tilde{f}^{(1)}, \tilde{f}^{(2)} \right) \leftrightarrow [\tilde{f}] \;\middle|\; \tilde{f} \in \tilde{\mathbf{K}}^L \right\}, \tag{2.20}$$

which constitutes a Hilbert space with the following inner product:

$$\left\langle [\tilde{f}] \middle| [\tilde{f}'] \right\rangle = \sum_{\alpha=1}^{2} \int_{V_0^+} \tilde{f}^{(\alpha)*}(k) \tilde{f}'^{(\alpha)}(k) d\Omega_0(k) \;, \qquad \tilde{f}, \tilde{f}' \in \tilde{\mathbf{K}}^L \,. \tag{2.21}$$

The representation induced by the representation in (2.5) on this latter Hilbert space is unitary. To establish that, we introduce the J-orthogonal projector of $\tilde{\mathbf{K}}^L$ onto the second component of the direct sum in (2.17),

$$\tilde{\mathbf{P}}_J^L : \tilde{f}(k) \;\mapsto\; \tilde{f}(k) - \tfrac{1}{2}\left(k_0^{-2} \sum_{\mu=0}^{3} k_\mu \tilde{f}_\mu(k) \right) k \,, \qquad \tilde{f} \in \tilde{\mathbf{K}}^L, \tag{2.22}$$

obtained by introducing in M^4 the Euclidean metric in which the frame $\{\boldsymbol{e}_\mu \mid \mu = 0,1,2,3\}$ gives rise to an orthonormal basis in that metric – so that the 4-vector $\boldsymbol{k}$ in (1.2) has the Euclidean norm $2^{1/2}k_0$. The J-orthogonal projection of the transformation in (2.5) onto that component is therefore given by

$$\tilde{U}_J(a,\Lambda) : \tilde{f}(k) \;\mapsto\; \exp(ia \cdot k) \left(\Lambda \tilde{f}(\Lambda^{-1}k) - \tfrac{1}{2} |\mathbf{k}|^{-2} \sum_{\mu=0}^{3} k_\mu \Lambda_\mu{}^\nu \tilde{f}_\nu(\Lambda^{-1}k) k \right). \tag{2.23}$$

Consequently, we obtain from (2.11)-(2.13) and (2.21) that

$$\left\langle [\tilde{f}]\middle|[\tilde{f}']\right\rangle = \left\langle [\tilde{U}_J(a,\Lambda)\tilde{f}]\middle|[\tilde{U}_J(a,\Lambda)\tilde{f}']\right\rangle , \qquad [\tilde{f}],[\tilde{f}'] \in \tilde{\mathbf{K}}^L/\tilde{\mathbf{K}}^0 . \tag{2.24}$$

The thus determined representation on $\tilde{\mathbf{K}}^L/\tilde{\mathbf{K}}^0$ is unitarily equivalent to the irreducible and unitary Wigner-type representation defined in (1.13). To establish that, let us replace the linear polarization 4-vectors in (2.14) with the circular polarization 4-vectors

$$\boldsymbol{\varepsilon}_{[\pm 1]}(k) := \mp\tfrac{1}{\sqrt{2}}\left(\boldsymbol{\varepsilon}_{(1)}(k) \pm i\boldsymbol{\varepsilon}_{(2)}(k)\right) , \qquad \boldsymbol{k} = k^\mu \boldsymbol{e}_\mu \in \boldsymbol{V}_0^+ , \tag{2.25}$$

in the complexified Minkowski space $(\mathbf{C}^4,\boldsymbol{\eta})$. It then follows from (1.13) that

$$\Lambda\, \boldsymbol{\varepsilon}_{[\pm 1]}(\Lambda^{-1}k) = e^{\pm i\theta(k,\Lambda)}\, \boldsymbol{\varepsilon}_{[\pm 1]}(k) + \left(\Lambda^0{}_\nu \varepsilon^\nu_{[\pm 1]}(\Lambda^{-1}k)/k_0\right)\boldsymbol{k} , \tag{2.26}$$

where the angle $\theta(k,\Lambda)$ is determined according to (1.6) and (1.14), so that

$$D_{\tilde{k}}(\hat{\Lambda}_k^{-1}\Lambda\hat{\Lambda}_{\Lambda^{-1}k}) = \begin{pmatrix} e^{+i\theta(k,\Lambda)} & 0 \\ 0 & e^{-i\theta(k,\Lambda)} \end{pmatrix} \in S_{\tilde{k}}(2) \tag{2.27}$$

In view of (2.20), we can represent the elements of $\tilde{\mathbf{K}}^L/\tilde{\mathbf{K}}^0$ by the wave functions

$$\tilde{f}_{\boldsymbol{u}}(k) = \tilde{f}^{(1)}(k)\boldsymbol{\varepsilon}_{(1)}(k) + \tilde{f}^{(2)}(k)\boldsymbol{\varepsilon}_{(2)}(k) = \tilde{f}^{[+1]}(k)\boldsymbol{\varepsilon}_{[+1]}(k) + \tilde{f}^{[-1]}(k)\boldsymbol{\varepsilon}_{[-1]}(k) . \tag{2.28}$$

It is then easily seen that the linear map

$$W_0 \;:\; [\tilde{f}_{\boldsymbol{u}}] \mapsto \hat{f} = \begin{pmatrix} \tilde{f}^{(+1)} \\ \tilde{f}^{(-1)} \end{pmatrix} \in \hat{\mathcal{H}}_0 , \qquad [\tilde{f}_{\boldsymbol{u}}] \in \tilde{\mathbf{K}}^L/\tilde{\mathbf{K}}^0 , \tag{2.29}$$

is unitary. Furthermore, on account of (2.25)-(2.29), it establishes the unitary equivalence

$$\tilde{U}_0(a,\Lambda) = W_0^{-1}\hat{U}_0(a,\Lambda)W_0 \;:\; [\tilde{f}_{\boldsymbol{u}}] \mapsto [\tilde{U}_J(a,\Lambda)\tilde{f}_{\boldsymbol{u}}] \tag{2.30}$$

of the Wigner-type representation in (1.13) to the representation obtained when the operators in (2.23), corresponding to $(a,\Lambda) \in \mathrm{ISO}_0(3,1)$, are deemed to act on the Hilbert space $\tilde{\mathbf{K}}^L/\tilde{\mathbf{K}}^0$ of equivalence classes in $\tilde{\mathbf{K}}^L$. This latter representation is defined for all $(a,\Lambda) \in \mathrm{ISO}^\uparrow(3,1)$, and it exhibits the fact that the intrinsic parity of the photon is -1. Indeed, for a frame $\boldsymbol{u}$ obtained by a space reflection the roles of the two circular polarization vectors in (2.25) are reversed, so that clockwise circular polarization becomes counterclockwise, and vice versa.

7.3. Typical Krein Fibres for Spin-1 Bosons

We saw in Sec. 7.1 that the Wigner-type representations for massive and massless particles of spin-1 cannot be readily compared, since they act on wave functions with different numbers of components – namely three components in the massive case, and only two components in the massless case. However, a massive counterpart of the Lorenz space in (2.9) is provided by the *Proca space*

$$\tilde{\mathbf{K}}^{(m)} = \left\{ \tilde{f}^{(m)} \middle| \sum_{\mu=0}^{3} \int_{V_m^+} \left| \tilde{f}_\mu^{(m)}(k) \right|^2 d\Omega_m(k) < +\infty, \; k^\mu \tilde{f}_\mu^{(m)}(k) = 0 \right\}, \tag{3.1}$$

consisting of all 4-tuples of functions defined on the forward mass hyperboloid $\boldsymbol{V}_m^+$ in the Minkowski space $(\mathbf{R}^4, \eta)$ which satisfy the conditions stipulated in (3.1). This space is a Krein space with the following J-inner product and indefinite inner product, respectively,

$$\left(\tilde{f}^{(m)} \middle| \tilde{f}'^{(m)} \right)_{J_m} = \sum_{\mu=0}^{3} \int_{V_m^+} \tilde{f}_\mu^{(m)*}(k) \; \tilde{f}_\mu'^{(m)}(k) \, d\Omega_m(k) = \left\langle \tilde{f}^{(m)} \middle| J_m \tilde{f}'^{(m)} \right\rangle, \tag{3.2}$$

$$\left\langle \tilde{f}^{(m)} \middle| \tilde{f}'^{(m)} \right\rangle = -\int_{V_m^+} \eta^{\mu\nu} \tilde{f}_\mu^{(m)*}(k) \; \tilde{f}_\nu'^{(m)}(k) \, d\Omega_m(k), \tag{3.3}$$

in which a massive counterpart of the pseudo-unitary representation in (2.5) is given by

$$\tilde{U}^{(m)}(a,\Lambda) : \; \tilde{f}_\mu^{(m)}(k) \mapsto \tilde{f}_\mu'^{(m)}(k) = \exp(ia \cdot k) \Lambda_\mu{}^\nu \tilde{f}_\nu^{(m)}(\Lambda^{-1}k) \tag{3.4}$$

The physical equivalence of this representation to the Wigner-type representation in (1.10) can be established adapting the method of Sec. 7.2 to the stability group (1.5) for the present massive case, and therefore using polarization tetrads attached to all the 4-tuples k corresponding to 4-momenta $\boldsymbol{k}$ on the forward mass hyperboloid $\boldsymbol{V}_m^+$:

$$\hat{\boldsymbol{\varepsilon}}_{(\alpha)}(k) = \Lambda_{k/m} \boldsymbol{e}_\alpha, \qquad \boldsymbol{k} = k^\mu \boldsymbol{e}_\mu \in \boldsymbol{V}_m^+, \qquad \alpha = 0,1,2,3. \tag{3.5}$$

Then, upon setting in accordance with (1.10b),

$$\Lambda = \Lambda_{k/m} R_{k,\Lambda} \Lambda^{-1}_{\Lambda^{-1}k/m}, \qquad \Lambda \in \mathrm{ISO}_0(3,1), \tag{3.6}$$

we see that (3.4) can be rewritten in the form

$$\tilde{U}^{(m)}(a,\Lambda) : \; \hat{f}^{(\alpha)}(k) \, \hat{\boldsymbol{\varepsilon}}_{(\alpha)}(k) \mapsto \exp(ia \cdot k) (R_{k,\Lambda} \hat{f})^{(\alpha)} (\Lambda^{-1}k) \, \hat{\boldsymbol{\varepsilon}}_{(\alpha)}(k), \tag{3.7}$$

so that the three spacelike polarization components transform as in (1.10b).[8]

To the elements of the Lorenz space $\tilde{\mathbf{K}}^L$ in (2.12) can be now assigned the functions

$$\tilde{f}^{(m)}(\sqrt{\mathbf{k}^2 + m^2}, k^1, k^2, k^3) = \tilde{f}(|\mathbf{k}|, k^1, k^2, k^3), \qquad \mathbf{k} = k^a \boldsymbol{e}_a, \quad \tilde{f} \in \tilde{\mathbf{K}}^L, \tag{3.8}$$

belonging to the Proca space $\tilde{\mathbf{K}}^{(m)}$. We do not obtain in this manner all the elements of $\tilde{\mathbf{K}}^{(m)}$, but we nevertheless arrive at a linear space which is dense in $\tilde{\mathbf{K}}^{(m)}$ in its majorant topology. Clearly, in the limit $m \to +0$ this dense space merges into the Lorenz space $\tilde{\mathbf{K}}^L$. However, the identification determined by (3.8) is not invariant under Lorentz boosts. This reflects the fact that the two stability groups in (1.5) and (1.6) are distinct, and can be related only by contractions which are not invariant under Lorentz boosts.

Indeed, as mentioned in Sec. 7.1, the contraction of $so(3)$ into $se(2)$ can be carried out by embedding a sphere of large radius R into a Euclidean 3-space, and then considering

[8] On account of the Lorenz-type gauge condition in (3.1), the timelike components of the Proca wave functions are dependent on the spacelike components. When that gauge condition is not imposed, then the representation in (3.4) becomes reducible, since it contains also a spin-0 subrepresentation.

around its "north pole" an area which is so small that it can be deemed to be almost flat, so that those rotations which move the locations of the axis of a Cartesian frame passing through that "pole" to a location within that area can be approximated with translations in the tangent plane at the "north pole." These approximations become exact relationships in the limit $R \to +\infty$, giving rise to the contraction of $so(3)$ into $se(2)$. Detailed studies (cf. Kim and Wigner, 1987) of the zero-mass and infinite-momentum contractions of the stability groups of massive particles confirm that this contraction procedure is dependent on the choice of global Lorentz frame $\boldsymbol{u} = \{\boldsymbol{e}_\mu \,|\, \mu = 0,1,2,3\}$ to the extent that it is not left invariant by boosts of that frame which disturb $\boldsymbol{e}_0$, since it depends on the choice of point on the forward hyperboloid for rest mass $m > 0$ which in the limit $m \to +0$ merges into the vertex of the forward light cone. In semiclassical terms, it depends on the choice of massive particles at rest which in that limit are formally deemed to transcend into mass-0 particles. However, a mass-0 particle has no special-relativistic rest frame, since from the classical point of view it travels with the same speed in relation to any global Lorentz frame. Thus, although the ensuing limit is mathematically well-defined, and is routinely invoked in quantum field theory[9] when dealing with mass-0 particles and their propagators, it is of importance for the later considerations to keep in mind that it represents only a mathematical limit, rather than an operationally realizable physical procedure.

On the other hand, although the concept of 4-velocity has no physical meaning for a photon, the concept of its 4-momentum is physically meaningful. This has been experimentally confirmed by the Compton effect, which is routinely used in photon detection and localization (cf., e.g., Gilmore, 1992), and which establishes 4-momentum conservation in all collision processes between photons and other elementary particles. The localization of photons can be therefore carried out operationally only at the instant of its collision with a massive particle, by observing the recoil of the latter. This indicates that the localizability of photons, as well as of other massless quantum entities, is a physically meaningful concept only when formulated in relation to quantum frames whose constituents are massive.

To arrive at a mathematically precise formulation of the concept of photon localization in relation to quantum frames in the limit $m \to +0$ for the rest masses of their constituents, let us first consider the probability amplitudes

$$f^{(m)}(\zeta) = \tilde{Z}_{\ell,m}^{-1/2} \int_{V_m^+} \exp(-i\zeta \cdot k) \tilde{f}^{(m)}(k)\, d\Omega_m(k) \ , \qquad \zeta = q - i\ell v \ , \tag{3.9}$$

of a spin-1 particle of rest mass $m > 0$ with respect to quantum frames consisting of spin-0 test particles of the same rest mass. The above wave functions satisfy a set of four Klein-Gordon equations and one Lorenz-type gauge condition,

$$\left(\partial^\mu \partial_\mu + m^2\right) f_\nu^{(m)}(\zeta) = 0 \ , \qquad \nu = 0,1,2,3 \ , \tag{3.10a}$$

$$\partial^\mu f_\mu^{(m)}(\zeta) = 0 \ , \qquad \partial_\mu = \partial / \partial q^\mu \ , \tag{3.10b}$$

which is equivalent to the following set of Proca equations (cf., e.g., Itzykson and Zuber,

[9] Cf., e.g., Secs. 3-2-3, 7-3-1, 8-4-2 in (Itzykson and Zuber, 1980). An upper limit of 4×10^{-48} g (i.e., approximately 3×10^{-15} eV) has been thus far experimentally established for the mass of the photon (*ibid.*, p. 138).

1980, p. 135):

$$\partial^{\mu} F_{\mu\nu}^{(m)}(\zeta) + m^2 f_{\nu}^{(m)}(\zeta) = 0 \ , \qquad F_{\mu\nu}^{(m)} = \partial_{\mu} f_{\nu}^{(m)} - \partial_{\nu} f_{\mu}^{(m)} \ . \tag{3.11}$$

Formally, in the limit $m \to +0$, these latter equations merge into the Maxwell equations *in vacuo*, whose general form is in the present context given by

$$\partial^{\mu} F_{\mu\nu}(\zeta) = \partial^{\mu}\partial_{\mu} f_{\nu}(\zeta) - \partial^{\nu}\partial_{\mu} f^{\mu}(\zeta) = 0 \ , \qquad F_{\mu\nu} = \partial_{\mu} f_{\nu} - \partial_{\nu} f_{\mu} \tag{3.12}$$

These latter equations are satisfied by the following four-component wave functions

$$f(\zeta) = (2\pi)^{-3/2} \int_{V_0^+} \exp(-i\zeta \cdot k) \tilde{f}(k)\, d\Omega_0(k) \ , \qquad \zeta = q - i\ell v \ , \tag{3.13}$$

defined in accordance with (3.9) from the elements of the Krein space $\tilde{\mathbf{K}}$ introduced in the last section. Moreover, since the 4-vector variable k in (3.13) lies on the light cone, the components of these wave functions satisfy the four wave equations

$$\partial^{\mu}\partial_{\mu} f_{\nu}(\zeta) = 0 \ , \qquad \nu = 0,1,2,3 \ , \tag{3.14}$$

which are normally satisfied by classical electromagnetic 4-potentials in the Lorenz gauge.

For any $\ell > 0$ and $m > 0$ the value of the normalization constant in (3.9) is as stipulated in (4.2.12). In the present context, that particular value is uniquely determined by the requirement that, in accordance with (4.2.2) and (4.2.3),

$$\left(f^{(m)} \middle| f'^{(m)}\right)_{J_m} := \sum_{\mu=0}^{3} \int f_{\mu}^{(m)*}(\zeta)\ f_{\mu}'^{(m)}(\zeta)\, d\Sigma(\zeta) = \left(\tilde{f}^{(m)} \middle| \tilde{f}'^{(m)}\right)_{\tilde{J}_m} \ , \tag{3.15}$$

$$\left\langle f^{(m)} \middle| f'^{(m)} \right\rangle := -\int \eta^{\mu\nu} f_{\mu}^{(m)*}(\zeta)\ f_{\nu}'^{(m)}(\zeta)\, d\Sigma(\zeta) = \left\langle \tilde{f}^{(m)} \middle| \tilde{f}'^{(m)} \right\rangle \ . \tag{3.16}$$

However, that value of the normalization constant in (3.9) diverges in the limit $m \to +0$:

$$\tilde{Z}_{\ell,m} = 8\pi^4 \, \mathrm{K}_2(2\ell m)/\ell m^2 = (4\pi^4/\ell^3 m^4) + O(m^{-2}) \xrightarrow[m \to +0]{} +\infty \ . \tag{3.17}$$

The same is true of the normalization constant

$$\hat{Z}_{\ell,m} = \mathrm{K}_2(2\ell m)/m\,\mathrm{K}_1(2\ell m) = (1/\ell m^2) + O(m^{-1}) \xrightarrow[m \to +0]{} +\infty \ , \tag{3.18}$$

which appears in the following counterparts of (5.4.7) for the above two inner products:

$$\left(f^{(m)} \middle| f'^{(m)}\right)_{J_m} = i\hat{Z}_{\ell,m} \sum_{\mu=0}^{3} \int f_{\mu}^{(m)*}(\zeta)\ \overleftrightarrow{\partial}_{\nu} f_{\mu}'^{(m)}(\zeta)\, d\sigma^{\nu}(q) d\Omega(v) \ , \tag{3.19}$$

$$\left\langle f^{(m)} \middle| f'^{(m)} \right\rangle = -i\hat{Z}_{\ell,m} \int \eta^{\mu\nu} f_{\mu}^{(m)*}(\zeta)\ \overleftrightarrow{\partial}_{\lambda}\ f_{\nu}'^{(m)}(\zeta)\, d\sigma^{\lambda}(q)\, d\Omega(v) \ . \tag{3.20}$$

These mathematical facts reflect the earlier mentioned impossibility of localizing in a physically sustained manner massless objects, which in the special relativistic regime move away with the same speed (namely, in classical terms, the speed c of light in vacuum) from other massive objects in *any* state of motion. Consequently, it cannot be expected on physical

grounds that Poincaré systems of covariance giving rise to normalized probability measures along reference hypersurfaces in Minkowski space can be formulated in relation to frames built of massless objects; neither can this be expected on mathematical grounds, given the fact that the contraction leading from the stability group (1.5) for massive particles to the stability group (1.6a) for massless particles is not invariant under Lorentz boosts.

Let us therefore denote by $\mathbf{K}$ the Krein space consisting of all the wave functions in (3.13) corresponding to all $\tilde{f} \in \tilde{\mathbf{K}}$, and with J-inner products and indefinite inner products defined by the following limits after the identifications in (3.8) are carried out (with $\tilde{f}$ and $\tilde{f}^{(m)}$ not necessarily in the Lorenz gauge):

$$\left(f\middle|f'\right)_J = \lim_{m\to 0}\left(f^{(m)}\middle|f'^{(m)}\right)_{J_m} = \left(\tilde{f}\middle|\tilde{f}'\right)_{\tilde{J}}\,, \tag{3.21}$$

$$\left\langle f\middle|f'\right\rangle = \lim_{m\to 0}\left\langle f^{(m)}\middle|f'^{(m)}\right\rangle = \left\langle \tilde{f}\middle|\tilde{f}'\right\rangle\,. \tag{3.22}$$

Of course, the two inner products on the left-hand sides of (3.21) and (3.22) could be directly identified with those in (2.1) and (2.2) for the Krein space $\tilde{\mathbf{K}}$ introduced in the last section. However, with the above definitions, they can be formally expressed in the form

$$\left(f\middle|f'\right)_J = \sum_{\mu=0}^{3}\int f_\mu^*(\zeta)\; f'^{\mu}(\zeta)\, d\tilde{\Sigma}(\zeta) = i\sum_{\mu=0}^{3}\int f_\mu^*(\zeta)\; \overleftrightarrow{\partial}_\nu f'^{\mu}(\zeta)\, d\hat{\sigma}^\nu(q) d\Omega(v), \tag{3.23}$$

$$\left\langle f\middle|f'\right\rangle = -\int \eta^{\mu\nu} f_\mu^*(\zeta)\; f'_\nu(\zeta)\, d\tilde{\Sigma}(\zeta) = -i\int \eta^{\mu\nu} f_\mu^*(\zeta)\; \overleftrightarrow{\partial}_\lambda f'_\nu(\zeta)\, d\hat{\sigma}^\lambda(q)\, d\Omega(v), \tag{3.24}$$

where the limits in (3.21) and (3.22) have been formally absorbed into the infinite renormalization of the measure elements of the integrals within those respective equations:

$$d\tilde{\Sigma}(\zeta) = (2\pi)^3 \lim_{m\to 0} \tilde{Z}_{\ell,m}^{-1}\, d\Sigma(\zeta)\;, \qquad d\hat{\sigma}^\nu(q) = \lim_{m\to 0}\left(\hat{Z}_{\ell,m}\middle/\tilde{Z}_{\ell,m}\right) d\sigma^\nu(q) \tag{3.25}$$

Indeed, the direct computation of the integrals in (3.19) and (3.20) with the wave functions in (3.9) replaced with those in (3.13) confirms that these latter integrals diverge. Hence, this limiting procedure is unavoidable for endowing such formal integrals with a well-defined mathematical meaning.

Such infinite renormalizations can be interpreted as wave function amplitude renormalizations. They are characteristic of zero mass, as they belong to the category of infrared renormalizations carried out in conventional QED (cf. Itzykson and Zuber, 1980, Sec. 5-1-7), as well as in non-Abelian quantum gauge field theories (cf. Curci and Ferrari, 1976). However, as opposed to these latter cases, the renormalization symbolically depicted by (3.25) is the outcome of well-defined limits that require no *ad hoc* subtractions, and it does not rely on formal regularization procedures. As such, it is a renormalization in the strict mathematical sense of the word; namely, it represents a change of norm resulting from the multiplication of an inner product by a positive number, followed by the subsequent taking of the well-defined limits in (3.17) and (3.18).

The pseudo-unitary representation in (2.5) of momentum wave functions is transformed by (3.13) into the following pseudo-unitary representation in the Krein space $\mathbf{K}$:

$$U(a,\Lambda)\;:\; f_\mu(q - i\ell v) \;\mapsto\; f'_\mu(q - i\ell v) = \Lambda_\mu{}^\nu f_\nu(\Lambda^{-1}(q-a) - i\ell\Lambda^{-1}v)\;. \tag{3.26}$$

This representation of the orthochronous Poincaré group $\mathrm{ISO}^{\uparrow}(3,1)$ mediates the construction of the single-photon bundle

$$\mathbf{E}^{\uparrow} = P^{\uparrow}\mathbf{M}(\boldsymbol{g}^{\mathrm{L}}) \times_{\mathbf{G}} \mathbf{K} \ , \qquad \mathbf{G} = \mathrm{ISO}^{\uparrow}(3,1) \ , \tag{3.27}$$

associated with the principal Poincaré frame bundle $P^{\uparrow}\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$, whose typical fibre is diffeomorphic to $\mathrm{ISO}^{\uparrow}(3,1)$, and therefore incorporates left-handed as well as right-handed spatial frames. The theory of propagators for the parallel transport of single photon states within this fibre bundle can be based (Prugovečki, 1988b) on the 4×4 matrix-valued function

$$iD^{(+)}(\zeta'';\zeta')\eta_{\mu\nu} = \eta_{\mu\nu}\int_{V_0^+} \exp\left[-i(\zeta''-\bar{\zeta}')\cdot k\right] d\Omega_0(k) \ , \tag{3.28}$$

obtained as the zero-mass limit of reproducing kernels in the corresponding spaces $\mathbf{K}^{(m)}$ of wave functions (3.9). The above k-integration can be performed explicitly, with the result

$$D^{(+)}(q''-i\ell v'';q'-i\ell v') = -2\pi/\left(q''-q'-i\ell(v''+v')\right)^2 \ , \qquad v',v'' \in \boldsymbol{V}^+ \ . \tag{3.29}$$

The above function displays the following formal reproducibility properties

$$\begin{aligned} iD^{(+)}(\zeta'';\zeta') &= -\int_{\Sigma} D^{(+)}(\zeta'';\zeta) D^{(+)}(\zeta;\zeta') d\tilde{\Sigma}(\zeta) \\ &= -i\int_{\Sigma} D^{(+)}(\zeta'';\zeta) \overleftrightarrow{\partial}_{\lambda} D^{(+)}(\zeta;\zeta') d\hat{\sigma}^{\lambda}(q)\, d\Omega(v) \ , \end{aligned} \tag{3.30}$$

if the q-integration is interpreted in accordance with (3.25), i.e., by taking, as in (3.21) and (3.22), zero-mass limits of the functions for the corresponding reproducing kernels in $\mathbf{K}^{(m)}$ Within the fibres above each $x \in \mathbf{M}$ in the single-photon bundle $\mathbf{E}^{\uparrow}$ in (3.27) this function then gives rise to the Poincaré gauge-invariant functions

$$D_x^{(+)}(\zeta'';\zeta') = -i\int_{V_0^+} \exp\left[-i(\zeta''-\bar{\zeta}')\cdot k\right] d\Omega_0(k) = 2\pi i/\left(\zeta''-\bar{\zeta}'\right)^2 \ , \tag{3.31}$$

which, by an extrapolation of the interpretation of (4.2.13b) and (4.2.17), can be viewed as local photon fluctuation amplitudes.

7.4. Gupta-Bleuler Bundles for Multi-Photon States

In the momentum representation of multi-photon states, the Gupta-Bleuler space is usually defined as the Krein space (cf. Bogolubov *et al.*, 1990, p. 420)

$$\tilde{\mathcal{K}} = \tilde{\bigoplus}_{n=0}^{\infty} \tilde{\mathcal{K}}_n \ , \qquad \tilde{\mathcal{K}}_n = \tilde{\mathbf{K}} \underset{S}{\otimes} \cdots \underset{S}{\otimes} \tilde{\mathbf{K}} \ . \tag{4.1}$$

However, the customary[10] Fock-type representation $\tilde{\boldsymbol{U}}(a,\Lambda)$ in such Gupta-Bleuler spaces as the one above does not remain bounded under Lorentz transformations that contain Lorentz boosts. Indeed, $\tilde{\boldsymbol{U}}(a,\Lambda)$ is constructed from the momentum space representation

[10] Cf., e.g., (Schweber, 1961), Sec. 9b. The fact the representation in (4.2) contains unbounded operators has remained largely unnoticed in the mathematically rigorous literature on the Gupta-Bleuler formalism. Only Strocchi (1978) allows for the possibility of J-unboundedness in his extension of the Wightman axioms to quantum gauge fields describing massless particles.

in (2.5) by taking a direct sum corresponding to the one in (4.1). When Λ is a Lorentz boost, such as the one in (4.4.13), then we have the inequalities

$$e^{-n\omega}\left\|\tilde{f}_n\right\|_{\tilde{J}} \leq \left\|\tilde{U}(a,\Lambda)\tilde{f}_n\right\|_{\tilde{J}} \leq e^{n\omega}\left\|\tilde{f}_n\right\|_{\tilde{J}}\,, \qquad \tilde{f}_n \in \tilde{\mathcal{K}}_n\,. \tag{4.2}$$

By considering in (2.6), for such Λ, sequences of J-normalized wave functions of increasingly small support around a given value of $k \neq 0$ on the forward light-cone, it can be established with the help of (4.4.13) that the above lower and upper bounds are actually infima and suprema, respectively, if $\tilde{f}_n$ is allowed to vary over all of n-photon wave functions. This implies that, whereas the restriction of $\tilde{U}(a,\Lambda)$ to the n-photon space in (4.2) is indeed J-bounded, that is not the case for the Gupta-Bleuler space in (4.1), constructed by taking the J-direct sum of such spaces for $n = 0,1,2,\ldots$, since in that case we have

$$\left\|\tilde{U}(a,\Lambda)\tilde{f}\right\|_J^2 = \sup_{\|\tilde{f}\|_J=1}\sum_{n=0}^{\infty} e^{2n\omega}\left\|\tilde{f}_n\right\|_{\tilde{J}}^2 = +\infty\,, \qquad \tilde{f} = \tilde{\bigoplus}_{n=0}^{\infty}\tilde{f}_n \tag{4.3}$$

This gives rise to mathematical difficulties, since the Gupta-Bleuler indefinite inner product in (4.1), which is derived from (2.2), does not provide a separable topology. Hence, a Gupta-Bleuler space with only such a Lorentz invariant but indefinite inner product would not contain a countable dense set that would enable expansions in countable bases – so that it could not be used to give meaning to infinite series defining families of coherent states for photons that would remain invariant under Poincaré transformations. Thus we are left with the only option of adopting a family of inequivalent majorant topologies corresponding to various J-inner products. A separate specification of a topology of this type would be then required each time an infinite superposition of various n-photon state vectors is considered for which the sum in (4.3) diverges in an already specified J-inner product.

This implies that the claim that "Lorentz covariance is manifest in the Gupta-Bleuler formalism" (Kaku, 1991, p. 13), frequently made in physics literature, cannot be maintained at a mathematically rigorous level as long as the Gupta-Bleuler formalism is intrinsically based on Krein spaces. Indeed, for such a manifest covariance to exist, it would be necessary to have a one-to-one correspondence between *all* the Gupta-Bleuler state vectors in relation to any two Lorentz frames in a state of relative motion. Such a one-to-one correspondence exists between the state vectors within the dense *core* of the Gupta-Bleuler space in (4.1) which consists of all *finite* linear superpositions of n-photon state vectors for various values of n. However, that one-to-one linear correspondence cannot be extended to the *entire* Gupta-Bleuler space in a J-independent manner by taking closures of the operators in (4.3). Indeed, if such closures in the majorant topology determined by a given J-inner product existed everywhere within the completion in a given J-norm of the core Gupta-Bleuler space, they would coincide with graph-closures of the unbounded operators in (4.3). This would contradict the closed-graph theorem in the theory of topological vector spaces in general, and Hilbert spaces in particular, since that theorem stipulates that such everywhere defined graph-closures exist *only* for *bounded* operators (cf., e.g., Prugovečki, 1981, p. 210; Bogolubov *et al.*, 1990, p. 24).

This situation does not represent a mere mathematical technicality. Rather, it is a reflection of the physical nature of massless gauge fields, which affects the manner in which

such fields are coupled to massive quantum fields. Clearly, its origins lie in the localizability features of massless particles and fields, whose peculiar problems we encountered in the last section. These problems fundamentally affect the manner in which Gupta-Bleuler bundles can be constructed in the quantum-geometric framework for gauge fields.

In the case of quantum-geometric electromagnetic fields, one way of handling such problems is to let these fields act in triply-fibrated bundles[11] based on indefinite metrics, and constructed in accordance with ideas suggested by geometric treatment of quantum Yang-Mills fields, discussed later in this chapter. A mathematically simpler approach can be, however, arrived at by adopting as a typical fibre the following algebraic[12] direct sum

$$\mathcal{K} = \breve{\bigoplus}_{n=0}^{\infty} \mathcal{K}_n \quad , \qquad \mathcal{K}_n = \mathbf{K} \underset{S}{\otimes} \cdots \underset{S}{\otimes} \mathbf{K} \ , \tag{4.4}$$

of n-photon spaces $\mathcal{K}_n$, obtained by taking symmetric tensor products of the Krein space $\mathbf{K}$ consisting of the photon wave functions in (3.13). This topologically incomplete typical fibre with indefinite metric is left invariant by all the operators

$$\boldsymbol{U}(a,\Lambda) = \breve{\bigoplus}_{n=0}^{\infty} U(a,\Lambda)^{\otimes n}, \qquad (a,\Lambda) \in \mathrm{ISO}^{\uparrow}(3,1) \ , \tag{4.5}$$

constructed from the operators belonging to the representation in (3.26). Consequently, the resulting *Gupta-Bleuler multi-photon bundle*

$$\mathcal{E}^{\uparrow} = P^{\uparrow}\mathbf{M}(\boldsymbol{g}^{\mathrm{L}}) \times_{\mathbf{G}} \mathcal{K} \ , \qquad \mathbf{G} = \mathrm{ISO}^{\uparrow}(3,1) \ , \tag{4.6}$$

is well-defined, and the soldering maps determined in it in accordance with (4.1.3),

$$\boldsymbol{\sigma}_x^{\boldsymbol{u}} : \ \boldsymbol{\Psi}_x \ \mapsto \ \Psi_x \in \mathcal{K} \ , \quad \boldsymbol{\Psi} \in \mathcal{K}_x \ , \quad \boldsymbol{u} = (\boldsymbol{a}, \boldsymbol{e}_i) \in \boldsymbol{\Pi}^{-1}(x) \subset P^{\uparrow}\mathbf{M}(\boldsymbol{g}^{\mathrm{L}}) \ , \tag{4.7}$$

assign to each local n-photon state vectors $\boldsymbol{\Psi}_{n;x} \in \mathcal{K}_{n;x}$ a coordinate wave function $\Psi_{n;x} \in \mathcal{K}_n$. This implies that the multi-photon coordinate mappings

$$\boldsymbol{\Psi}_{n;x} \ \mapsto \ \Psi_{n;x}(\zeta_1, i_1, \ldots, \zeta_n, i_n) \ , \qquad i_1, \ldots, i_n = 0,1,2,3, \tag{4.8}$$

are also well-defined, and can be used to introduce within the fibres of the Gupta-Bleuler multi-photon bundles in (4.6) the photon creation operators defined by means of local photon fluctuation amplitudes in (3.31) as follows:

$$\begin{aligned} &\left(A_i^{(+)}(x;\zeta)\Psi_{n;x}\right)_{n+1}(\zeta_1, i_1, \ldots, \zeta_{n+1}, i_{n+1}) \\ &= -i(n+1)^{-1/2} \sum_{r=1}^{n+1} \eta_{ii_r} D_x^{(+)}(\zeta_r;\zeta)\, \Psi_{n;x}(\zeta_1, i_1, \ldots, \zeta_{r-1}, i_{r-1}, \zeta_{r+1}, i_{r+1}, \ldots, \zeta_{n+1}, i_{n+1}) \end{aligned} \tag{4.9}$$

The photon annihilation operators adjoint to these operators then satisfy the relations

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[11] Cf. Sec. 9.4 in (Prugovečki, 1992). The present approach is, however, mathematically more straightforward, and makes easier the treatment of states which do not satisfy Gupta-Bleuler subsidiary condition in (4.12).

[12] In an algebraic direct sum of infinitely many vector spaces, the sequences of vectors from the constituent spaces, which represent elements of that algebraic direct sum, contain only finite numbers of non-zero components. Thus, an *algebraic* direct sum of an infinite number of Hilbert spaces is only a pre-Hilbert space; whereas the corresponding *Hilbert* direct sum is equal to its metric completion (cf., e.g., Prugovečki, 1981, pp. 132-136).

$$\left(A_i^{(-)}(x;\zeta)\Psi_{n;x}\right)_{n-1}(\zeta_1,i_1,\ldots,\zeta_{n-1},i_{n-1}) = \sqrt{n}\,\Psi_{n;x}(\zeta,i,\zeta_1,i_1,\ldots,\zeta_{n-1},i_{n-1}) \ , \quad (4.10)$$

$$\left[A_i^{(-)}(x;\zeta),A_{i'}^{(+)}(x;\zeta')\right] = -i\eta_{ii'}\,D_x^{(+)}(\zeta;\zeta') \ , \quad \left[A_i^{(\pm)}(x;\zeta),A_{i'}^{(\pm)}(x;\zeta')\right] = 0 \ . \quad (4.11)$$

In terms of them, the well-known Gupta-Bleuler subsidiary condition (cf., e.g., Schweber, 1961, p. 246) can be expressed in the present context as follows:

$$\partial^i A_i^{(-)}(x;\zeta)\Psi_x = \mathbf{0} \ , \qquad \Psi_x \in \mathcal{K}_x^L \ , \qquad \partial^i = \eta^{ij}\,\partial/\partial q^j \ . \quad (4.12)$$

This subsidiary condition singles out within each fibre $\mathcal{K}_x$ all those multi-photon state vectors which are constructed exclusively from single-photon state vectors in the Lorenz gauge – namely from single-photon wave functions which obey $\partial^\nu f_\nu(\zeta) \equiv 0$ in (3.14).

The photon creation operators in (4.9) can be used to construct indefinite-metric coherent states which are formally analogous to the ones in (5.3.8):

$$\Phi_f = \exp\left[-\tfrac{1}{2}\langle f|f\rangle - \int f^i(\zeta)A_i^{(+)}(x;\zeta)\,d\tilde{\Sigma}(\zeta)\right]\Psi_{0;x} \ , \quad f \in \mathbf{K}_x \quad (4.13)$$

However, in order to determine the convergence of the counterpart of the power series in (5.3.8) for the above exponential one requires the use of majorant topologies – which, as we saw earlier, give rise to frame dependent completions of the fibre $\mathcal{K}_x$. Moreover, when the Berezin method of functional integration of Sec. 5.3 is adapted to (4.13), the presence of the timelike polarization modes in (4.13) leads to divergencies, and compensating factors have to be introduced for their elimination.[13] These extraneous factors destroy the manifest Poincaré covariance of the formalism. Furthermore, in the end it becomes necessary to carry out in Gupta-Bleuler subfibres a decomposition

$$\Phi_f = \Phi_{f^{\mathrm{T}}} + \Phi_f^{\perp} \in \mathcal{K}_x^L \ , \qquad \partial^i A_i^{(-)}(x;\zeta)\Phi_f = \mathbf{0} \quad (4.14)$$

corresponding to a similar decomposition of the single-photon state vectors given by

$$f = f^{\mathrm{T}} + f^{\perp} \in \mathbf{K}_x^L \ , \quad f^{\mathrm{T}} = f^{(1)} + f^{(2)} \in \mathbf{K}_{x,J}^{\mathrm{T}} \ , \quad f^{\perp} = f^{(0)} + f^{(3)} \in \mathbf{K}_x^0 \ , \quad (4.15a)$$

$$f^{(s)}(\zeta) = (2\pi)^{-3/2}\int_{V_0^+}\exp(-i\,\zeta\cdot k)\tilde{f}^{(s)}(k)\,\boldsymbol{\varepsilon}_{(s)}(k)\,d\Omega_0(k) \ , \quad s = 0,1,2,3 \ , \quad (4.15b)$$

– where no summation over s is to be carried out on the right-hand side of (4.15b). This decomposition supplies the physically meaningful part of the coherent states in (4.13),

$$\Phi_{f^{\mathrm{T}}} = \exp\left[-\tfrac{1}{2}\left\langle f^{\mathrm{T}}\middle|f^{\mathrm{T}}\right\rangle + \int\left[f^{(1)}(\zeta)+f^{(2)}(\zeta)\right]\cdot A^{(+)}(x;\zeta)\,d\tilde{\Sigma}(\zeta)\right]\Psi_{0;x} \ , \quad (4.16)$$

which is determined exclusively by the transversal polarization modes, and a remainder

$$\Phi_f^{\perp} = \sum_{n=1}^{\infty}(n!)^{-1}\left(\int\left[f^{(0)}(\zeta)+f^{(3)}(\zeta)\right]\cdot A^{(+)}(x;\zeta)\,d\tilde{\Sigma}(\zeta)\right)^n\Phi_{f^{\mathrm{T}}} \ , \quad (4.17)$$

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[13] Cf. Eq. (3.28) in Sec. 9.3 of (Prugovečki, 1992). Exactly the same situation is encountered in the treatment of coherent states within the conventional Gupta-Bleuler framework (cf. Gomatam, 1971).

which contains (unphysical) timelike and longitudinal polarization modes. It is for the definition of this remainder that the majorant topology determined by the J-inner product that implicitly occurs in (4.15a) is required, but which has no physical significance, and is of no mathematical consequence either, since it is easily seen that

$$\left\langle \boldsymbol{\Phi}_{\boldsymbol{f}^{\mathrm{T}}} \middle| \boldsymbol{\Phi}_{\boldsymbol{f}^{\mathrm{T}}} \right\rangle = \left\langle \boldsymbol{\Phi}_{\boldsymbol{f}} \middle| \boldsymbol{\Phi}_{\boldsymbol{f}} \right\rangle = 1 \quad , \qquad \left\langle \boldsymbol{\Phi}_{\boldsymbol{f}}^{\perp} \middle| \boldsymbol{\Phi}_{\boldsymbol{f}}^{\perp} \right\rangle = \left\langle \boldsymbol{\Phi}_{\boldsymbol{f}}^{\perp} \middle| \boldsymbol{\Phi}_{\boldsymbol{f}^{\mathrm{T}}} \right\rangle = 0 \ . \tag{4.18}$$

In view of all this, we can carry out Berezin-type functional integration within the physical subfibre of each fibre $\mathcal{K}_x$ by adopting within the single-photon fibre $\mathbf{K}_x$ a J-orthonormal basis consisting of state vectors having only one non-zero polarization component. We can then expand any $\boldsymbol{f} \in \mathbf{K}_x$ with respect to such a basis,

$$\boldsymbol{f} = \sum_{s=0}^{3} \sum_{\alpha=1}^{\infty} z_{\alpha}^{(s)} \boldsymbol{w}_{\alpha}^{(s)} \in \mathbf{K}_x^{(0)} \underset{J}{\oplus} \mathbf{K}_x^{(1)} \underset{J}{\oplus} \mathbf{K}_x^{(2)} \underset{J}{\oplus} \mathbf{K}_x^{(3)} \ , \qquad \boldsymbol{w}_{\alpha}^{(s)} \in \mathbf{K}_x^{(s)} \ , \tag{4.19}$$

and introduce the annihilation and creation operators

$$a_{\alpha}^{(s)} = -\int \overline{\boldsymbol{w}}_{\alpha}^{(s)}(\zeta) \cdot \boldsymbol{A}^{(-)}(x;\zeta) d\tilde{\Sigma}(\zeta) \ , \quad a_{\alpha}^{(s)\dagger} = -\int \boldsymbol{w}_{\alpha}^{(s)}(\zeta) \cdot \boldsymbol{A}^{(+)}(x;\zeta) d\tilde{\Sigma}(\zeta) \ , \tag{4.20}$$

which, on account of (4.11), satisfy the following commutation relations:

$$[a_{\alpha}^{(r)}, a_{\beta}^{(s)\dagger}] = -\delta_{\alpha\beta} \eta^{rs}, \qquad [a_{\alpha}^{(r)}, a_{\beta}^{(s)}] = [a_{\alpha}^{(r)\dagger}, a_{\beta}^{(s)\dagger}] = 0 \ . \tag{4.21}$$

Then the physical components of the coherent states in (4.13) can be expressed in the form

$$\boldsymbol{\Phi}_{\boldsymbol{f}^{\mathrm{T}}} = \exp\left[\sum_{\alpha=1}^{\infty} \sum_{s=1}^{2} \left(-\tfrac{1}{2} \left| z_{\alpha}^{(s)} \right|^2 + z_{\alpha}^{(s)} a_{\alpha}^{(s)\dagger} \right) \right] \boldsymbol{\Psi}_{0;x} \ \in \ \mathcal{K}_{x,J}^{\mathrm{T}} \ , \tag{4.22}$$

where the convergence of the power series to the above exponential can be taken to be in the Gupta-Bleuler metric, since according to (2.16) and (3.16)

$$\langle \boldsymbol{f} | \boldsymbol{f}' \rangle = \sum_{s=1}^{2} \int f^{(s)*}(\zeta) f'^{(s)}(\zeta) d\tilde{\Sigma}(\zeta) \ , \qquad \boldsymbol{f}, \boldsymbol{f}' \in \mathbf{K}_{x,J}^{\mathrm{T}} \ , \tag{4.23}$$

so that this indefinite metric becomes positive definite when it is restricted to multi-photon states containing only transversal polarization modes. Consequently, we can now apply the Berezin method in the same manner as in (5.3.27), to derive the resolution of the identity

$$\int_{\mathbf{K}_{x,J}^{\mathrm{T}}} \left| \boldsymbol{\Phi}_{\boldsymbol{f}} \right\rangle d\boldsymbol{f} d\bar{\boldsymbol{f}} \left\langle \boldsymbol{\Phi}_{\boldsymbol{f}} \right| = \mathrm{w}\text{-}\lim_{n \to \infty} \pi^{-2n} \int_{\mathbf{R}^{4n}} \left| \boldsymbol{\Phi}_{\hat{z}_n} \right\rangle \mathrm{d}\hat{z}_n \mathrm{d}\hat{z}_n^* \left\langle \boldsymbol{\Phi}_{\hat{z}_n} \right| \ = \ \mathbf{1}_{x,J}^{\mathrm{T}} \ , \tag{4.24a}$$

$$\boldsymbol{\Phi}_{\hat{z}_n} = \exp\left[\sum_{\alpha=1}^{n} \sum_{s=1}^{2} \left(-\tfrac{1}{2} \left| z_{\alpha}^{(s)} \right|^2 + z_{\alpha}^{(s)} a_{\alpha}^{(s)\dagger} \right) \right] \boldsymbol{\Psi}_{0;x} \ \in \ \mathcal{K}_{x,J}^{\mathrm{T}} \ , \tag{4.24b}$$

within the physical subfibre of the closure $\mathcal{K}_{x,J}$ of $\mathcal{K}_x$ in the majorant topology to which the J-inner product associated with a given Poincaré frame $\boldsymbol{u}$ gives rise. This physical subfibre corresponds to the first term in the J-orthogonal decomposition

$$\mathcal{K}_{x,J}^{L} = \mathcal{K}_{x,J}^{\mathrm{T}} \underset{J}{\oplus} \mathcal{K}_{x,J}^{0} \ , \qquad \mathbf{K}_x^{L} = \mathbf{K}_{x,J}^{\mathrm{T}} \underset{J}{\oplus} \mathbf{K}_x^{0} \ , \tag{4.25}$$

of the closure of the Lorenz subfibre in $\mathcal{K}_x$ with respect to that topology.

We note that although the decompositions in (4.25) are not left invariant by Lorentz boosts, the majorant topologies on all the physical subfibres are equivalent. Indeed, those subfibres consist of states containing only transversal polarization modes, so that on them all the J-inner products that give rise to majorant topologies coincide with the Gupta-Bleuler inner product, which is Poincaré invariant. Clearly, therefore, the earlier discussed topological difficulties, which are reflected in the unboundedness of the operators in (4.5) that follows from (4.3), stem from the behavior of the majorant topologies on the unphysical component in the J-orthogonal decomposition of the typical Lorenz fibre $\mathcal{K}^L$ into a transversal (physical) component $\mathcal{K}^T$ and a null (unphysical) component $\mathcal{K}^0$.

By executing the decomposition in (4.14), and then adopting the resolution of the identity in (4.24) as the effective basis of *physical* multi-photon frames, the source of these mathematical difficulties are removed from the end product of present Gupta-Bleuler framework, without any loss to the physically significant aspects of this framework. Moreover, on account of (2.30), the equivalence classes to which those physical Gupta-Bleuler states give rise when they are identified modulo null vectors, transform in a covariant manner under changes of Poincaré moving frames. Consequently, the present adaptation of the Gupta-Bleuler framework is Poincaré gauge covariant at the physically significant level.

7.5. Quantum-Geometric Propagation in Gupta-Bleuler Bundles

The parallel transport within the Gupta-Bleuler bundle (4.6) gives rise to the operators

$$\tau_\gamma(x'',x') = \left(\sigma^{u''}_{x''}\right)^{-1} \circ \sigma^{u'}_{x'} \; : \quad \mathcal{K}_{x'} \rightarrow \mathcal{K}_{x''} , \qquad u'' = \tau_\gamma(x'',x')u' , \tag{5.1}$$

for the parallel transport along a smooth curve γ joining x' and x'' in the same manner as in any other bundle associated with the Poincaré frame bundle $P^\uparrow\mathbf{M}(\boldsymbol{g}^L)$. From this the values

$$\boldsymbol{\nabla} = \partial - i\tilde{\boldsymbol{\theta}}^i_s \boldsymbol{P}_{i;s} + \tfrac{i}{2}\tilde{\boldsymbol{\omega}}^s_{jk}\boldsymbol{M}^{jk}_s \tag{5.2}$$

of the quantum connection associated with any section s of $P^\uparrow\mathbf{M}(\boldsymbol{g}^L)$ can be obtained basically as in Sec. 4.4. Indeed, the only fundamental difference between these two cases emerges from the values of the generators for infinitesimal Poincaré transformations, since those in (5.2) are peculiar to the parallel transport of local photon states. The values of these infinitesimal generators can be derived from the expressions

$$\boldsymbol{P}_{j;s} = -i\int \eta^{ik} A^{(+)}_i(x;\zeta)\partial_j A^{(-)}_k(x;\zeta)\, d\tilde{\Sigma}(\zeta) , \qquad \partial_j = \partial/\partial q^j , \tag{5.3}$$

for the infinitesimal generators of spacetime translations with respect to any given choice $\boldsymbol{s}$ of Poincaré moving frame. These expressions can be easily verified by applying the operators they define to n-photon state vectors $\Psi_{n;x} \in \mathcal{K}_{n;x}$, and then using (4.9) and (4.10).

To arrive at a form of these operators which reveals the presence of a stress-energy tensor for the quantum-geometric electromagnetic field, we shall proceed in basically the same manner as in the derivation of (5.4.14) from (5.4.6). Thus, upon adopting the form (3.24) of the inner product in the single-photon fibres, and then choosing hypersurfaces Σ

corresponding to $q^0 = 0$ values with respect to the Poincaré frames in $\boldsymbol{s}$, we can recast (5.3) in the form

$$\boldsymbol{P}_{j;\boldsymbol{s}} = \int_{\Sigma} d\hat{\sigma}^0(\mathbf{q}) d\Omega(v)\, \eta^{ik}\left[A_i^{(+)}(x;\zeta)\,\partial_0\partial_j A_k^{(-)}(x;\zeta) - \partial_0 A_i^{(+)}(x;\zeta)\,\partial_j A_k^{(-)}(x;\zeta)\right] . \quad (5.4)$$

On account of (3.14), the components of all the n-photon wave functions in (4.8) satisfy the wave equation in each one of their q-variables. Consequently, the photon annihilation and creation operators satisfy the following four wave equations:

$$\partial^j\partial_j A_i^{(\pm)}(x;\zeta) = 0 \ , \qquad i = 0,1,2,3 \ . \quad (5.5)$$

We can use these equations to remove the second derivative in q^0 from the square bracket in the expression (5.4) for $\boldsymbol{P}_{0;\boldsymbol{s}}$. Upon carrying out an integration by parts[14] with respect to the $\mathbf{q}$-variable we obtain

$$\boldsymbol{P}_{0;\boldsymbol{s}} = -\int_{q^0=0} d\hat{\sigma}^0(\mathbf{q}) d\Omega(v) \sum_{j=0}^{3} \partial_j A_i^{(+)}(x;\zeta)\, \eta^{ik}\partial_j A_k^{(-)}(x;\zeta) \ . \quad (5.6)$$

On the other hand, according to the type of argument used in deriving (5.4.10), we have

$$\int_{q^0=0} d\hat{\sigma}^0(\mathbf{q}) d\Omega(v) \sum_{j=0}^{3} \partial_j A_i^{(\pm)}(x;\zeta)\, \eta^{ik}\partial_j A_k^{(\pm)}(x;\zeta) = 0 \ , \quad (5.7)$$

so that upon introducing the photon frame field whose components are given by

$$A_j(x;\zeta) = A_j^{(+)}(x;\zeta) + A_j^{(-)}(x;\zeta) \ , \qquad j = 0,1,2,3 \ , \quad (5.8)$$

we can express (5.6) in a form containing normally ordered products of that field:

$$\boldsymbol{P}_{0;\boldsymbol{s}} = -\int_{q^0=0} d\hat{\sigma}^0(\mathbf{q}) d\Omega(v) :\tfrac{1}{2}\sum_{j=0}^{3} \partial_j A_k(x;\zeta)\,\partial_j A^k(x;\zeta): \ . \quad (5.9)$$

In the case of the 3-momentum components, corresponding to $j = a = 1,2,3$ in (5.4), we can integrate in the $\mathbf{q}$-variable with the result that, in simplified notation,

$$\boldsymbol{P}_{a;\boldsymbol{s}} = \int_{q^0=0} d\hat{\sigma}^0(\mathbf{q}) d\Omega(v)\, \eta^{ik}\left[\partial_a A_i^{(+)}\,\partial_0 A_k^{(-)} + \partial_0 A_i^{(+)}\,\partial_a A_k^{(-)}\right] \quad (5.10)$$

Since we can again claim that, on the same grounds as in (5.7),

$$\int_{q^0=0} d\hat{\sigma}^0(\mathbf{q}) d\Omega(v)\, \eta^{ik}\left[\partial_a A_i^{(\pm)}\,\partial_0 A_k^{(\pm)} + \partial_0 A_i^{(\pm)}\,\partial_a A_k^{(\pm)}\right] \equiv 0 \ , \quad (5.11)$$

we can express (5.10) as an integral over normally ordered products of the field in (5.8):

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[14] Procedures with integrals for the mass-0 case that involve the formal limit in (3.25) can be carried out in a mathematically rigorous manner by first carrying them out on the corresponding integrals for the massive Proca case in (3.9)-(3.11), and then taking the mass-0 limit. In the present situation this procedure involves applying the massive counterpart of the Bochner integral in (5.6) to massive counterparts of n-photon state vectors, then taking the inner products with other similar state vectors for the Proca case, and eventually letting m converge to zero after the indicated integrations by parts are performed. The rapid decrease at infinity of the coordinate wave functions in the $\mathbf{q}$-variables ensures that the surface terms resulting from the integration by parts contribute zero.

$$\boldsymbol{P}_{\alpha;\boldsymbol{s}} = \int_{q^0=0} d\hat{\sigma}^0(\mathbf{q}) d\Omega(v) : \partial_0 A^k(x;\zeta)\, \partial_\alpha A_k(x;\zeta) : \ . \tag{5.12}$$

When (5.9) and (5.12) are taken in conjunction, these relations can be formulated in the manifestly Poincaré covariant form

$$\boldsymbol{P}_{j;\boldsymbol{s}} = \int : T_{jk}[A(x;\zeta)] : d\hat{\sigma}^k(q)\, d\Omega(v) \ , \quad j = 0,1,2,3 \ , \tag{5.13}$$

based on the normal ordering of the stress-energy tensor[15]

$$T_{ij}[A] = \tfrac{1}{2}\eta_{ij} A_{k,l} A^{k,l} - A_{k,i} A^k{}_{,j} \ , \qquad A_{i,j} = \partial A_i / \partial q^j \ . \tag{5.14}$$

Moreover, when the type of procedure leading from (5.3) to (5.12) is applied to the infinitesimal generators for spatial rotations and for Lorentz boosts, it yields the integrals

$$\boldsymbol{M}^{ij}_{\boldsymbol{s}} = \int : M^{ijk}_{\boldsymbol{s}}[A(x;\zeta)] : d\hat{\sigma}_k(q) d\Omega(v) \ , \tag{5.15}$$

whose integrands incorporate, in addition to spin operators, the same stress-energy tensor:

$$M^{ijk}_{\boldsymbol{s}}[A] = Q^i_{\boldsymbol{s}} T^{jk}[A] - Q^j_{\boldsymbol{s}} T^{ik}[A] + A^\dagger S^{ijk}_{\boldsymbol{s}} A \ . \tag{5.16}$$

With the fundamental notions of parallel transport and stress-energy tensor in Gupta-Bleuler bundles thus established, we can formulate the quantum-geometric propagation of the electromagnetic field

$$\boldsymbol{A}(x,v) := A^i(x; -i\ell v)\boldsymbol{e}_i(x) \ , \quad v = v^i \boldsymbol{e}_i(x) \in \boldsymbol{V}^+_x \ , \quad x \in \mathbf{M} \ , \tag{5.17}$$

by following the the same general procedure as in the treatment of the quantum-geometric propagation of the boson field in (5.5.1). Thus, let us consider a foliation of $(\mathbf{M}, \boldsymbol{g}^{\mathrm{L}})$ into synchronous reference hypersurfaces Σ_t with future-oriented normal 4-vectors $\boldsymbol{n}(x) \in \Sigma_t$. Upon introducing, as in (5.5.3), the proper energy density operator based on the stress-energy operator in (5.14),

$$\boldsymbol{\rho}(x) = \int_{V^+_x} : n^j(x) n^k(x) T_{jk}[\boldsymbol{A}(x,v)] : \ d\Omega(v) \ , \tag{5.18}$$

we can proceed as in Secs. 5.5 and 5.6, and arrive at the following counterpart of (5.6.10):

$$K(A(x'');A(x')) = \lim_{\varepsilon \to +0} \prod_{n=N}^{1}{}' \int_{\mathbf{E}^{\mathrm{T}}_{\boldsymbol{s}}(\Sigma_{t_n})} \mathcal{D}A_n \left\langle \hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{A_n(x_n)} \middle| e^{-i\varepsilon \int d\sigma(x_n) \rho(x_n)} \ \hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{A_{n-1}(x_n)} \right\rangle . \tag{5.19}$$

In a Poincaré moving frame $\boldsymbol{s}$ adapted to the family of reference hypersurface Σ_t the matrix elements in (5.19) assume values formally similar to those in (5.6.13):

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[15] This stress-energy tensor is Poincaré gauge covariant, but not $U(1)$-gauge invariant. The same is the case with some of the momentum-energy tensors which can be introduced in classical electromagnetism (cf., e.g., Itzykson and Zuber, 1980, Sec. 1-2-2). This reflects a breaking of gauge invariance that is intrinsic to the second-quantization of all gauge field theories – cf. (Schweber, 1961), p. 241 and (Itzykson and Zuber, 1980), Eq. (3-120). Such a gauge breaking occurs during the construction of the single-photon wave function in (3.13), since those wave functions satisfy the wave equations in (3.14) even when the Lorenz gauge condition is not satisfied.

$$\exp\Big\{-i\delta^4 x_n \int_{V^+} d\Omega(v) T_{00}[A_n^*(x_n,v) + A_{n-1}(x_n,v)]\Big\} \Big\langle \hat{\Phi}^s_{A_n(x_n)} \Big| \hat{\Phi}^s_{A_{n-1}(x_n)} \Big\rangle \ . \qquad (5.20)$$

A fundamental difference between the massive and the massless case emerges, however, when trying to express the inner products on the right-hand side of the following relation,

$$\Big\langle \hat{\Phi}^s_{A_n(x_n)} \Big| \hat{\Phi}^s_{A_{n-1}(x_n)} \Big\rangle = \exp\Big[i\varepsilon \tfrac{i}{2} \Big(\Big\langle \hat{A}_n(x_n) \Big| \dot{A}_n(x_n) \Big\rangle - \Big\langle \dot{A}_n(x_n) \Big| \hat{A}_{n-1}(x_n) \Big\rangle \Big) \Big] \ , \qquad (5.21)$$

by means of integrals analogous to those in (5.5.17), namely as

$$\langle f | f' \rangle = - Z_{\ell,0} \int \eta^{\mu\nu} \hat{f}_\mu^*(v) \ \hat{f}'_\nu(v) \, d\Omega(v) = \Big\langle \hat{f} \Big| \hat{f}' \Big\rangle \qquad (5.22)$$

Upon using the expression in (5.5.15) for the renormalization constant we find that in the present case of $m = 0$ we have

$$Z_{\ell,0}^{-1} = (8\pi^2/\ell^2) \int_1^\infty \Big(\sqrt{x-1}/(1+x)^{3/2} \Big) dx \ , \qquad (5.23)$$

where the above integral is obviously logarithmically divergent. Consequently, although the quantum-geometric propagator of the electromagnetic field is well-defined by (5.19)-(5.21) as it stands, the ensuing counterparts of (5.6.14)-(5.6.19) leading to action integrals for the quantum-geometric propagator of the electromagnetic field require the infinite wave-function renormalization obtained, as in (3.22), by interpreting (5.22) as a mass-zero limit.

7.6. The Geometric Interpretation of Yang-Mills Gauge Fields

At the classical level, Yang-Mills gauge fields can be viewed as non-Abelian generalizations of electromagnetic 4-potentials. Indeed, a classical Yang-Mills gauge field defined on the Minkowski space M^4 can be represented by a 4-tuple of functions A_μ that assume values within the n-dimensional Lie algebra $L(\mathbf{G}_0)$ of a compact group $\mathbf{G}_0$. If we introduce within the tangent space $T_e\mathbf{G}$ a basis $\{\mathcal{Y}_a \mid a = 1,...,n\}$ as in (2.5.3), the Lie product in $L(\mathbf{G}_0)$ is determined by the structure constants $C_{ab}{}^c$ of $\mathbf{G}_0$, since they give rise to the Lie brackets

$$[\mathcal{Y}_a, \mathcal{Y}_b] = C_{ab}{}^c \mathcal{Y}_c \ , \qquad a,b,c = 1,...,n \ , \quad n = dim \, L(\mathbf{G}_0) \qquad (6.1)$$

With the Lie brackets thus determined, one can define the Yang-Mills field strength $F_{\mu\nu}$ by

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \ . \qquad (6.2)$$

In the absence of matter, this field is then required to obey the following generalizations of the Maxwell equations in (3.12) (cf., e.g., Itzykson and Zuber, 1980, Sec. 12-1):

$$\partial^\mu F_{\mu\nu} + [A^\mu, F_{\mu\nu}] = 0 \ , \qquad A^\mu = \eta^{\mu\nu} A_\nu \qquad (6.3)$$

The common geometric origins of electromagnetic fields and Yang-Mills fields can be traced to the work of H. Weyl, who was the first to introduce the concept of "gauge field," and relate it to the theory of connections on a principal bundle with a "gauge group" (i.e., structure group) $\mathbf{G}_0$. However, Weyl dealt with the case of electromagetism, where that

gauge group is U(1), and therefore it is Abelian; whereas in Yang-Mills theories $\mathbf{G}_0$ is, in general, a compact but non-Abelian Lie group – with typical examples being provided by the SU(N) groups with $N = 2,3,\ldots$.[16]

To arrive at this geometric interpretation of a Yang-Mills gauge field, let us consider a connection form $\boldsymbol{\omega}$ defined in accordance with (2.5.3)-(2.5.7) on a principal bundle $\mathbf{P}(M^4,\mathbf{G}_0)$ over the Minkowski space M^4. If ∇ is a Koszul connection (i.e., covariant differentiation operator form) defined as in (2.5.18) on an associated vector bundle $\mathbf{E}$, then for any given section $\boldsymbol{s}$ of $\mathbf{P}(M^4,\mathbf{G}_0)$ and any given global Poincaré frame $\{(\boldsymbol{a},\boldsymbol{e}_\mu)| \mu = 0,1,2,3\}$, we can rewrite the relations in (2.5.18c) for its gauge potential $\boldsymbol{A}^{\boldsymbol{s}}$ in the form

$$\boldsymbol{A}^{\boldsymbol{s}} = \mathbf{A}_{\alpha;\boldsymbol{s}} A^\alpha_\mu \, dx^\mu \;:\; \boldsymbol{X} \mapsto \mathbf{A}_{\alpha;\boldsymbol{s}} A^\alpha_\mu X^\mu \,, \qquad \boldsymbol{X} = X^\mu \boldsymbol{e}_\mu \tag{6.4}$$

In that case we immediately see that, under a change of global Poincaré frame effected by $(b,\Lambda) \in \mathrm{ISO}^\uparrow(3,1)$, we will have:

$$A^\alpha_\mu(x) \;\mapsto\; A'^\alpha_\mu(x) = \Lambda_\mu{}^\nu A^\alpha_\nu(\Lambda^{-1}(x-b)) \,, \qquad \alpha = 1,\ldots,n \,. \tag{6.5}$$

On the other hand, under a change of Yang-Mills gauge, given by a change (2.5.20) of section $\boldsymbol{s}$ of $\mathbf{P}(M^4,\mathbf{G}_0)$, we obtain according to (2.5.21) the following transformation law:

$$A_\mu = A^\alpha_\mu \mathbf{A}_{\alpha;\boldsymbol{s}} \;\mapsto\; A'_\mu = U^{-1}(g)A_\mu U(g) + U^{-1}(g)\partial_\mu U(g), \qquad \mu = 0,1,2,3 \,. \tag{6.6}$$

For "infinitesimal" Yang-Mills gauge transformations the map in (6.6) assumes the form

$$A_\mu \;\mapsto\; A'_\mu = A_\mu + D_\mu \lambda \,, \qquad D_\mu = \partial_\mu + A^\alpha_\mu \mathbf{A}_{\alpha;\boldsymbol{s}} \,, \tag{6.7}$$

which, in the special case of an electromagnetic field on M^4, coincides with the well-known form of the gauge transformations that leave the equations in (3.12) invariant.

In general, one can associate with any connection form $\boldsymbol{\omega}$ on any principal bundle $\mathbf{P}(\mathbf{M},\mathbf{G}) = (\mathbf{P}, \boldsymbol{\Pi}, \mathbf{M},\mathbf{G})$ a curvature 2-form $\boldsymbol{\Omega}^{\boldsymbol{\omega}} := \mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\omega}$, which is defined by means of the exterior covariant derivative $\mathbf{D}^{\boldsymbol{\omega}}$ determined by $\boldsymbol{\omega}$.[17] For any pair of vectors $\mathbf{X},\mathbf{Y} \in T_u\mathbf{P}$ tangent to the total space $\mathbf{P}$ of the principal bundle $\mathbf{P}(\mathbf{M},\mathbf{G})$ this curvature 2-form assumes values within the Lie algebra $L(\mathbf{G})$ of the structure group $\mathbf{G}$, and satisfies Cartan structural equations which can be expressed in the form

$$\boldsymbol{\Omega}^{\boldsymbol{\omega}}(\mathbf{X},\mathbf{Y}) = \mathbf{d}\boldsymbol{\omega}(\mathbf{X},\mathbf{Y}) + [\boldsymbol{\omega}(\mathbf{X}),\boldsymbol{\omega}(\mathbf{Y})] \,, \quad \mathbf{X},\mathbf{Y} \in T_u\mathbf{P} \,, \tag{6.8}$$

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[16] The original 1918 article by Weyl was reprinted in the 1923 edition of *The Principle of Relativity* (reissued by Dover Publications in 1952) in which Einstein's 1905 and 1916 also appear, and was also reported in various editions of Weyl (1923). In 1929 Weyl pinpointed U(1) as a "gauge group" in the quantum regime, and subsequently O. Klein (1939) considered a non-Abelian gauge theory for the first time. However, these facts were apparently unknown to Yang and Mills (1954) when they made their proposal to replace global SU(2) isospin invariance with local gauge invariance. Although the geometric origins of their proposal had eluded them at the time, the purely geometric nature of "gauge symmetries" became clear after this idea gained popularity amongst elementary particle physicists, as it led to the Weinberg-Salam model of electroweak interactions, and ultimately to QCD and the "standard model" which at present dominates the theory of strong interactions. Eventually Wu and Yang (1975) established the full equivalence between the concepts used in elementary particle physics and their mathematical equivalents in the theory of connections on principal fibre bundles.

[17] Cf., e.g. (Isham, 1989), or (Nakahara, 1990). Whenever later required, such standard definitions and results will be reformulated in a form more amenable to direct physical interpretation. The main purpose of the present short review is to make the reader aware of the underlying geometric framework for these ideas.

as well as Bianchi identities, which can be schematically written in the form

$$\mathbf{D}^{\omega}\mathbf{\Omega}^{\omega} = \mathbf{d}\mathbf{\Omega}^{\omega} + [\omega\,,\,\mathbf{\Omega}^{\omega}] \;=\; \mathbf{0} \tag{6.9}$$

The Yang-Mills field strength in (6.2) then emerges as the curvature tensor corresponding to the curvature 2-form $\mathbf{\Omega}^{\omega}$ for the connection with the Yang-Mills gauge potential in (6.4). Furthermore, the field equations (6.3) are then seen to coincide with the Bianchi identities that follow from (6.9) for curvature tensors associated with curvature 2-forms.

Despite the common geometric origins of electromagnetic and Yang-Mills classical fields, the Gupta-Bleuler method of quantization does not extend in its original form to the latter. Thus, it was noticed by Feynman (1963) that in the case where the gauge groups are non-Abelian the Gupta-Bleuler formalism leads to violations of the unitarity of the physical S-matrix (cf. Kugo and Ojima, 1979). To deal with this problem, Faddeev and Popov (1967) introduced in the Yang-Mills theory so-called "ghost" fields, which assume values in Grassmann algebras. Their idea was inspired by a method of Dirac (1958) for the treatment of Hamiltonian systems with constraints, and enabled the formulation of special relativistic Yang-Mills quantum field theories which are renormalizable in the conventional sense (cf. Faddeev and Slavnov, 1980; Gitman and Tyutin, 1990; Nakanishi and Ojima, 1990). The introduction of Faddeev-Popov ghost fields also paved the way for the independent discoveries by Tyutin (1975) and by Becchi, Rouet and Stora (1976) of global symmetries associated with what eventually became known as "BRST charges." In turn, these developments led to the BRST method of quantization, which came to be extensively used by superstring theory (cf. Green *et al.*, 1987; Brink and Henneaux, 1988; Kaku, 1989, 1991), and therefore became very popular by the mid-1980s, when superstring theory reached the pinnacle of fashion in elementary particle physics.

A geometric interpretation of the Faddeev-Popov "ghost" fields as left-invariant forms on a group of gauge transformations was first proposed by Stora (1976). It was later noticed by Thierry-Mieg (1980) that the constraint equations satisfied by the BRST operators bear a formal resemblance to the Maurer-Cartan structural equations

$$\mathbf{d}\mathbf{\Theta}_g(\mathbf{X},\mathbf{Y}) + [\mathbf{\Theta}_g(\mathbf{X})\,,\mathbf{\Theta}_g(\mathbf{Y})]=\mathbf{0}\;\;,\;\;\; \mathbf{X},\mathbf{Y}\in T_g\mathbf{G}\;\;,\;\; g\in\mathbf{G}\;, \tag{6.10}$$

which are in general satisfied by the Maurer-Cartan 1-form $\mathbf{\Theta}$ of a Lie group $\mathbf{G}$ (cf., e.g., Nakahara, 1990, p.176) – where, by definition, this $L(\mathbf{G})$-valued 1-form maps the values of any left-invariant vector field in $\mathbf{G}$ into the value it assumes in $T_e\mathbf{G}$, namely

$$\mathbf{\Theta}_g \;:\; \mathbf{X} \mapsto L_g{}^{*}\mathbf{X} \in T_e\mathbf{G}\;\;, \qquad \mathbf{X}\in T_g\mathbf{G}\;\;,\;\; g\in\mathbf{G}\;. \tag{6.11}$$

Hence, Thierry-Mieg speculated that BRST operators can be interpreted as the generators for the parallel transport along the fibres of the principal bundle $\mathbf{P}(M^4,\mathbf{G}_0)$ with which the given Yang-Mills fields in (6.4) are associated, and that therefore it is $\mathbf{P}(M^4,\mathbf{G}_0)$ rather than just M^4 that actually serves as the base manifold for the bundles in which the matter fields interacting via Yang-Mills gauge fields are defined in conventional QFT in Minkowski space. However, it was subsequently pointed out by Leinaas and Olaussen (1982) that Faddeev-Popov ghost fields acted on infinite-dimensional spaces; whereas, if they were identified, as suggested by Thierry-Mieg, with differential forms on such principal fibre

bundles, those spaces would have to be finite-dimensional spaces, since the exterior algebra generated by such forms is finite-dimensional (cf., e.g., Nakahara, 1990, p. 159).

The solution to this problem emerged from the replacement of the finite-dimensional gauge groups $\mathbf{G}$ of the first kind, which mathematically play the role of structure groups of principal bundles $\mathbf{P}(\mathbf{M},\mathbf{G})$, with generically infinite-dimensional[18] gauge groups $\mathcal{G}(\mathbf{P})$ of the second kind, whose elements are mathematically represented by vertical automorphisms of $\mathbf{P}(\mathbf{M},\mathbf{G})$. Hence, in the course of the 1980s the ensuing geometric interpretation of BRST symmetries became a very useful tool in the perturbative treatment of Yang-Mills gauge fields (cf., e.g., Baulieu *et al.*, 1985-1991).

This usefulness extends, however, beyond the realm of perturbation theory for Yang-Mills fields. We shall, therefore, present the fundamental geometric ideas underlying the theory of gauge groups of the second kind in a general setting, which will make them equally well applicable to the quantum-geometric treatment of gravity in the next chapter.

7.7. Global Gauge Transformations on Principal Bundles

In applications to physical theories the structure group $\mathbf{G}$ of the principal bundle $\mathbf{P}(\mathbf{M},\mathbf{G})$ is usually referred to as a "gauge group." Thus, U(1) is said to be the gauge group of electromagnetism, SU(2)×U(1) the gauge group in the Weinberg-Salam model of electroweak interactions, and SU(3)×SU(2)×U(1) the gauge group in the standard QCD model, which incorporates strong interactions (cf., e.g., Nachtman, 1990; Huang, 1993). However, the family of all smooth maps from the total space $\mathbf{P}$ into the structure group $\mathbf{G}$,

$$g : \boldsymbol{u} \mapsto g(\boldsymbol{u}) \in \mathbf{G} , \qquad \boldsymbol{u} \in \mathbf{P}, \tag{7.1a}$$

that are defined globally on $\mathbf{P}$ and for which the relations

$$g(\boldsymbol{u}\cdot h) = h^{-1}g(\boldsymbol{u})\, h , \qquad \boldsymbol{u} \in \mathbf{P}, \quad h \in \mathbf{G}, \tag{7.1b}$$

are satisfied, can be also rendered into a group $\mathcal{G}(\mathbf{P})$ in which the operations of group multiplication and of taking the inverse are defined in the following natural manner:

$$g_1 g_2 : \boldsymbol{u} \mapsto g_1(\boldsymbol{u})g_2(\boldsymbol{u}) \in \mathbf{G} , \quad g^{-1} : \boldsymbol{u} \mapsto g(\boldsymbol{u})^{-1} \in \mathbf{G}, \quad \boldsymbol{u} \in \mathbf{P} . \tag{7.1c}$$

We shall demonstrate that the elements of this group are intimately related to changes of local sections of the principal bundle $\mathbf{P}(\mathbf{M},\mathbf{G})$ which give rise to the gauge potential transformations in (2.5.19). Hence, we shall call $\mathcal{G}(\mathbf{P})$ a *gauge group of the second kind*, and forthwith refer to the structure group $\mathbf{G}$ as a *gauge group of the first kind.*

We recall from Sec. 2.4 that on any fibre bundle one can define the local trivialization maps in (2.4.1), which in the case of a principal bundle $\mathbf{P}(\mathbf{M},\mathbf{G})$ assume the form

$$\boldsymbol{\Phi}_\alpha : \boldsymbol{u} \mapsto (x, \boldsymbol{\varphi}_\alpha(\boldsymbol{u})) \in \mathbf{M}_\alpha \times \mathbf{G} , \quad \boldsymbol{u} \in \Pi^{-1}(\mathbf{M}_\alpha) \subset \mathbf{P} , \tag{7.2}$$

that coincides with the one in (2.4.7). On account of the defining properties of principal

[18] Cf. (Atiyah *et al.*, 1978). The terminology of *gauge groups of the first kind* and *of the second kind* was first proposed by Trautman (1981). Research work following in the footsteps of Thierry-Mieg (1980) was subsequently carried out by Bonora *et al.* (1981–1983), Hoyos *et al.* (1982), Mayer *et al.* (1981, 1983), and many others.

bundles (cf. Sec. 2.3) we have that $\varphi_\alpha(\boldsymbol{u}\cdot h) = \varphi_\alpha(\boldsymbol{u})\cdot h$ for all $h \in \mathbf{G}$, so that the map

$$\boldsymbol{s}_\alpha : x \mapsto \boldsymbol{u}\cdot\varphi_\alpha^{-1}(\boldsymbol{u}) \in \Pi^{-1}(x), \qquad x \in \mathbf{M}_\alpha , \tag{7.3}$$

is $\boldsymbol{u}$-independent. Since by one of those properties this map is also smooth, it represents a local section of $\mathbf{P}(\mathbf{M},\mathbf{G})$. Obviously, this section assigns to each $x \in \mathbf{M}_\alpha$ the element $\boldsymbol{u}_\alpha \in \Pi^{-1}(x)$ in the principal bundle $\mathbf{P}(\mathbf{M},\mathbf{G})$ for which $\varphi_\alpha(\boldsymbol{u}_\alpha) = e$.

In general, we can cover $\mathbf{P}(\mathbf{M},\mathbf{G})$ with a family of local trivialization maps constituting a bundle atlas for which any two of these local trivializations with intersecting domains $\mathbf{M}_\alpha$ and $\mathbf{M}_\beta$ give rise, in accordance with (2.4.2) and (2.4.9), to the transition functions

$$\mathbf{g}_{\beta\alpha}(x) = \varphi_\beta(\boldsymbol{u})\circ\varphi_\alpha^{-1}(\boldsymbol{u}) : \mathbf{G} \to \mathbf{G} , \qquad \boldsymbol{u} \in \Pi^{-1}(x) , \quad x \in \mathbf{M}_\alpha \cap \mathbf{M}_\beta . \tag{7.4}$$

Since these transition functions satisfy the compatibility relations in (2.4.6), the maps

$$\hat{g}_\alpha = \varphi_\alpha \circ \boldsymbol{s}_\alpha : x \mapsto \varphi_\alpha(\boldsymbol{s}_\alpha(x)) \in \mathbf{G} , \qquad x \in \mathbf{M}_\alpha , \tag{7.5}$$

will satisfy the corresponding compatibility relations

$$\hat{g}_\beta(x) = \mathbf{g}_{\beta\alpha}(x)\hat{g}_\alpha(x)\mathbf{g}_{\beta\alpha}^{-1}(x) , \qquad x \in \mathbf{M}_\alpha \cap \mathbf{M}_\beta . \tag{7.6}$$

We can therefore use (7.1b) as a means of extending the collection of all such maps,

$$\hat{g} = \left\{ \hat{g}_\alpha \;\middle|\; \hat{g}_\alpha : \mathbf{M}_\alpha \to \mathbf{G} , \; \bigcup\nolimits_\alpha \mathbf{M}_\alpha = \mathbf{M} \right\} , \tag{7.7}$$

into a transformation g defined as in (7.1a) globally on $\mathbf{P}(\mathbf{M},\mathbf{G})$, and belonging to $\mathcal{G}(\mathbf{P})$. Moreover, for any given transformation $g \in \mathcal{G}(\mathbf{P})$, we can cast all $g\circ\boldsymbol{s}_\alpha$ into the role of the family of local maps that obey (7.6) (cf. Göckeler and Schücker, 1987, p. 160), so that a correspondence between the elements of the gauge group of the second kind $\mathcal{G}(\mathbf{P})$ and the coverings of $\mathbf{P}(\mathbf{M},\mathbf{G})$ with local trivialization maps is thereby obtained.

The elements of a gauge group of the second kind can be also identified with certain types of automorphisms of $\mathbf{P}(\mathbf{M},\mathbf{G})$. In general, an *automorphism of the principal bundle* $\mathbf{P}(\mathbf{M},\mathbf{G})$ is defined as a pair (ϕ,ψ) of diffeomorphisms $\phi : \mathbf{P} \to \mathbf{P}$ and $\psi : \mathbf{M} \to \mathbf{M}$ which are such that $\Pi\circ\phi = \psi\circ\Pi$, and for which we have that $\phi(\boldsymbol{u}\cdot h) = \phi(\boldsymbol{u})\cdot h$ for all $\boldsymbol{u} \in \mathbf{P}$ and all $h \in \mathbf{G}$. The family $\mathrm{Aut}\mathbf{P}$ of all automorphisms of $\mathbf{P}(\mathbf{M},\mathbf{G})$ is a group under the group multiplication law provided by the composition of the respective diffeomorphisms:

$$((\phi_1,\psi_1),(\phi_2,\psi_2)) \mapsto (\phi_1\circ\phi_2 , \psi_1\circ\psi_2) . \tag{7.8}$$

A *vertical automorphism* can be then defined as an automorphism which induces in $\mathbf{M}$ the identity transformation, i.e., for which $\Pi(\boldsymbol{u}) = \Pi(\phi(\boldsymbol{u}))$, so that for such automorphisms ψ coincides with the identity map on the base manifold $\mathbf{M}$. The family $\mathrm{Aut}_\mathbf{M}\mathbf{P}$ of all vertical automorphisms of $\mathbf{P}(\mathbf{M},\mathbf{G})$ obviously constitutes a subgroup of $\mathrm{Aut}\mathbf{P}$. By associating with each $\phi \in \mathrm{Aut}_\mathbf{M}\mathbf{P}$ a gauge transformation $g_\phi \in \mathcal{G}(\mathbf{P})$ which is such that

$$\phi : \boldsymbol{u} \mapsto \boldsymbol{u}\cdot g_\phi(\boldsymbol{u}) , \qquad g_\phi(\boldsymbol{u}\cdot h) = h^{-1}g_\phi(\boldsymbol{u})\,h , \quad \boldsymbol{u} \in \mathbf{P}, \; h \in \mathbf{G} , \tag{7.9}$$

it can be established (cf. Bleecker, 1981, Theorem 3.2.2) that the group $\mathrm{Aut}_\mathbf{M}\mathbf{P}$ is isomor-

phic to $\mathcal{G}(\mathbf{P})$. Consequently, vertical automorphisms of $\mathbf{P}(\mathbf{M},\mathbf{G})$ are sometimes referred to in physics literature[19] as *global gauge transformations of the principal bundle* $\mathbf{P}(\mathbf{M},\mathbf{G})$.

This terminology is also justified by the fact that $\mathcal{G}(\mathbf{P})$, and therefore also $\mathrm{Aut}_{\mathbf{M}}\mathbf{P}$, can be identified with the family $\Gamma(ad\,\mathbf{P})$ of all global sections of the bundle

$$ad\,\mathbf{P} = \mathbf{P}\times_{\mathrm{ad}}\mathbf{G} = \{\,\lceil \boldsymbol{u},g\rceil \mid \boldsymbol{u}\in\mathbf{P}\,,\, g\in\mathbf{G}\,\}\quad , \tag{7.10}$$

associated with the principal bundle $\mathbf{P}(\mathbf{M},\mathbf{G})$ by the adjoint action $\mathrm{ad}_h : g \mapsto hgh^{-1}$ of $\mathbf{G}$ onto itself. Indeed, according to the general definition of a $\mathbf{G}$-product given in Sec. 4.1, the elements of $ad\,\mathbf{P}$ are equal to the equivalence classes

$$\lceil \boldsymbol{u},g\rceil = \{\,(\boldsymbol{u}\cdot h, \mathrm{ad}_{h^{-1}}g) \mid h\in\mathbf{G}\,\}\quad ,\qquad \mathrm{ad}_{h^{-1}}g = h^{-1}gh \tag{7.11}$$

Hence, to each $g_\phi \in \mathcal{G}(\mathbf{P})$ corresponds a cross-section $s_\phi \in \Gamma(ad\,\mathbf{P})$ defined by the map

$$s_\phi : \boldsymbol{u} \mapsto \lceil \boldsymbol{u}, g_\phi(\boldsymbol{u})\rceil \in \mathbf{P}\times_{\mathrm{ad}}\mathbf{G}\quad ,\quad \boldsymbol{u}\in\mathbf{P} \tag{7.12}$$

The isomorphic gauge groups $\mathcal{G}(\mathbf{P})$, $\mathrm{Aut}_{\mathbf{M}}\mathbf{P}$ and $\Gamma(ad\,\mathbf{P})$ are infinite-dimensional Lie groups (Atiyah *et al.*, 1978), in the sense that they display the structure of infinite-dimensional Banach manifolds,[20] and that their manifold structure is compatible with their group structure. In this context we note that many of the basic definitions and results presented in Sec. 2.1 for finite-dimensional manifolds can be generalized in a straightforward manner to the case of infinite-dimensional Banach manifolds by extending the concept of chart and atlas to that case. This can be accomplished by replacing in (2.1.1) and (2.1.2) $\mathbf{R}^4$ with any real or complex and finite or infinite dimensional Banach space (i.e., complete normed space) $\mathbf{B}$, and requiring that the corresponding coordinate transformation maps in (2.1.3) are smooth in the sense of being infinitely many times differentiable in accordance with the following definition: a map F from an open subset $\boldsymbol{O}$ of $\mathbf{B}$ into $\mathbf{B}$ is *differentiable* at a point $f\in\boldsymbol{O}$ if there is a linear map $DF(f)$ of $\mathbf{B}$ into $\mathbf{B}$ (called the Fréchet differential of F at f) which is continuous in the norm $\|\cdot\|$ of $\mathbf{B}$, and for which we have $F(f+h)-F(f) = DF(f)h + R(h)$, where the remainder $R(h)$ is such that $\|R(h)\|/\|h\| \to 0$ when $\|h\| \to 0$. Under certain technical conditions, atlases of charts which are smooth in this sense can be constructed from bundle atlases on $\mathbf{P}(\mathbf{M},\mathbf{G})$ and manifold atlases on $\mathbf{M}$ and $\mathbf{G}$.[21]

To define the Lie algebra $L(\mathcal{G})$ of the infinite-dimensional Lie group $\mathcal{G}(\mathbf{P})$ in terms of the Lie algebra $L(\mathbf{G})$ of the finite-dimensional Lie group $\mathbf{G}$, let us consider the bundle

$$Ad\,\mathbf{P} = \mathbf{P}\times_{\mathrm{Ad}}L(\mathbf{G}) = \{\,\lceil \boldsymbol{u},f\rceil \mid \boldsymbol{u}\in\mathbf{P},\ f\in L(\mathbf{G})\,\} \tag{7.13}$$

associated with $\mathbf{P}(\mathbf{M},\mathbf{G})$. The elements of this bundle are the equivalence classes

$$\lceil \boldsymbol{u},f\rceil := \{\,(\boldsymbol{u}\cdot g, \mathrm{Ad}_{g^{-1}}f) \mid g\in\mathbf{G}\,\}\,,\qquad \mathrm{Ad}_g : f\mapsto (\mathrm{ad}_g)_* f\,, \tag{7.14}$$

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[19] Cf., e.g., (Bleecker, 1981) or (Isham, 1989). However, Trautman (1976, 1980, 1981) considers also gauge transformations of the second kind given by automorphisms of P(M,G) that are not vertical, and refers to those which are vertical as "pure" gauge transformations of the second kind.

[20] Cf. (Lang, 1972) and (Choquet-Bruhat *et al.*, 1987, Chapter VII).

[21] Cf. (Mitter and Viallet, 1981) and (Socolovsky, 1991). The particulars of this construction are not absolutely necessary, however, for an understanding of the subsequent exposition.

determined by the adjoint representation of the group $\mathbf{G}$ on its Lie algebra $L(\mathbf{G})$. In turn, the elements of this representation are the push-forwards Ad_g of those within the corresponding representation in (7.11). Hence, the bundle $Ad\,\mathbf{P}$ becomes a Lie algebra if its Lie bracket is defined for each pair of its elements by the corresponding Lie brackets in $L(\mathbf{G})$:

$$[\lceil \boldsymbol{u}, f \rceil, \lceil \boldsymbol{u}, f' \rceil] := \lceil \boldsymbol{u}, [f, f'] \rceil \quad . \tag{7.15}$$

Moreover, due to the very mode of construction of $Ad\,\mathbf{P}$, the family

$$\mathrm{Exp}(t \lceil \boldsymbol{u}, f \rceil) := \lceil \boldsymbol{u} \, , \exp(tf) \rceil \, , \quad t \in \mathbf{R}^1 \, , \tag{7.16}$$

consists of cross-sections of the bundle $ad\,\mathbf{P}$ in (7.10), and is an additive one-parameter subgroup of $\Gamma(ad\,\mathbf{P})$. Hence, the Lie algebra $L(\mathcal{G})$ can be indeed identified with the infinite-dimensional vector space $\Gamma(A\,d\,\mathbf{P})$ consisting of all cross-sections of the associated bundle $Ad\,\mathbf{P}$ in (7.13) (cf. also Bleecker, 1981, Theorems. 3.2.10 and 3.2.11).

7.8. BRST and Anti-BRST Operators on Spaces of Connections

The introduction of the gauge group $\mathcal{G}(\mathbf{P})$ of the second kind affords an approach to the theory of connections on principal bundles $\mathbf{P}(\mathbf{M},\mathbf{G})$ which opens new perspectives in the theory of Yang-Mills fields as well as in general relativity, and leads to a conceptualization of gauge transformations from an "active" point of view, which is complementary to the "passive" point of view adopted in Chapter 2.

In physics the *active* conceptualization of an observable effect is the one in which that effect is achieved by submitting the observed system to a physical operation – for example, to a rotation or a boost; whereas, the *passive* conceptualization is the one in which the same effect is achieved by executing a complementary physical operation on the "observer" and his "apparatus" – such as the performance of an opposite rotation or boost on the system of reference. This conceptual distinction can be extended also to the mathematical constructs that play a role in physical theories. Thus, in the case of manifolds, the passive point of view is the one that was adopted in Sec. 2.1. From that passive perspective, the map in (2.1.2b) was viewed as representing a change in the coordinates of a given point $x \in \mathbf{M}$; whereas, from an active perspective the same map is viewed as a transformation that takes a point $x \in \mathbf{M}$ having the unprimed coordinates in a *fixed* chart into the point $x' \in \mathbf{M}$ having the primed coordinates in that same chart. Similarly, for tensors the passive point of view is the one whereby the transformation in (2.1.13) is viewed as corresponding to the change of tensor components above $x \in \mathbf{M}$; whereas, according to the corresponding active point of view, the very same transformation is deemed to supply a map that takes the tensor $\boldsymbol{T}$ in (2.1.12) into a new tensor $\boldsymbol{T}'$ having the primed coordinates defined in (2.1.13) with respect to the holonomic frames and coframes that appear in (2.1.13).

The same two complementary attitudes can be adopted towards gauge transformations: whereas the transformation in (2.5.19) was viewed in Sec. 2.5 from a passive point of view as a change in gauge potential due to a change of section of $\mathbf{P}(\mathbf{M},\mathbf{G})$ for a given connection form ω, from an active point of view it is considered to be due to a change of connection, which takes us from the connection form ω to another connection form ω'.

Since from the active point of view gauge transformations give rise to changes of connection forms, let us consider the family $C(\mathbf{P})$ of all connection forms on a given principal bundle $\mathbf{P}(\mathbf{M},\mathbf{G})$. This family is an affine space, since for any fixed $\omega_0 \in C(\mathbf{P})$ we have that (cf. Bleecker, 1981, Theorem 3.2.6)

$$C(\mathbf{P}) = \{\ \omega = \omega_0 + \tau \mid \tau \in \Lambda^1(\mathbf{P},L(\mathbf{G}))\ \}\ , \tag{8.1}$$

where $\Lambda^1(\mathbf{P},L(\mathbf{G}))$ is a linear space.[22] This affine space is convex, since for any two connection forms $\omega',\omega'' \in C(\mathbf{P})$ we obviously have that $(1-t)\omega'+t\omega'' \in C(\mathbf{P})$ for all $t \in [0,1]$. It can also be established by the method used in deriving (2.5.21) that for any $\omega \in C(\mathbf{P})$ and any $\phi \in \mathrm{Aut}_{\mathbf{M}}\mathbf{P}$, we have

$$\phi^*\omega = \mathrm{Ad}_{g^{-1}}\omega + g_\phi{}^*\Theta\ , \qquad g_\phi \in \mathcal{G}(\mathbf{P})\ , \tag{8.2}$$

where g_ϕ is defined by (7.9), and Θ is the Maurer-Cartan form of $\mathbf{G}$ defined in (6.11). In the case where $\mathbf{G}$ is a group of matrices or operators, as is normally the case in physics, this relation can be written in the form (cf. Bleecker, 1981, p. 31; Isham, 1989, p. 161)

$${}^g\omega := \phi^*\omega = \mathrm{Ad}_{g^{-1}}\omega + g^{-1}dg\ , \qquad g = g_\phi\ , \tag{8.3}$$

since we can then express the elements of dg in terms of matrices or operators, so that

$$g^*\Theta = g^{-1}dg\ , \quad g \in \mathcal{G}(\mathbf{P})\ , \quad dg \in \Gamma(Ad\,\mathbf{P})\ . \tag{8.4}$$

This is exactly the situation encountered in the theory of Yang-Mills fields. For the case considered in Sec. 7.6 we can make the transition from the passive to the active point of view by extending the "matter field" Ψ on which the Yang-Mills field in (6.5) acts from the Minkowski base space M^4 to the total space $\mathbf{P}$ of the principal bundle $\mathbf{P}(M^4,\mathbf{G}_0)$. This extension can be effected by introducing for any bundle atlas of $\mathbf{P}(M^4,\mathbf{G}_0)$, with transition functions defined as in (7.4), the Yang-Mills gauges provided by the local sections of $\mathbf{P}(M^4,\mathbf{G}_0)$ in (7.3), and then defining for each local section $\{\Psi_x \mid x \in \mathbf{M}^s\}$ of the vector bundle $(\mathbf{E},\pi,\mathbf{M},\mathbf{F})$ for "matter" fields the corresponding push-forwards

$$\Psi(u) = \Psi_{x,0} = s_\alpha{}^*\Psi_x\ , \quad u = s_\alpha(x)\ , \quad x \in \mathbf{M}^s \cap \mathbf{M}_\alpha\ . \tag{8.5}$$

The extension of these push-forwards into a generalized Higgs field[23] Ψ on $\mathbf{P}(M^4,\mathbf{G}_0)$ is then completed by taking advantage of the condition that such a field has to be equivariant under the action of the structure group $\mathbf{G}_0$, namely that it has to satisfy

$$\Psi(u\cdot g) = U(g^{-1})\Psi(u)\ , \qquad u \in \mathbf{P}\ , \qquad g \in \mathbf{G}_0\ , \tag{8.6}$$

to extend each push-forward in (8.5) to the domain $\Pi^{-1}(\mathbf{M}^s \cap \mathbf{M}_\alpha)$ within $\mathbf{P}(M^4,\mathbf{G}_0)$.

Within the resulting active framework, the Koszul connection ∇^ω with the gauge

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[22] An outline of the constructions of graded Lie algebras of k-forms, including this space of $L(\mathrm{G})$-valued 1-forms, can be found in Sec. 10.3 of (Prugovečki, 1992). However, in the sequel we shall work with Koszul connections for which the counterparts of these 1-forms are tensor fields over $\mathbf{M}$, so that such constructions can be by-passed.

[23] We follow here the terminology of Trautman (1980, 1981). The original idea of Higgs (1964), independently arrived at also by Englert and Brout (1964), pertained to massive vector fields that emerged from massless gauge fields by the mechanism of spontaneous symmetry breaking – cf. (Itzykson and Zuber, 1981, Sec 3-5-3).

potential in (6.4) that corresponds to a given connection form $\omega \in C(\mathbf{P})$ has to be extended along the fibres of $\mathbf{P}(M^4,\mathbf{G}_0)$, so that it can mediate parallel transport in arbitrary directions within $\mathbf{P}$. Since, according to (7.9), we can reach from a given point in $\mathbf{P}$ all other points in the same fibre by means of various gauge transformations g of $\mathbf{P}(M^4,\mathbf{G}_0)$, such an extension results from (8.3), and according to (2.5.21), its gauge potential is represented by

$$ {}^{g}\mathcal{A}^{\omega} = (\mathrm{Ad}_{g^{-1}}\,\omega + g^{-1}\boldsymbol{d}g)^{a}\,\mathbf{A}_a \quad , \qquad g \in \mathcal{G}(\mathbf{P}) \ . \tag{8.7} $$

This construction therefore results in a family $C(\mathbf{P},\mathbf{F})$ of Koszul connections represented by covariant differentiation operators acting on generalized Higgs fields, and constitutes an infinite-dimensional manifold that is diffeomorphic to the family $C(\mathbf{P})$ of all Ehresmann connections, i.e., to the family of all connection forms on $\mathbf{P}(M^4,\mathbf{G}_0)$.

As we mentioned in Sec. 7.6, it was originally conjectured by Thierry-Mieg (1980) that the Maurer-Cartan form $\boldsymbol{\Theta}$ of $\mathbf{G}_0$ which appears in (8.2) can be related to the Faddeev-Popov ghost fields, which are usually deemed to emerge from the Yang-Mills classical action (cf. Babelon and Viallet, 1981, Sec. 2). In view of the above considerations, it now becomes clear that the identification

$$ c := (g^{-1}\boldsymbol{d}g)^{a}\,\mathbf{A}_a \quad \leftrightarrow \quad \boldsymbol{\Theta}^{\mathcal{G}} = g^{-1}\boldsymbol{d}g \ , \tag{8.8} $$

embodying this idea has to be carried out from the point of view of the infinite-dimensional gauge group $\mathcal{G}(\mathbf{P})$ of the second kind, rather than from that of the gauge group $\mathbf{G}_0$ of the first kind. It then becomes justified to refer to c in (8.8), regarded as a field defined on the total space $\mathbf{P}$ of the principal bundle $\mathbf{P}(M^4,\mathbf{G}_0)$, as a *ghost field*. The global gauge transformations on $\mathbf{P}(M^4,\mathbf{G}_0)$ which are "infinitesimal" then give rise to the transformations

$$ \mathcal{A}^{\omega} \mapsto \mathcal{A}^{\omega} + s\mathcal{A}^{\omega} \ , \quad s = \boldsymbol{d}g^{a}\,\partial_a \ , \quad \partial_a = \partial U(g)/\partial g^{a} \ , \tag{8.9} $$

that can be expressed in terms of the following mappings

$$ \mathcal{A}^{\omega} \mapsto \mathcal{A}^{\omega} + \boldsymbol{D}^{\omega}c \ , \qquad \boldsymbol{D}^{\omega}c \ \leftrightarrow \ \mathbf{D}^{\omega}\boldsymbol{\Theta}^{\mathcal{G}} \ , \tag{8.10} $$

computed at $g(\boldsymbol{u}) = e$. The exterior differential form s that appears in (8.9) is called in the present context of Yang-Mills field theories a *BRST operator*.

We note that $\boldsymbol{D}^{\omega}c$ assumes values equal to those of $s\mathcal{A}^{\omega}$ for vectors that are tangent to $\mathbf{P}$, and which point in directions that are horizontal in relation to the connection form ω (cf. the decomposition in (2.5.1) in Sec. 2.5). On the other hand, the Maurer-Cartan structural equations resulting from (6.10) are satisfied for all vertical directions within $T\mathbf{P}$. This means that the external covariant derivative of the Maurer-Cartan form in (8.10) has to vanish in such directions. Hence, in accordance with (6.8), we shall have that

$$ sc + \tfrac{1}{2}[c,c] = 0 \quad , \qquad s^2 = 0 \ , \tag{8.11} $$

where the second of these equations reflects the fact that repeated external differentiation of any forms always produces a null result, so that the BRST operator s is nilpotent.

In order to underline the formal analogy with the conventional quantum theory of Yang-Mills fields, where additional "antighost" fields make their appearance, some authors

(Quirós *et al.*, 1981; Baulieu and Thierry-Mieg, 1982; Bonora and Cotta-Ramusino, 1983) have introduced such fields also in the classical context – although, as pointed out by Mayer (1983), that is not absolutely necessary in that case. On the other hand, antighost fields are encountered in Yang-Mills theories during the course of gauge fixing procedures.

To give the notion of gauge fixing a precise mathematical meaning, we observe that, according to (8.1) and (8.2), the elements of the gauge group $\mathcal{G}(\mathbf{P})$ map the family $\mathcal{C}(\mathbf{P})$ of connection forms onto itself. Since $\mathcal{C}(\mathbf{P})$ is diffeomorphic to $\mathcal{C}(\mathbf{P},\mathbf{F})$, the same is true of the action of $\mathcal{G}(\mathbf{P})$ on $\mathcal{C}(\mathbf{P},\mathbf{F})$. Consequently, the elements of $\mathcal{C}(\mathbf{P},\mathbf{F})$ fall naturally into equivalence classes called *gauge orbits*. We can then construct a principal bundle

$$(\mathcal{C}(\mathbf{P},\mathbf{F}),\pi_{\mathcal{C}},\mathcal{C}_{\mathcal{G}}(\mathbf{F}),\mathcal{G}(\mathbf{P}))\,, \qquad \mathcal{C}_{\mathcal{G}}(\mathbf{F}) = \mathcal{C}(\mathbf{P},\mathbf{F})/\mathcal{G}(\mathbf{P})\,, \tag{8.12}$$

for which the family $\mathcal{C}_{\mathcal{G}}(\mathbf{F})$ of all such gauge orbits supplies the base space, $\mathcal{C}(\mathbf{P},\mathbf{F})$ the total space, $\mathcal{G}(\mathbf{P})$the structure group, and where $\pi_{\mathcal{C}}$ is the natural projection of $\mathcal{C}(\mathbf{P},\mathbf{F})$ onto $\mathcal{C}_{\mathcal{G}}(\mathbf{F})$. A *gauge-fixing* procedure is tantamount to a choice of a unique representative from each gauge orbit, and if it can be carried out in a smooth manner, then a cross-section

$$\phi_{\mathcal{C}} \;:\; \mathcal{C}_{\mathcal{G}}(\mathbf{F}) \;\rightarrow\; \mathcal{C}(\mathbf{P},\mathbf{F})\,, \tag{8.13}$$

of the principal bundle in (8.12) will obviously result.

The standard gauge-fixing method in the theory of Yang-Mills fields consists of the introduction of an auxiliary scalar field b that indexes all the gauge orbits in (8.12), and of the specification of the values of additional parameters required for the pinpointing of a unique representative $\mathcal{A}^{\omega}$ within each gauge orbit. However, if this procedure is meant to produce the smooth map $\phi_{\mathcal{C}}$ in (8.13), then such a gauge fixing is precluded by the "no-go" theorem of Singer (1978), which rigorously confirmed the existence of Gribov (1978) "ambiguities" in SU(N) Yang-Mills theories. Its central conclusion is that for such theories the principal bundle in (8.12) does not possess global sections, since the family $\mathcal{C}(\mathbf{P},\mathbf{F})$ cannot supply the total space of a principal bundle with structure group $\mathcal{G}(\mathbf{P})$ if the center $Z(\mathbf{G})$ of $\mathbf{G}$ is not trivial. Indeed, $Z(\mathbf{G})$ consists of all those elements of $\mathbf{G}$ which commute with everything in $\mathbf{G}$, so that if $Z(\mathbf{G})$ is not trivial then the adjoint action in (8.12) is not free, as required by the general definition of principal bundles (cf. Sec. 2.3). Hence, to cope with this problem, instead of the family $\mathcal{C}(\mathbf{P})$ of all connection forms one has to use the subfamily $\mathcal{C}_0(\mathbf{P})$ of all connections which are irreducible[24], and instead of $\mathcal{G}(\mathbf{P})$ one has to use the quotient $\mathcal{G}_0(\mathbf{P})$ of $\mathcal{G}(\mathbf{P})$ by the family of $Z(\mathbf{G})$-valued sections of the adjoint representation.[25] On the other hand, in perturbative approaches, such problems are handled by arguing that within a sufficiently small neighborhood of the unit element of the gauge group $\mathcal{G}(\mathbf{P})$ the Gribov ambiguity does not occur (cf. Itzykson and Zuber, 1980, p. 576).

Each element g of the gauge group $\mathcal{G}(\mathbf{P})$ is given by a map (7.1a), so that it can be

[24] A connection is *irreducible* if its holonomy group is equal to the entire structure group G. In general, the *holonomy group* $H(x)$ of a connection at a point x in the base manifold M is, by definition, the group of all isomorphisms of the fibre in P(M,G) above x that are obtained by carrying out the parallel transport of that fibre along all possible closed piecewise smooth curves in M that begin and end at x – cf. (Nakahara, 1990, p. 339).

[25] Cf. Sec. VI of (Daniel and Viallet, 1980), which also contains a very lucid summary of all these difficulties.

identified with a section of the trivial principal bundle $\mathbf{P}\times\mathbf{G}_0$. Hence, the gauge potentials in (8.9) and (8.10) can be considered to be defined on the principal bundle $\mathbf{P}_1=\mathbf{P}\times\mathbf{G}_0$. In order to introduce the geometric counterparts of the aforementioned antighost fields, the domains of definition of these gauge potentials have to be further extended to the principal bundle $\mathbf{P}_2=(\mathbf{P}\times\mathbf{G}_0)\times\mathbf{G}_0$. This results in a triply-fibrated manifold structure, which is based on projection maps for which $\mathbf{P}_2\to\mathbf{P}_1\to\mathbf{P}$. The infinitesimal gauge transformation in (8.10) is then extended by adding to (8.9) another term,

$$\mathcal{A}^{\omega}\mapsto\ \mathcal{A}^{\omega}+s\mathcal{A}^{\omega}+\bar{s}\,\mathcal{A}^{\omega}\,,\qquad \bar{s}=\boldsymbol{d}\bar{g}^{\alpha}\bar{\partial}_{\alpha}\,,\qquad \bar{\partial}_{\alpha}=\partial U(\bar{g})/\partial\bar{g}^{\alpha}\,, \tag{8.14}$$

where $\bar{s}$ is called an *anti-BRST operator*. This kind of transformation can be expressed by means of the mapping that incorporates an "infinitesimal" gauge fixing field b, so that it then assumes the form

$$\mathcal{A}^{\omega}\mapsto\ \mathcal{A}^{\omega}+\boldsymbol{D}^{\omega}c\ +\boldsymbol{D}^{\omega}\bar{c}\ \,,\qquad s\bar{c}\ =b\ \,, \tag{8.15}$$

where $\bar{c}$ is referred to as an *antighost field.*

We can now use any local section $\boldsymbol{s}$ of $\mathbf{P}(\mathbf{M},\mathbf{G})$ to pull $\mathcal{A}^{\omega}$ back to the domain $\mathbf{M}^{\boldsymbol{s}}$ in $\mathbf{M}$ of that section. We then obtain a Yang-Mills gauge field $\boldsymbol{s}_*\mathcal{A}^{\omega}$ that can be expressed in the form

$$A(x,g,\bar{g})=A_{\mu}(x,g,\bar{g})\,\boldsymbol{dx}^{\mu}+c(x,g,\bar{g})+\bar{c}(x,g,\bar{g})\,, \tag{8.16}$$

$$c=c_{\alpha}(x,g,\bar{g})\boldsymbol{d}g^{\alpha},\quad \bar{c}=\bar{c}_{\alpha}(x,g,\bar{g})\boldsymbol{d}\bar{g}^{\alpha}\,,\quad x\in\mathbf{M}^{\boldsymbol{s}},\quad g,\bar{g}\in\mathcal{G}(\mathbf{P})\,, \tag{8.17}$$

and corresponds to a covariant differentiation operator

$$\overline{\nabla}=\boldsymbol{d}+s+\bar{s}+A\ \,,\qquad \boldsymbol{d}=\boldsymbol{dx}^{\mu}\partial_{\mu}\,, \tag{8.18}$$

defined on the triply fibrated vector bundle $\mathbf{E}_2\to\mathbf{E}_1\to\mathbf{E}$ that is associated to a triply-fibrated principal bundle $\mathbf{P}_2\to\mathbf{P}_1\to\mathbf{P}$ – where $\mathbf{E}$ is the bundle with typical fibre $\mathbf{F}$ that originally carried the matter fields. The relations in (8.11) are now extrapolated into

$$sc=-\tfrac{1}{2}[c,c]\ \,,\qquad \bar{s}\bar{c}=-\tfrac{1}{2}[\bar{c},\bar{c}]\ \,, \tag{8.19}$$

$$s^2=0\ \,,\qquad s\bar{s}+\bar{s}s=0\ \,,\qquad \bar{s}^2=0\ \,, \tag{8.20}$$

where the second equality in (8.20) follows from the fact that $s+\bar{s}$ is nilpotent, since it is a one-form. Furthermore, the Maurer-Cartan structural equations on $\mathbf{G}_0\times\mathbf{G}_0$ imply that in addition to (8.20) we must also have (cf. Baulieu and and Thierry-Mieg, 1982)

$$s\bar{c}+\bar{s}c=-[\bar{c},c]\ . \tag{8.21}$$

On the other hand, the fact that $\boldsymbol{D}^{\omega}c$ assumes values equal to those of $s\mathcal{A}^{\omega}$ for those vectors within $T\mathbf{P}$ which point in directions that are horizontal in relation to the connection form $\boldsymbol{\omega}$ in (8.14) and (8.15), implies that

$$sA_{\mu}=D_{\mu}c\ \,,\qquad \bar{s}A_{\mu}=D_{\mu}\bar{c}\ \,,\qquad Dc:=\boldsymbol{s}^{*}\boldsymbol{D}^{\omega}c\ \,; \tag{8.22}$$

whereas, when the second of the equations in (8.15) is taken in conjunction with the equations in (8.20) and (8.21), it implies that

$$s\bar{c} = b\,, \qquad s\bar{c} = -[\bar{c},c] - b\,, \qquad sb = 0\,, \qquad \bar{s}b = -[\bar{c},b]\,. \tag{8.23}$$

In this manner a purely geometric derivation is obtained for all the equations which are usually derived in the Lagrangian approach to the theory of Yang-Mills fields (cf. Kugo and Ojima, 1979). When such gauge fields act as external fields on matter fields Ψ representing wave functions of quantum particles, then the following subsidiary conditions are imposed in order to ensure that those matter fields are physical (cf. Baulieu, 1985):

$$s\Psi = -c\Psi\,, \qquad \bar{s}\Psi = -\bar{c}\Psi \tag{8.24}$$

The relations in (8.16)-(8.18) indicate that, in geometric terms, these subsidiary conditions amount to the request that all those generalized Higgs fields that represent *physical* matter fields should have vanishing covariant derivatives in all vertical directions.

The physical significance of this last restriction is very clear: whereas mathematically the parallel transport of generalized Higgs fields can be carried out along arbitrary smooth or piecewise smooth curves within the total space $\mathbf{P}$ of the principal bundle $\mathbf{P}(M^4,\mathbf{G}_0)$, the parallel transport that is physical can be carried out only along those curves in $\mathbf{P}$ which are the horizontal lifts (cf. Sec. 2.5) of paths within the base manifold M^4 of that bundle. In other words, the vertical degrees of freedom introduced by extending the domain of definition of a matter field Ψ from M^4 to $\mathbf{P}$ cannot play any role as far as the physically observable phenomena are concerned. This fact is also reflected in the choice of terminology, which labels as "ghost" and "antighost" fields those additional components of gauge fields that are introduced while such a transition from M^4 to $\mathbf{P}$ is carried out.

7.9. The Quantum-Geometric Framework for Yang-Mills Fields

Ghost and antighost fields, as well as of BRST and anti-BRST operators, are usually presented in the Lagrangian approach to the quantum theory of Yang-Mills fields.[26] However, it is the geometric formulation presented in the preceding section that provides the basis of the quantum-geometric framework for such fields. The fundamental ideas of this framework will be extrapolated in the next chapter to gravitational fields, and will be presented in considerable detail in that context. Hence, in this section we shall limit ourselves to indicating what these ideas are, without venturing into all the technical details of their implementation, which can be easily surmised from the considerations in the next chapter.

The quanta described by Yang-Mills fields are massless and of spin one (such as the gluons in QCD). Therefore, the Krein space formalism of Sec. 7.2 can be applied to them in the momentum representation, in which their state vectors consist of $L(\mathbf{G}_0)$-valued wave functions of the 4-vector variable k that assumes values on the forward light-cone in $(\mathbf{R}^4,\eta)$. If these wave functions are expanded in any $L(\mathbf{G}_0)$-basis $\{\mathcal{Y}_a \mid a = 1,\ldots,n\}$,

$$\tilde{f}_\mu(k) = \tilde{f}^a_\mu(k)\mathcal{Y}_a \in L(\mathbf{G}_0)\,, \qquad k \in V_0^+\,, \qquad \mu = 0,1,2,3\,, \tag{9.1}$$

then their components must transform according to (2.5) under all Poincaré transformations $(a,\Lambda) \in \mathrm{ISO}^\uparrow(3,1)$ (cf. the Appendix of Kugo and Ojima, 1979):

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[26] Cf.(Itzykson and Zuber, 1980, Chapter 12), (Faddeev and Slavnov, 1980), (Popov, 1983), (Gitman and Tyutin, 1990), (Nakanishi and Ojima, 1990).

$$\tilde{U}(a,\Lambda) \; : \; \tilde{f}^b_\mu(k) \; \mapsto \; \exp(ia\cdot k)\Lambda_\mu{}^\nu \tilde{f}^b_\nu(\Lambda^{-1}k) \; , \qquad b = 1,\ldots,n \; . \tag{9.2}$$

Moreover, in any global Lorentz frame in M^4 the J-inner product of these wave functions

$$\left(\tilde{f}\middle|\tilde{f}'\right)_J = \sum_{\mu=0}^{3} \int_{V_0^+} \tilde{f}^*_\mu(k) \bullet \tilde{f}'_\mu(k)\, d\Omega_0(k) \tag{9.3}$$

must exists, so that their Poincaré-invariant indefinite inner product is then given by

$$\left\langle \tilde{f}\middle|\tilde{f}'\right\rangle = -\int_{V_0^+} \eta^{\mu\nu} \tilde{f}^*_\mu(k) \bullet \tilde{f}'_\nu(k)\, d\Omega_0(k) \; . \tag{9.4}$$

In the cases of primary interest in physics $\mathbf{G}_0$ is a group of real matrices, so that the elements of the Lie algebra $L(\mathbf{G}_0)$ can be also represented by real matrices, and we have

$$\tilde{f}^*_\mu(k) \bullet \tilde{f}'_\nu(k) := \mathrm{Tr}[\tilde{f}^*_\mu(k)\, \tilde{f}'_\nu(k)], \qquad k \in V_0^+, \quad \mu,\nu = 0,1,2,3 \; , \tag{9.5}$$

in terms of trace of those matrices. In the generic situation, the $L(\mathbf{G}_0)$-scalar product that appears on the left-hand side of (9.5) has to be constructed from some non-degenerate scalar product that is left invariant by the adjoint map in (7.14).[27] This then ensures that, in general, the following representation of $\mathbf{G}_0$,

$$\tilde{\mathrm{U}}(g) : \; \tilde{f}_\mu(k) \quad \mapsto \; \mathrm{Ad}_{g^{-1}} \tilde{f}_\mu(k) \in L(\mathbf{G}_0) \; , \qquad g \in \mathbf{G}_0 \, , \tag{9.6}$$

preserves the inner products in (9.3) and (9.4).

We shall appropriately refer to the Krein space $\tilde{\mathbf{W}}$ of all the wave function for which the J-norm determined by (9.3) exists as a *Weyl-Klein space*. Indeed, as was pointed out in footnote 16 of this chapter, it was Weyl (1918, 1929) who first introduced in physics the concept of gauge field for the Abelian case, and it was Klein (1939) who first extended that idea to non-Abelian gauge fields.

The Gupta-Bleuler framework relies very much on the Lorenz gauge condition in (2.9) but, as we mentioned in Sec. 7.6, it is not entirely suitable in the case of non-Abelian quantum fields. Consequently, we shall consider within $\tilde{\mathbf{W}}$ various *generalized Lorenz gauge* conditions

$$k_\mu \tilde{f}^\mu(k) = \tilde{b}(k) \; , \qquad \tilde{b}(k) = \tilde{b}^a(k) \mathcal{Y}_a \in L(\mathbf{G}_0) \; , \tag{9.7}$$

provided by, in general, various non-vanishing gauge-fixing fields $\tilde{b}$ whose components $\tilde{b}^a$ behave as scalar fields under Poincaré transformations. This means that the generically affine subspaces of $\tilde{\mathbf{W}}$ that correspond to these generalized Lorenz gauges, and can be described as follows in terms of the linear polarization modes determined by the polarization tetrads in (2.11),

$$\tilde{\mathbf{W}}(\tilde{b}) = \left\{ \tilde{f} = \tilde{f}^{(\alpha)} \boldsymbol{\varepsilon}_{(\alpha)} \in \tilde{\mathbf{W}} \middle| \tilde{f}^{(0)}(k) = \tilde{f}^{(3)}(k) + \tilde{b}(k) \right\} \; , \tag{9.8}$$

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[27] Such a scalar product can be always obtained from the metric tensor of the Killing form of G_0 – cf. (Choquet-Bruhat *et al.*, 1987, p. 168).

will be left invariant by the representation in (9.2), but not, in general, by the representation in (9.6). This turns out to be the root cause for the emergence of antighost fields in the ensuing framework for quantum-geometric Yang-Mills fields.

The construction of the *Weyl-Klein typical fibre* $\mathbf{W}$ can be effected as in the Abelian case treated in Sec. 7.3, namely by adopting

$$f_\mu(\zeta) = (2\pi)^{-3/2}\int_{V_0^+} \exp(-i\zeta\cdot k)\tilde{f}_\mu(k)\, d\Omega_0(k) \in L(\mathbf{G}_0)\ , \qquad \mu = 0,1,2,3\ , \tag{9.9}$$

as the generalizations of (3.13) that determine all the elements of $\mathbf{W}$ as $\tilde{f}$ ranges over $\tilde{\mathbf{W}}$. The J-inner product and the indefinite inner product in (9.3) and (9.4) therefore assume for each a-component f^a of $f \in \mathbf{W}$ the respective forms in (3.21)-(3.24). On the other hand, if we express elements of the Weyl-Klein typical fibre $\mathbf{W}$ in terms of linear polarization tetrads, these inner products assume the form

$$(f|f')_J = \sum_{\alpha=0}^{3}\int_\Sigma \bar{f}^{(\alpha)}(\zeta)\bullet f'^{(\alpha)}(\zeta)\, d\tilde{\Sigma}(\zeta)\ , \qquad f,f' \in \mathbf{W}\ , \tag{9.10}$$

$$\langle f|f'\rangle = -\int_\Sigma \eta_{\alpha\beta}\bar{f}^{(\alpha)}(\zeta)\bullet f'^{(\beta)}(\zeta)\, d\tilde{\Sigma}(\zeta)\ , \qquad f,f' \in \mathbf{W} \tag{9.11}$$

The generalized Lorenz subfibres corresponding to (9.8) are given by

$$\mathbf{W}(b) = \left\{ f \in \mathbf{W} \,\middle|\, \partial_\mu f^\mu(\zeta) = b(\zeta) \right\} \subset \mathbf{W}\ , \qquad \partial_\mu = \partial/\partial q^\mu\ , \tag{9.12}$$

as b ranges over the space $\mathbf{W}_B$ of all gauge-fixing field values

$$b(\zeta) = (2\pi)^{-3/2}\int_{V_0^+} \exp(-i\zeta\cdot k)\tilde{b}(k)\, d\Omega_0(k) \in L(\mathbf{G}_0)\ , \qquad \zeta = q - i\ell v\ , \tag{9.13}$$

corresponding to all the feasible $\tilde{b}$ in (9.7). The Weyl-Lorenz typical subfibre $\mathbf{W}(0)$ of the Weyl-Klein typical fibre $\mathbf{W}$ obviously corresponds to $b \equiv 0$. For any given fundamental symmetry J in $\mathbf{W}$, we can decompose $\mathbf{W}(0)$ into a counterpart

$$\mathbf{W}(0) = \mathbf{W}_J(0) \underset{J}{\oplus} \mathbf{W}^0\ , \qquad \mathbf{W}_J(0) = \left\{ f \in \mathbf{W}(0) \,\middle|\, f^{(0)} = f^{(3)} = \mathbf{0} \right\}\ , \tag{9.14}$$

of the J-direct sum in (2.17), in which the subfibre $\mathbf{W}_J(0)$ consists of all state vectors that possess only transversal polarization modes; whereas the space $\mathbf{W}^0$ consist of null state vectors that possess only longitudinal and timelike polarization modes. On the other hand, it is obvious from (9.8) that an arbitrary element of the Weyl-Klein typical fibre $\mathbf{W}$ can be expressed as a sum whose first component lies within $\mathbf{W}(0)$, whereas the second is a representative element h of $\mathbf{W}$ that obeys the generalized Lorenz subsidiary condition in (9.12). Hence, for any choice of global Lorentz frame in $(\mathbf{R}^4,\boldsymbol{\eta})$, and therefore also of fundamental symmetry J in $\mathbf{W}$, we can unambiguously represent any element of $\mathbf{W}$ by a direct sum as follows:

$$f \oplus h \oplus b\ , \qquad f \in \mathbf{W}_J(0)\ , \quad h \in \mathbf{W}^0\ , \quad b \in \mathbf{W}_B\ . \tag{9.15}$$

Let us now single out the fundamental symmetry J_0 corresponding to the canonical basis $\boldsymbol{u}_0$ in $\mathbf{R}^4$ described in (5.5.12), and use the corresponding decomposition in (9.15) as

the basis for the definition of the *typical Yang-Mills fibre*

$$\mathcal{W} = \mathcal{W}^{\mathrm{P}} \otimes \Big(\mathcal{F}^{\mathrm{G}} \underset{A}{\otimes} \overline{\mathcal{F}}^{\mathrm{G}} \Big) \,. \tag{9.16}$$

This typical fibre consists of a physical fibre

$$\mathcal{W}^{\mathrm{P}} = \bigoplus_{J_0}^{\infty}{}_{n=0} \mathcal{W}_n^{\mathrm{P}} \,, \qquad \mathcal{W}_n^{\mathrm{P}} = \mathbf{W}_{J_0}(0) \underset{S}{\otimes} \cdots \underset{S}{\otimes} \mathbf{W}_{J_0}(0) \,, \tag{9.17}$$

of a Faddeev-Popov ghost fibre

$$\mathcal{F}^{\mathrm{G}} = \bigoplus_{J_0}^{\infty}{}_{n=0} \mathcal{F}_n^{\mathrm{G}}, \qquad \mathcal{F}_n^{\mathrm{G}} = \mathbf{F}^{\mathrm{G}} \underset{A}{\otimes} \cdots \underset{A}{\otimes} \mathbf{F}^{\mathrm{G}} \,, \tag{9.18}$$

constructed from the single-ghost fibre $\mathbf{F}^{\mathrm{G}}$ consisting of n-tuples $\mathbf{c}$ of functions set in correspondence with the "pure gauge" components h in (9.15),

$$h = h^a \mathcal{Y}_a \;\leftrightarrow\; \mathbf{c} = (c^1,\ldots,c^n) \in \mathbf{F}^{\mathrm{G}} \,, \qquad h_i^a = \partial c^a / \partial q^i \,, \qquad i = 0,1,2,3 \,, \tag{9.19}$$

and of a conjugate antighost fibre $\overline{\mathcal{F}}^{\mathrm{G}}$ constructed in a similar manner. Within the fibre $\mathcal{W}$ the annihilation operators of Yang-Mills gauge quantum are, as usual, determined by

$$\Big(A_i^{(-)}(\zeta) \Psi_{k;m;n} \Big)_{k-1} (\zeta_1, i_1, \ldots, \zeta_{k-1}, i_{k-1}) = \sqrt{k}\, \Psi_{k;m;n}(\zeta, i, \zeta_1, i_1, \ldots, \zeta_{k-1}, i_{k-1}) \,, \tag{9.20}$$

for states with k Yang-Mills quanta, m Faddeev-Popov "ghosts" and n "antighosts." The corresponding creation operators are their pseudo-adjoints, so that the typical Yang-Mills quantum frame field is given by

$$A_i(\zeta) = A_i^{(+)}(\zeta) + A_i^{(-)}(\zeta) \,, \qquad A_i^{(+)}(\zeta) = A_i^{(-)\dagger}(\zeta) \,. \tag{9.21}$$

As on previous occasions, this field determines the Yang-Mills quantum frames

$$\Phi_f = \exp\Big[-\tfrac{1}{2} \langle f | f \rangle - \int f(\zeta) * A^{(+)}(\zeta) d\tilde{\Sigma}(\zeta) \Big] \Psi_0 \,, \tag{9.22a}$$

$$f * g := f^i \bullet g_i = \sum_{a=1}^{n} \eta^{ij} f_i^a g_j^a \,, \qquad f, g \in \mathbf{W} \,, \tag{9.22b}$$

within the typical Yang-Mills fibre $\mathcal{W}$ in (9.16).

The annihilation operators of Faddeev-Popov "ghost" quanta are determined by

$$\begin{aligned} &\Big(C^{(-)}(\zeta) \Psi_{k;m;n} \Big)_{m-1} (\zeta_1, i_1, \ldots, \zeta_k, i_k; \zeta_{k+1}, \ldots, \zeta_{k+m-1}; \ldots, \zeta_{k+m+n}) \\ &= \sqrt{m}\, \Psi_{k;m;n}(\zeta_1, i_1, \ldots, \zeta_k, i_k; \zeta, \zeta_{k+1}, \ldots, \zeta_{k+m-1}; \ldots, \zeta_{k+m+n}) \,, \end{aligned} \tag{9.23}$$

so that the typical Faddeev-Popov quantum frame field is given by

$$C(\zeta) = C^{(+)}(\zeta) + C^{(-)}(\zeta) \,, \qquad C^{(+)}(\zeta) = C^{(-)\dagger}(\zeta) \tag{9.24}$$

The corresponding typical Faddeev-Popov coframe field is given by

$$\overline{C}(\zeta) = \overline{C}^{(+)}(\zeta) + \overline{C}^{(-)}(\zeta) \quad , \qquad \overline{C}^{(+)}(\zeta) = -\overline{C}^{(-)\dagger}(\zeta) \; , \tag{9.25}$$

where the anti-hermiticity is imposed in accordance with the stipulation in (Kugo and Ojima, 1978a, pp. 1871-1873), which in the special relativistic QFT regime emerges from the requirement of a (formally) pseudo-unitary S-matrix.

Upon introducing in single-ghost fibre $\mathbf{F}^{\mathrm{G}}$ a J_0-orthonormal basis $\{\mathbf{c}_1, \mathbf{c}_2, ...\}$, and a conjugate one within single-antighost fibre $\overline{\mathbf{F}}^{\mathrm{G}}$, by the method of Sec. 6.2 one can transform these fibres into the respective superfibres $\mathbf{B}^{\mathrm{G}}$ and $\overline{\mathbf{B}}^{\mathrm{G}}$,

$$\mathbf{B}^{\mathrm{G}} = \left\{ \boldsymbol{c} = \theta^{\alpha}\mathbf{c}_{\alpha} \,\middle|\, \theta^{\alpha} = \left(\mathbf{c}_{\alpha}|\mathbf{c}\right)_{J_0}\vartheta_{\alpha} \; , \;\; \alpha = 1,2,... \; , \;\; \mathbf{c} \in \mathbf{F}^{\mathrm{G}} \right\} , \tag{9.26a}$$

$$\overline{\mathbf{B}}^{\mathrm{G}} = \left\{ \overline{\boldsymbol{c}} = \overline{\theta}^{\alpha}\overline{\mathbf{c}}_{\alpha} \,\middle|\, \overline{\theta}^{\alpha} = \left(\overline{\mathbf{c}}_{\alpha}|\overline{\mathbf{c}}\right)_{J_0}\overline{\vartheta}_{\alpha} \; , \;\; \alpha = 1,2,... \; , \;\; \overline{\mathbf{c}} \in \overline{\mathbf{F}}^{\mathrm{G}} \right\} , \tag{9.26b}$$

in which ϑ_{α} and $\overline{\vartheta}_{\alpha}$ are the generators of the Grassmann algebra Λ described in Sec. 6.2. From these single-ghost and single-antighost superfibres one can construct, by the same method as in Sec. 6.3, the standard Berezin-Faddeev-Popov (BFP) superfibre $\mathcal{B}^{\mathrm{G}} \otimes \overline{\mathcal{B}}^{\mathrm{G}}$ over the Grassmann algebra Λ of supernumbers in (6.2.7), and the following standard BFP ghost and antighost superframes:

$$\Phi_{\boldsymbol{c}} = \exp\!\left(-\tfrac{1}{2}\int \overline{\boldsymbol{c}}(\zeta) * \overleftarrow{\partial}_k \boldsymbol{c}(\zeta) d\hat{\sigma}^k(q) d\Omega(v)\right) \exp\!\left(C^{(+)}(\boldsymbol{c})\right) \Psi_0 \; , \tag{9.27a}$$

$$\Phi_{\overline{\boldsymbol{c}}} = \exp\!\left(-\tfrac{1}{2}\int \boldsymbol{c}(\zeta) * \overleftarrow{\partial}_k \overline{\boldsymbol{c}}(\zeta) d\hat{\sigma}^k(q) d\Omega(v)\right) \exp\!\left(\overline{C}^{(+)}(\overline{\boldsymbol{c}})\right) \Psi_0 \; . \tag{9.27b}$$

These superframes, together with their adjoints, supply resolutions of the identity in the FP ghost and antighost typical fibres in the general sense explained in Sec. 6.3.

On account of (9.9), the representations in (9.2) and (9.6) give rise within the Weyl-Klein typical fibre $\mathbf{W}$ to a representation $U(a,\Lambda,g) = U(a,\Lambda) \otimes U(g)$ of $\mathrm{ISO}^{\uparrow}(3,1) \times \mathbf{G}_0$. However, when trying to extend this representation to the entire Yang-Mills typical fibre in (9.16), we encounter the type of domain problems discussed in Sec. 7.4. Consequently, we shall consider instead within that fibre the representation

$$\boldsymbol{U}(a,R,g) = \bigoplus\nolimits_{n=0}^{\infty} U(a,R,g)^{\otimes n} \; , \qquad (a,R) \in \mathrm{T}^4 \wedge \mathrm{O}(3), \quad g \in \mathbf{G}_0 \; , \tag{9.28}$$

and use the $\mathbf{G}$-product method of Sec. 4.1 to construct the quantum bundle

$$\mathcal{E}_0 = \mathbf{P}(\mathbf{M},\mathbf{G}) \times_{\mathbf{G}} \mathcal{W} \; , \qquad \mathbf{G} = \left(\mathrm{T}^4 \wedge \mathrm{O}(3)\right) \times \mathbf{G}_0 \; , \tag{9.29}$$

whose structure group $\mathbf{G}$ does not incorporate Lorentz boosts.

The incorporation of transformations that incorporate Lorentz boosts can be effected by the supergroup method expounded in detail in Secs. 8.6-8.8 of the next chapter. Since the main steps of that technique are basically the same in the present context of Yang-Mills fields, we shall present them only in broad outline for a Lorenz gauge fixing.

The representation $U(a,\Lambda,g)$ of $\mathrm{ISO}^{\uparrow}(3,1) \times \mathbf{G}_0$ can be used for the construction of

$$\mathbf{E}_1 = \mathbf{P}(\mathbf{M},\mathbf{G}_1) \times_{\mathbf{G}_1} \mathbf{W} \; , \qquad \mathbf{G}_1 = \mathrm{ISO}^{\uparrow}(3,1) \times \mathbf{G}_0 \; . \tag{9.30}$$

The soldering maps (4.1.3) resulting from this **G**-product can be used, in the same general manner as in Chapter 6, for mapping the J_0-orthonormal basis $\{\mathbf{c}_1, \mathbf{c}_2, ...\}$ into J_u-orthonormal bases $\{\mathbf{c}_{u;1}, \mathbf{c}_{u;2}, ...\}$ within the single-ghost fibres $\mathbf{F}^{\mathrm{G}}{}_x$, which are associated with the various Poincaré frames $\boldsymbol{u} \in P^{\uparrow}\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ above each given $x \in \mathbf{M}$; a similar construction can be carried out for the the single-antighost fibres $\overline{\mathbf{F}}^{\mathrm{G}}{}_x$.

For Lorentz boosts Λ_v the action of $U(0,\Lambda_v,g)$ within $\mathbf{F}^{\mathrm{G}}{}_x$ and $\overline{\mathbf{F}}^{\mathrm{G}}{}_x$ can be transferred (cf. (8.6.5)) into transformations of the counterparts of the Grassmann coordinates in (9.26) within the corresponding superfibres $\mathbf{B}^{\mathrm{G}}{}_x$ and $\overline{\mathbf{B}}^{\mathrm{G}}{}_x$. It can be then combined with the action of the structure group **G** of the quantum bundle (9.29) into gauge transformations given by action from the right upon the quantum coframes

$$\boldsymbol{\Phi}^{\dagger}_{f,\bar{c},c} = \boldsymbol{\Phi}^{\dagger}_{f} \otimes \boldsymbol{\Phi}_{\bar{c}} \otimes \boldsymbol{\Phi}^{\dagger}_{c}\,, \qquad \sigma^{\mu}_{x}\, f \in \mathbf{W}_{J_0}(0), \qquad \sigma^{\mu}_{x}\, \bar{c} \in \overline{\mathbf{B}}^{\mathrm{G}}\,, \quad \sigma^{\mu}_{x}\, c \in \mathbf{B}^{\mathrm{G}} \tag{9.31}$$

This results into a gauge group $\mathcal{G}_0$ of the second kind which is a supergroup that acts upon cross-sections of principal bundles of such superframes – cf. (8.6.24)-(8.6.34), where a generalized Lorenz gauge fixing is used that can be also incorporated into the present considerations.

The quantum connection in the quantum fibre bundle $\mathcal{E}_0$, based on the gauge group **G**, can be then extended as in Sec. 8.7 into a quantum connection in a quantum superfibre bundle $\mathcal{E}_1$ over a base supermanifold[28] $\boldsymbol{M}$, whose coordinate charts incorporate the earlier mentioned Grassmann coordinates. Superfield analogues of (8.16)-(8.23) can be then derived as in Chapter 8 – cf. (8.7.23).

The *physical* parallel transport which underlies the extrapolation of the quantum-geometric mode of propagation to the present regime is then effected along lifts to the base supermanifold $\boldsymbol{M}$ of geodesic arcs in the base manifold **M** (cf. Sec. 8.8). This entails the imposition of subsidiary conditions analogous to those in (8.24), which represent the geometric counterparts of the Kugo-Ojima (1979) subsidiary conditions for Yang-Mills fields in conventional QFT – cf. (8.8.27).

The presence of nonlinear terms in the field equations (6.3) for classical Yang-Mills fields gives rise in the quantum regime to self-interactions of the quantum-geometric Yang-Mills field. During the ensuing quantum-geometric propagation these self-interactions manifest themselves via terms that have to be added to the proper energy density operator obtained by inserting into (5.18) the counterpart of (5.14) for that field. The construction of quantum-geometric propagators for Yang-Mills field then proceeds along the same general lines as the quantum-geometric fields treated on previous occasions, once allowances are made for the fact that the functional integrations have to be carried out over the real as well as the Grassmannian base degrees of freedom – cf. Sec. 8.9.

[28] For reasons discussed in Sec. 8.7, $\boldsymbol{M}$ is, strictly speaking, only a pseudo-supermanifold.

Chapter 8

Geometric Quantum Gravity

A geometric framework for quantum gravity emerges naturally from an extrapolation of the semi-classical framework for massless gauge fields in the preceding chapter. This extrapolation is based on the postulate that the interactions between a quantum gravitational field and the fields describing matter and nongravitational radiation is mediated by gravitons of spin 2 and zero-mass. Such interactions give rise to a quantum-geometric evolution in which the mean values of the quantum gravitational field created by them manifest themselves as metrizations of the quantum frame bundles of all the fields partaking in that evolution; whereas, the quantum gravitational fluctuations around those mean values manifest themselves as superlocal multigraviton states within superfibres over a quantum spacetime supermanifold. The Grassmann variables in the coordinate charts of that supermanifold are related to unphysical degrees of freedom of graviton polarization. The quantum-geometric evolution giving rise to a quantum spacetime is governed by a connection based on a quantum gravitational gauge supergroup incorporating those Grassmann variables. This supergroup represents a quantum counterpart of the semidirect product of the group of Poincaré gauge transformations with a group of vertical automorphisms. The latter is isomorphic to the diffeomorphism group that plays a central role in active formulations of general covariance in classical general relativity.

8.1. Diffeomorphism and Poincaré Gauge Invariance in CGR

At the macroscopic level, the existence of a classical general relativistic spacetime metric within our universe is an indisputable physical fact. Indeed, its existence has been directly confirmed by a multitude of terrestrial experiments and observations. Furthermore, it is also supported by astronomical observations in all extra-terrestrial settings.

On the other hand, as was pointed out in Secs. 1.6 and 2.2, a classical general relativistic spacetime metric cannot be identified with a single Lorentzian metric. Rather, the CGR principle of general covariance – which is nowadays suitably formulated from an active point of view (cf. Sec. 7.8) – imposes the requirement of diffeomorphism invariance on all CGR models (cf., e.g., Ashtekar and Stachel, 1991). This active interpretation of general covariance stipulates that if the Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ is a suitable CGR model for spacetime, and if $\psi : \mathbf{M} \to \mathbf{M}$ is any diffeomorphism of $\mathbf{M}$ onto itself, then the Lorentzian manifold $(\mathbf{M},\boldsymbol{g}'^{\mathrm{L}})$ provides a *physically* equivalent model if we set, in accordance with (2.2.1),

$$\boldsymbol{g}'^{\mathrm{L}} = \psi_* \boldsymbol{g}^{\mathrm{L}} , \quad \boldsymbol{T}'_\alpha = \psi_* \boldsymbol{T}_\alpha , \quad x'_\beta(t) = \psi(x_\beta(t)) , \tag{1.1}$$

for all the matter and radiation tensor fields $\boldsymbol{T}_\alpha$, as well as for all causal test-particle world-

lines $\{x_\beta(t) \in \mathbf{M}\}$ within that model. In other words, if $Riem^{\mathrm{L}}\mathbf{M}$ denotes the family of all Lorentzian metrics in $\mathbf{M}$, and if $Diff\mathbf{M}$ denotes the family of all diffeomorphisms of $\mathbf{M}$, then a *classical spacetime metric* in $\mathbf{M}$ is actually represented by an equivalence class

$$g^{\mathrm{L}} = \{ \psi_* g^{\mathrm{L}} \mid \psi \in Diff\mathbf{M} \} \in Riem^{\mathrm{L}}\mathbf{M}/Diff\mathbf{M} , \tag{1.2}$$

of Lorentzian metrics in $\mathbf{M}$ (cf. Hawking and Ellis, 1973, Sec. 7.1; Straumann, 1984, Sec. 2.7). In this context we note that if the composition of diffeomorphisms of $\mathbf{M}$ is adopted as the operation of their group multiplication, then $Diff\mathbf{M}$ becomes an infinite-dimensional Lie group (cf. Sec. 7.7) – which is usually referred to as the *diffeomorphism group* over $\mathbf{M}$. Hence, g^{L} is an orbit within $Riem^{\mathrm{L}}\mathbf{M}$ under the left action of the diffeomorphism group $Diff\mathbf{M}$ that is determined by (1.2), and the resulting mappings $Diff\mathbf{M} \rightarrow Riem^{\mathrm{L}}\mathbf{M} \rightarrow Riem^{\mathrm{L}}\mathbf{M}/Diff\mathbf{M}$ give rise to a principal bundle that is considered to be the fundamental CGR "gauge bundle" over $\mathbf{M}$ (cf., e.g., Isham, 1889, pp. 132-133). We shall refer, therefore, to $(\mathbf{M}, g^{\mathrm{L}})$ as a *CGR gauge orbit* over $\mathbf{M}$.

As we expounded in Sec. 2.2, the adoption of a specific Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ from the equivalence class in (1.2) mediates the reduction of the general linear frame bundle $GL\mathbf{M}$ to the Lorentz frame bundle $L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$ over $\mathbf{M}$. Moreover, as we saw there, in CGR this mathematical reduction can be physically interpreted in terms of a metrization procedure operationally performed on classical frames that are physically realizable objects. Hence, the special relativistic invariance that is actually always *locally* observed in Nature, manifests itself as the Lorentz gauge invariance of CGR models, which is partly reflected in the fact that the Lorentz group $\mathrm{SO}(3,1)$ is the structure group $\mathbf{G}$ (i.e., gauge group of the first kind) of the principal bundle $\mathbf{P}(\mathbf{M},\mathbf{G}) \cong L\mathbf{M}(\boldsymbol{g}^{\mathrm{L}})$, which is basic to CGR. As it transpires from the ensuing operational formulation of Lorentz gauge transformations, this choice of gauge group of the *first* kind (cf. Sec. 7.7) does *not* merely express a "commitment to a certain style of formulating physical theories" (Friedman, 1983), but it also reflects some of the physical substance of CGR. Indeed, *local* Lorentz covariance is as empirically verifiable in CGR as *global* Lorentz covariance is verifiable in special relativity, since both these forms of Lorentz covariance deal with *operationally* defined coordinates, rather than with coordinates that are mere mathematical labels; namely with Minkowski coordinates in special relativity, and with the coordinates with respect to local Lorentz frames $\{\boldsymbol{e}_0(x),\boldsymbol{e}_1(x),\boldsymbol{e}_2(x),\boldsymbol{e}_3(x)\}$ in tangent spaces $T_x\mathbf{M}$ in general relativity.

This is to be expected, because the effects of gravitation are long-range and manifest themselves throughout the universe, since no screening against them is feasible – as is the case with electromagnetic interactions. Actually, the fact that no screening can *ever* be achieved against gravitational influences was one of the main factors which motivated Einstein to reexamine the physical role of the concept of "inertial frame," and to ultimately identify it in CGR with the concept of "free-falling frame." Hence, strictly speaking, it is only *local* Lorentz covariance that can be ever actually verified in experimental praxis. The Levi-Civita connection that is uniquely determined (cf. Sec. 2.6) by the Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ of any CGR model is, therefore, another reflection of the empirically verifiable physical substance, rather than of the mere style, of the local Lorentz gauge invariance of such models. Indeed, in CGR it reflects the implicit choice of Lorentz group $\mathrm{SO}(3,1)$ as gauge

group of the first kind; moreover, according to the geodesic postulate, it unambiguously determines the physically observable worldlines of all test particles in free fall.

On purely physical grounds, not all elements of $Riem^{L}\mathbf{M}$ are suitable candidates for classical Lorentzian metrics. Indeed, first of all, only space and time orientable Lorentzian manifolds are deemed to be physically acceptable (cf. Sec. 2.6) – so that $L\mathbf{M}(\boldsymbol{g}^{L})$ can be further reduced to the Lorentz frame bundle $L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$ having the orthochronous Lorentz group $SO^{\uparrow}(3,1)$ as its structure group.[1] And, second, in view of Geroch's (1968) theorem, in order to be able to accommodate in the semi-classical regime also quantum systems which display half-integer spin states, the Lorentz frame bundle $L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$ corresponding to a physically acceptable metric has to possess global sections, i.e., it has to be "trivial" in the fibre-theoretical sense of the word.

As discussed in Sec. 4.3, the extrapolation of Lorentz covariance based on $SO^{\uparrow}(3,1)$ into Poincaré gauge covariance based on $ISO^{\uparrow}(3,1)$ can be always mathematically effected by extending the principal bundle $L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$ of linear Lorentz frames into the Poincaré frame bundle $P\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$ of corresponding affine frames. Furthermore, according to the standard theorems in fibre bundle theory that were mentioned there, one can always associate with any linear Levi-Civita connection various affine Riemann-Cartan connections that are still compatible with the adopted Lorentzian metric on $\mathbf{M}$, but might not display zero torsion. Such Riemann-Cartan connections give rise to covariant differentiation operators of the general form displayed in (4.3.3), which contain, in general, canonical forms provided by the duals of the Lorentzian frames $\{\boldsymbol{e}_0,\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\} \in L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$. However, as we saw in Sec. 4.3, one can always choose from amongst all the resulting Riemann-Cartan connection forms in $P\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$ those which give rise to Levi-Civita covariant differentiation operators on those sections of $P\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$ that are identifiable with sections of $L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$.

It should be noted at this stage that some researchers in quantum gravity insist on "the replacement of the Poincaré group by the diffeomorphism group as the governing symmetry [gauge] group of [any nonperturbative quantization of gravity] theory," on the grounds that "diffeomorphism invariance is a local gauge symmetry of [classical] general relativity" (Smolin, 1991 pp. 442-443). Others, however, point out that "the diffeomorphism group is not a [CGR] gauge group and [that the Levi-Civita connection] Γ is not its connection" (Göckeler and Schücker, 1987, p. 76).

The analysis presented thus far in this section makes it clear that the diffeomorphism invariance of CGR models is a *mathematically* necessary condition, resulting from the mathematical "style" (cf. Sec. 1.6) adopted by the CGR framework, and from the physical interpretation of that framework; but that it is not a *physically* meaningful condition, of the kind originally envisaged by Einstein (1916) for general covariance. Actually, diffeomorphism invariance essentially represents a reformulation, from the active point of view, of the invariance under coordinate changes originally imposed in CGR by Einstein from the passive point of view. Hence, it does not possess the experimentally verifiable physical substance exhibitted by Lorentz or Poincaré gauge invariance. Indeed, an automorphism

[1] For the massive cases considered in Sec. 2.6, the further reduction to the Lorentz frame bundle $L\mathbf{M}_0(\boldsymbol{g}^{L})$ of right-handed Lorentz frames was performed, which has the restricted Lorentz group $SO_0(3,1)$ as its structure group. On the other hand, $L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$ contains right-handed as well as left-handed (but future oriented) Lorentz frames. Its use is required for the incorporation of all helicity states of massless excitons, such as photons and gravitons.

(cf. Sec. 7.7) of the Lorentz frame bundle $L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$ that corresponds to some diffeomorphism $\psi \in \mathit{Diff}\mathbf{M}$ is obviously supplied by a pair (ϕ,ψ) of diffeomorphisms, in which the diffeomorphism $\phi : L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L}) \rightarrow L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$ of its total space is simply the identity map in each one of its fibres $\Pi^{-1}(x)$ (namely the map which leaves each Lorentz frame in $\Pi^{-1}(x)$ unchanged), so that $\psi : \mathbf{M} \rightarrow \mathbf{M}$ simply provides a smooth relabeling of the fibres of $L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$ by means of points in the base manifold $\mathbf{M}$. In CGR, where the elements $\boldsymbol{u}$ of $L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$ are associated with physical objects that are classical Lorentz frames (cf. Sec. 2.2), this merely entails a *mathematical* relabeling of those frames that has no observable consequences. And this fact remains true if $L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$ is replaced by $P\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$.

On the other hand, according to the general definitions in Sec. 7.7, an element of a gauge group of the *second* kind of the Lorentz frame bundle $L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$ is an automorphism (ϕ,ψ) of $\mathbf{P}(\mathbf{M},\mathbf{G})$, $\mathbf{G} = \mathrm{SO}^{\uparrow}(3,1)$, for which ψ is the identity automorphism in $\mathbf{M}$, so that it is represented exclusively by the vertical automorphism $\phi : \mathbf{P}(\mathbf{M},\mathbf{G}) \rightarrow \mathbf{P}(\mathbf{M},\mathbf{G})$. Hence, for any given $\boldsymbol{s}$ of section of $\mathbf{P}(\mathbf{M},\mathbf{G})$, the Lorentz transformations $\phi(\boldsymbol{s}(x)) \in \mathbf{P}(\mathbf{M},\mathbf{G})$ can reflect a *local* operational procedure that takes us from a local Lorentz frame $\boldsymbol{u} \in L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$ *above* some point $x \in \mathbf{M}^{g}$ to another local Lorentz frame above the *same* point x – as is the case with the inertial frames which Einstein employed in his formulation of the strong equivalence principle. Hence, an operational interpretation can be assigned to the gauge transformations of the second kind performed on the Lorentz frame bundle $L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L}) \cong \mathbf{P}(\mathbf{M},\mathbf{G})$. Moreover, this statement remains true if $L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$and $\mathrm{SO}^{\uparrow}(3,1)$ are replaced in the above considerations by $P\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$ and $\mathrm{ISO}^{\uparrow}(3,1)$, respectively.

Thus, the principle of general covariance displays in CGR *two* gauge-theoretical facets, which are clearly brought out by the fibre-theoretical approach to CGR models: 1) a purely mathematical one exhibited under the action of the diffeomorphism group $\mathit{Diff}\mathbf{M}$, which is reflected by the physical equivalence depicted in (1.1) for CGR models formulated over the same base manifold $\mathbf{M}$; 2) an empirically verifiable one, which is reflected by the choice of the Lorentz or the Poincaré group as gauge groups of the first kind, once a representative metric $\boldsymbol{g}^{L}$ has been selected from a CGR class $\mathcal{g}^{L}$ in (1.2) of diffeomorphically equivalent metrics. In case the Poincaré frame bundle $P\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$ is chosen in the beginning, and is then reduced to the Lorentz frame bundle $L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{L})$, then the latter yields the Levi-Civita connection as the unique torsionless connection governing the physical parallel transport of local linear frames that is compatible with such a choice of metric. This fact underlies the original formulation of the geodesic postulate, and is therefore closely related to the strong equivalence principle.

Both these aspects of general relativistic covariance can be integrated into a form of gauge invariance under a subgroup of the group $\mathrm{Aut}_{\mathbf{M}}\mathbf{P}$ of vertical automorphism of the general linear frame bundle $\mathbf{P} = GL\mathbf{M}$, achieved by mapping all infinitesimal diffeomorphisms in $\mathbf{M}$ into infinitesimal gauge transformations of the second kind that take place along the fibres of $GL\mathbf{M}$. This reinterpretation of the general covariance principle was actually foreshadowed by Einstein's ultimate conceptualization of the relationship between spacetime as a physical entity and as a mathematical framework.

Its most essential aspect is best reflected in Einstein's epistemic observation that: "There is no such thing as empty space, *i.e.* space without a [metric] field. Space-time does not claim existence on its own, but only as a structural quality of the [metric] field."

(Einstein, 1961, p. 155.) In other words, in CGR a spacetime manifold $\mathbf{M}$ does *not* represent some physically "empty space," to whose points could be therefore assigned a physical meaning which would be independent of the presence of a Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$; rather, the presence of a Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ is absolutely necessary for the identification of points within $\mathbf{M}$ with physical events, and any subsequent mathematical manipulation which leaves all resulting metric relationships invariant – such as the one provided by the maps in (1.1) – has no physically observable consequences. Hence, as discussed in Sec. 2.2, from the point of view of principal frame bundles, it is an element $\boldsymbol{u}(x)$ in a cross-section $\boldsymbol{s}$ of a Lorentz frame bundle $L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$ that represents a classical event. Since each such cross-section of $L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$ can be identified with a cross-section (4.3.2) of the Poincaré frame bundle $P\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$ for which $\boldsymbol{a} \equiv 0$, the same conclusion can be expressed within the bundle $P\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$.

To re-express the notion of diffeomorphism invariance in terms of vertical automorphisms on the general affine frame bundle $GA\mathbf{M}$,[2] let us consider the following action of the diffeomorphism group $Diff\mathbf{M}$ on all (r,s)-tensor fields $\boldsymbol{T}$ in $\mathbf{M}$:

$$\mathrm{D}^{r,s}(\psi) \;:\; \boldsymbol{T} \;\mapsto\; \psi_{*}(\boldsymbol{T} \circ \psi^{-1}) \;, \qquad \psi \in Diff\mathbf{M} \tag{1.3}$$

Since $(\boldsymbol{T} \circ \psi^{-1})(x) = \boldsymbol{T}(\psi^{-1}(x))$ at all $x \in \mathbf{M}$ within the range of ψ, the push-forward in (1.3), which is defined in accordance with (2.2.2)-(2.2.4), gives rise to a new (r,s)-tensor field $\mathrm{D}(\psi)\boldsymbol{T}$ that has the same domain as $\boldsymbol{T}$. Hence, if $\boldsymbol{s}$ is any section of $GA\mathbf{M}$ representing a moving frame in the sense of Cartan (cf. Sec. 2.3), we can define for any $\psi \in Diff\mathbf{M}$ a corresponding Cartan gauge transformation

$$\mathbf{D}(\psi) \;:\; \boldsymbol{s} = \{(\boldsymbol{a},\boldsymbol{e}_i)\} \;\mapsto\; \boldsymbol{s}' = \{(\mathrm{D}^{1,0}(\psi)\boldsymbol{a},\mathrm{D}^{1,0}(\psi)\boldsymbol{e}_i)\} \tag{1.4}$$

to another moving frame $\boldsymbol{s}'$ with the same domain as that of $\boldsymbol{s}$. Hence, upon covering $GA\mathbf{M}$ with a bundle atlas of local trivializations associated, in accordance with (7.7.3), with local sections of $GA\mathbf{M}$, we can assign to each $\psi \in Diff\mathbf{M}$ a vertical automorphism $\boldsymbol{\psi} \in \mathrm{Aut}_{\mathbf{M}}\mathbf{P}$, $\mathbf{P} \cong GA\mathbf{M}$, in accordance with (7.7.4)-(7.7.9) and (1.4), thus arriving at an isomorphism between $Diff\mathbf{M}$ and a subgroup $\mathrm{G}_A(Diff\mathbf{M})$ of $\mathrm{Aut}_{\mathbf{M}}GA\mathbf{M}$.

For each $\boldsymbol{\psi} \in \mathrm{G}_A(Diff\mathbf{M})$ the action of a diffeomorphism ψ on the base manifold of the Poincaré frame bundle $P\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$, viewed as a subbundle of general affine bundle $GA\mathbf{M}$, has been transformed into a vertical bundle automorphism $\boldsymbol{\psi}: GA\mathbf{M} \to GA\mathbf{M}$ which maps a given cross-section $\boldsymbol{s}$ of $GA\mathbf{M}$, that is also a cross-section of $P\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$, into another cross-section $\boldsymbol{s}'$ of $GA\mathbf{M}$. In general, $\boldsymbol{s}'$ is a cross-section of the Poincaré frame bundle $P\mathbf{M}^{\uparrow}(\boldsymbol{g}'^{\mathrm{L}})$ corresponding to the Lorentzian metric $\boldsymbol{g}'^{\mathrm{L}}$ in (1.1), and it is the outcome of the push-forward of $\boldsymbol{g}^{\mathrm{L}}$ under the diffeomorphism ψ.[3] Consequently, let us introduce the *metrized coframe bundle* $P^{*}\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$ which, by definition, consists of all the affine coframes $\{\boldsymbol{a}^{*}(x),\boldsymbol{\theta}^{i}(x)\} \in GA^{*}\mathbf{M}$ for which $\{\boldsymbol{\theta}^{i}(x)\}$ is dual to a Lorentz frame $\{\boldsymbol{e}_i(x)\} \in$

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[2] Cf. Sec. 4.3. From the point of view of CGR alone, all the subsequent considerations in this section could be carried out with linear frame bundles and Lorentz frame subbundles. The use of affine frame bundles and Poincaré frame subbundles is dictated by the requirements of QGR, that will emerge later in this chapter.

[3] Since bundles possessing diffeomorphic base manifolds are deemed equivalent, this gauge-theoretical formulation of a general covariance in CGR can be immediately extended to the more general case where $\mathbf{M} \neq \mathbf{M}'$ in (2.2.1) – but that does not lead to any additional physical insights.

$L\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$ for some Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ belonging to a CGR gauge orbit $(\mathbf{M},\mathcal{g}^{\mathrm{L}})$ over $\mathbf{M}$, defined in (1.2).

We shall call a cross-section of $P^{*}\mathbf{M}^{\uparrow}(\mathcal{g}^{\mathrm{L}})$ a *physical moving coframe* in $\mathbf{M}$ if it coincides with a cross-section $\boldsymbol{s}$ of one of its subbundles $P^{*}\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$. We shall denote such a physical moving coframe by $(\boldsymbol{s},\boldsymbol{g}^{\mathrm{L}})$, and we shall refer to the corresponding cross-section $\{\boldsymbol{\theta}^{i}(x)\,|\,x\in\mathbf{M}\}$ of $L^{*}\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$ as a *vierbein* (cf. Sec. 2.4). We can define on any physical moving coframe $(\boldsymbol{s},\boldsymbol{g}^{\mathrm{L}})$ the following combination,

$$(\boldsymbol{\psi},\phi):\ (\boldsymbol{s},\boldsymbol{g}^{\mathrm{L}})\ \mapsto\ (\boldsymbol{s}\circ\phi\circ\boldsymbol{\psi},\psi_{*}\boldsymbol{g}^{\mathrm{L}}),\quad \boldsymbol{\psi}\in \mathrm{G}_{A}(\mathit{Diff}\mathbf{M}),\ \phi\in \mathrm{Aut}_{\mathbf{M}}P\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}}),\quad (1.5)$$

of vertical actions from the right of the gauge group $\mathrm{Aut}_{\mathbf{M}}P\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$ (which is isomorphic to the group $\mathcal{G}(\mathbf{P}(\mathbf{M},\mathrm{ISO}^{\uparrow}(3,1))$ of Poincaré gauge transformations of the second kind) and the subgroup $\mathrm{G}_{A}(\mathit{Diff}\mathbf{M})$ of $\mathrm{Aut}_{\mathbf{M}}GA\mathbf{M}$.

According to (7.7.9), the action of vertical automorphisms on elements of principal bundles is from the right, so that (1.5) leads to the following group multiplication law

$$(\boldsymbol{\psi},\boldsymbol{\phi})(\boldsymbol{\psi}',\boldsymbol{\phi}')=((\boldsymbol{\psi}\circ\phi')\circ\boldsymbol{\psi}',\boldsymbol{\phi}\boldsymbol{\phi}'),\quad \boldsymbol{\phi}=\mathcal{g}_{\phi},\ \boldsymbol{\phi}'=\mathcal{g}_{\phi'}\in\mathcal{G}(\mathbf{P}(\mathbf{M},\mathrm{ISO}^{\uparrow}(3,1))).\quad (1.6)$$

Hence, the resulting combination of Poincaré gauge invariance with the diffeomorphism invariance of CGR models can be embedded into the *CGR gauge group* of the second kind given by the semidirect product

$$\mathcal{G}(\mathbf{M},\mathcal{g}^{\mathrm{L}})=\mathrm{G}_{A}(\mathit{Diff}\mathbf{M})\wedge\mathcal{G}(\mathbf{P}(\mathbf{M},\mathrm{ISO}^{\uparrow}(3,1))),\quad (1.7)$$

corresponding to an equivalence class $\mathcal{g}^{\mathrm{L}}$ of physically equivalent Lorentzian metrics on $\mathbf{M}$. It is the pair consisting of the metrized coframe bundle $P^{*}\mathbf{M}^{\uparrow}(\mathcal{g}^{\mathrm{L}})$ and the CGR gauge group $\mathcal{G}(\mathbf{M},\mathcal{g}^{\mathrm{L}})$ interrelating its physical moving coframes that can be legitimately regarded as a classical geometry for a general relativistic spacetime whose quantum counterpart we shall derive later in this chapter.

In Sec. 2.7 we described the algorithmic procedure for computing a maximal Cauchy development $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ for a CGR spacetime by sequentially computing the Lorentzian segments $(\mathbf{S}_{n},\boldsymbol{g}^{\mathrm{L}})$, $n=0,\pm1,\pm2,\ldots$, which in the limit $\max(t_{n}-t_{n-1})\to+0$ will give rise to $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$. We now see that this procedure can be viewed as a method of deriving a representative $P^{*}\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$ by a sequential metrization of the bundle segments $GA^{*}\mathbf{S}_{n}$ of affine coframes, which describes the geometrodynamic evolution of a classical spacetime in *physically* observable terms that are subject to empirical verification. At the ontological level, such a conceptualization views spacetime as a physical entity undergoing a perpetual evolutionary process of *creation* of geometry in mutual interaction with with all matter and radiation fields, rather than as an object in a temporal state of global existence, which as such *exists* in its entirety only in a Platonic world of ideas.

It is this kind of evolutionary conceptualization of spacetime, whereby spacetime undergoes a *perpetual process of creative evolution*, rather than *being* in a temporal state of global existence, that can be most naturally extrapolated to the quantum regime.

8.2. Basic Aspects of Geometric Quantum Gravity

In Chapters 5-7 we have seen how a geometrodynamically evolving external gravitational field represented by a Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ within segments $\mathbf{S}_n$, $n = 0,1,2,...$, affects the evolution of various types of all matter and nongravitational radiation fields – to which we shall refer, as customary, by the collective term of "matter" fields.

In covariant quantum gravity the states of all these "matter" fields affect, in turn, the states of a quantum gravitational field $\boldsymbol{g}^{\mathrm{quan}}$, which incorporates the multi-graviton states of quantum gravitational radiation. On the other hand, the "covariant" approach to quantum gravity assumes (cf. Sec. 1.6) the existence of a prescribed Lorentzian manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{back}})$ carrying a background metric $\boldsymbol{g}^{\mathrm{back}}$ – with the most frequent assumption[4] being that $(\mathbf{M},\boldsymbol{g}^{\mathrm{back}})$ coincides with the Minkowski space $(\mathbf{R}^4,\boldsymbol{\eta})$. The "covariant" quantum gravitational field is then presented on $\mathbf{M}$ in the form $\boldsymbol{g}^{\mathrm{quan}} = \boldsymbol{g}^{\mathrm{back}} + \boldsymbol{g}^{\mathrm{cor}}$, where the "correction" $\boldsymbol{g}^{\mathrm{cor}}$ to the classical background field $\boldsymbol{g}^{\mathrm{back}}$ is a quantum field giving rise to gravitational radiation in the form of multi-graviton states defined globally on $(\mathbf{M},\boldsymbol{g}^{\mathrm{back}})$. The resulting approach then relies on the conventional renormalization techniques developed for Yang-Mills fields to compute terms in a formal S-matrix perturbation expansion describing the scattering of gravitons.[5] However, these techniques were eventually proven (Goroff and Sagnotti, 1986) to lead to divergences even at the two-loop level.

In addition to the violation of diffeomorphism invariance, the splitting of the quantum gravitational field $\boldsymbol{g}^{\mathrm{quan}}$ into an *a priori* given classical background metric $\boldsymbol{g}^{\mathrm{back}}$ and a quantum "correction" $\boldsymbol{g}^{\mathrm{cor}}$ gives rise to the conceptual difficulty as to what $\boldsymbol{g}^{\mathrm{back}}$ actually represents from a physical point of view. Indeed, the most natural assumption would be that it represents a mean metric that is macroscopically observable in Nature. However, that assumption is at odds with $\boldsymbol{g}^{\mathrm{back}}$ being *a priori* prescribed (and, in practice, on pragmatic computational grounds alone), since any such mean metric would obviously have to *physically* emerge during the process of a geometrodynamic evolution, which is has to perpetually take into account the influence of the actually prevailing states of the quantum "matter" fields upon the states of the quantum gravitational field $\boldsymbol{g}^{\mathrm{quan}}$, as well as the reaction of that gravitational field upon those "matter" fields.

The resolution to this conceptual dilemma presents itself most naturally if the *quantum* geometrodynamic evolution is viewed as follows: the presence of a specific state of the quantum gravitational field $\mathbf{g}^{\mathrm{Q}}$ within a base-segment $\mathbf{S}_n$ dictates the quantum-geometric evolution of all the quantum "matter" fields from the inflow hypersurface $\Sigma_{t_{n-1}}$ to the outflow hypersurface Σ_{t_n} of that base-segment; Σ_{t_n} then becomes the inflow hypersurface of the next base-segment $\mathbf{S}_{n+1}$; the values of the local quantum energy-momentum density operators of all the "matter" fields in the resulting states of those fields along the inflow hypersurface Σ_{t_n} of $\mathbf{S}_{n+1}$ subsequently determine the formation of the state of the quantum

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[4] This assumption has no deeper physical justification beyond the ensuing easy availability of momentum space techniques which are very familiar "to anyone who is primarily trained in conventional particle physics" (Isham, 1975, p. 29). It not made, however, in DeWitt's (1967) seminal papers on quantum gravity.

[5] A formal treatment of the scattering of gravitons, which underlines the similarities with that of photons, was presented by Weinberg (1964, 1965). However, at a mathematically rigorous level, the later work of Strocchi (1968) revealed basic technical difficulties in combining the Einstein equations for weak gravitational fields with the conventional formulations of quantum field theory.

gravitational field $\mathbf{g}^{\mathrm{Q}}$ within that base-segment, by *creating* its mean value $\bar{\mathbf{g}}^{\mathrm{Q}}$ within $\mathbf{S}_{n+1}$, and by singling out within the quantum gravitational fibres above its points x the multi-graviton states of quantum gravitational radiation. This process then repeats itself from segment to segment, with the limit to "infinitely-thin" segments being eventually taken.

Naturally, the presence of constraints and quantum gravitational self-interaction has to be taken into account before this very sketchy picture of quantum-geometric gravitational evolution can be completed into a legitimate extrapolation to the quantum regime of Einstein's formulation of the theory of gravity. This will be ultimately achieved in this chapter by extending the base-segments $\mathbf{S}_n$ into supermanifolds $\boldsymbol{S}_n$ over which a quantum gravitational superbundle can be defined, and on whose sections a quantum gravitational gauge supergroup representing an extrapolation of (1.7) to the quantum regime can act. This action determines the physically equivalent representations of a quantum spacetime related to various physically equivalent mean values $\bar{\mathbf{g}}^{\mathrm{Q}}$ of a quantum gravitational superfield $\mathbf{g}^{\mathrm{Q}}$.

However, in order to describe the most basic ideas of geometric quantum gravity, and compare them with those in the covariant and canonical quantum gravity, let us temporarily ignore these superfield aspects, and regard the quantum gravitational field $\mathbf{g}^{\mathrm{Q}}$ as defined over base-segments $\mathbf{S}$ that are differential manifolds. If we make the fundamental assumption that the mean values $\bar{\mathbf{g}}^{\mathrm{Q}}$ of that field in any physical configuration of local states in $\mathbf{S}$ are equal to those of a Lorentzian metric $\mathbf{g}^{\mathrm{M}}$, then we can carry out in any physical moving frame of the affine frame bundle $GA\mathbf{S}$ (cf. Sec. 4.3) the decomposition

$$\mathbf{g}^{\mathrm{Q}} = \mathbf{g}^{\mathrm{M}} + \mathbf{g} \ , \quad \mathbf{g}^{\mathrm{M}} = \eta_{ij}\, \boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j \ , \quad \{\boldsymbol{\theta}^i\} \in L^*\mathbf{S}^{\uparrow}(\bar{\mathbf{g}}^{\mathrm{Q}}) \ , \tag{2.1}$$

by taking into account the fact that the metrization induced in $\mathbf{S}$ by $\mathbf{g}^{\mathrm{M}}$ mediates the reduction of the linear frame bundle $GL\mathbf{S}$ to the Lorentz frame bundle $L\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$, as well as of the affine frame bundle $GA\mathbf{S}$ to the Poincaré frame bundle $P\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$. We shall refer to $\mathbf{g}$ as the *quantum gravitational radiation field* (cf. Sec. 8.8) over the base-segment $\mathbf{S}$, since its role and mode of construction resembles that of the other quantum frame fields constructed in the preceding three chapters.

Physically, the metrization of a base-segment $\mathbf{S}$ obviously results from the inflow into $\mathbf{S}$ of quantum matter and radiation in the course of their quantum-geometric propagation. It is clear, however, that only those metrizations of a base-segment $\mathbf{S}$ can be deemed physical for which its boundary $\partial\mathbf{S}$ emerges as spacelike, and for which the time-orientation of the coframes $P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$ is such that the outflow hypersurface Σ'' lies in the future of the inflow hypersurface Σ', so that the causal (in relation to $\mathbf{g}^{\mathrm{M}}$) futures of all points in Σ' intersect Σ''; whereas the causal pasts of all points in Σ'' intersect Σ'. This is a most essential aspect of causality in geometric quantum gravity, and the quantum-geometric propagation of gravitational as well as all "matter" has to abide by it.

This aspect of quantum-geometric causality does not represent a topologically trivial feature. In the case of classical spacetime manifolds, two 3-manifolds Σ' and Σ'' are called *cobordant* if there is a 4-manifold $\mathbf{S}$ whose boundary is the disjoint union of Σ' and Σ''. Furthermore, two cobordant manifolds are called *Lorentz cobordant* if there is a Lorentzian metric on $\mathbf{S}$ so that Σ' and Σ'' are both spacelike. Thus, mathematically the existence of cobordancy cannot be taken for granted. However, a theorem by Lickorish (1963) has established it for any two 3-manifolds, and a theorem by Reinhart (1963) has extended this

result by showing that any two such 3-manifolds are actually Lorentz cobordant. Further generalizations of these results to the case of n-dimensional manifolds Σ' and Σ'' were later established by Sorkin (1986) for n even, and by Friedman (1991) for n odd.

This means that, without any loss of generality, we can always assume that inflow and outflow surfaces of any initial-data base-segment are Lorentz cobordant, and that the 4-manifold $\mathbf{S}_0$ interpolated between them does carry at least one Lorentzian metric that makes them both spacelike. For all the other base-segments $\mathbf{S}$ in the causal future of $\mathbf{S}_0$ this feature will then follow from the very nature of quantum-geometric propagation.

Thus, as in covariant quantum gravity, we shall take it for granted that the states of any quantum gravitational radiation field $\mathbf{g}$ are represented by superpositions of multigraviton states, where a graviton will be assumed be massless[6] and to have spin[7] two. This means that the manifestly covariant wave functions of gravitons are supplied by spin-2 and mass-0 indefinite-metric representations of the Poincaré group $\mathrm{ISO}^{\uparrow}(3,1)$, which acts on the Poincaré coframes $\boldsymbol{u}$ in various bundle-segments $P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$.[8] Hence, the quantum gravitational radiation field $\mathbf{g}$ in (2.1) can be decomposed into creation and annihilation operators,

$$\mathbf{g} = \mathrm{g}_{ij}\,\boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j\,, \qquad \mathrm{g}_{ij} = \mathrm{g}_{ij}^{(+)} + \mathrm{g}_{ij}^{(-)}\,, \tag{2.2}$$

that depend on the choice of a physical moving frame given by a cross-section $\boldsymbol{s}$ of a Poincaré coframe bundle $P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$, as well as on Grassmannian gauge variables θ and $\bar{\theta}$. However, in contrast to the situation prevailing in covariant quantum gravity, no background metric is assumed in geometric quantum gravity, so that the decompositions in (2.1) and (2.2) are, in the geometric sense, purely (super)local. Hence, so are its multigraviton states, which will be therefore represented by state vectors that, as mathematical objects, exist in graviton fibres *above* the various points $x \in \mathbf{S}$ in any Poincaré gauge given by a cross-section of $P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$, once a metrization of $\mathbf{S}$ has emerged as a consequence of the quantum-geometric evolution of a gravitational field configuration.

A quantum-geometric evolution has at its disposal *all* the physically acceptable Lorentzian metrics over the base-segment $\mathbf{S}$, in the form of mean metrics $\mathbf{g}^{\mathrm{M}}$ that are space-compatible with the boundary of $\mathbf{S}$ in the sense of making that boundary spacelike. These mean Lorentzian metrics over $\mathbf{S}$ result from Riemannian 3-metrics created along the inflow boundary Σ' of $\mathbf{S}$, and from the components of extrinsic curvature which supply, in accordance with (2.7.20), their rates of change in the timelike directions orthogonal to Σ' In this respect, therefore, the quantum-geometric approach to gravity resembles the "true" canonical approach to quantum gravity (cf. Isham, 1975, Sec. 5). The crucial difference is that, from amongst all the Riemannian 3-metrics and extrinsic curvatures on Σ', quantum-geometric propagation first selects, on the basis of the inflowing quantum information, an entire QGR gauge orbit $(\mathbf{S}, \mathcal{g}^{\mathrm{M}})$ of Lorentzian mean metrics $\boldsymbol{g}^{\mathrm{M}}$ in $\mathbf{S}$, and that the quantum-gravitational metric fluctuations around those mean metrics are taking place *locally* and *stochastically*; whereas, in canonical gravity, they are taking place globally and coherently.

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[6] A non-zero graviton mass would contradict the observed bending of light (van Dam and Veltman, 1970).

[7] A spin-1 graviton would give rise, as in the electromagnetic case, to repulsive forces; whereas a spin-0 graviton would contradict the observed bending of light (Seielstad *et al.*, 1970; Muckelman *et al.*, 1970; Sramek, 1971).

[8] Such representations were first derived by Bracci and Strocchi (1972, 1975), and by Bertrand (1978).

The physically implications of the latter type of fluctuations are most clearly manifested in the superspace approach (cf. Isham, 1975, Sec. 6) to the canonical quantization of gravity, where a physical interpretation is assigned to the "wave function" $\Psi(\gamma,\phi)$ over the superspace of all Riemannian 3-metrics γ in the initial-data 3-manifold Σ_0 (cf. Sec. 2.7), and over all participating "matter" fields, collectively denoted by ϕ. This means that in this conceptualization quantum metric fluctuations manifest themselves "simultaneously" across the entire reach of our Universe. However, such an epistemic hypothesis is impossible to verify empirically in the absence of a global "time" observable that would provide the operational instructions as to *when* the measurement of a given γ should be performed – a basic source of conceptual difficulties related to the "issue of time in [canonical] quantum gravity" (cf. Ashtekar and Stachel, 1961, Part II).

It might be thought that this problem can be by-passed by making the transition to the Euclidean variant of the superspace approach to quantum gravity (Hawking, 1979-88), since that approach does not involve the concept of time. Rather, it assumes that the transition amplitude from a state with 3-metric γ' and "matter" fields ϕ' on a 3-surface Σ' to a state with 3-metric γ'' and "matter" fields ϕ'' on a 3-surface Σ'' can be represented in the form of the "Euclidean path integral"

$$\langle \gamma'', \phi'', \Sigma'' | \gamma', \phi', \Sigma' \rangle = \int \mathcal{D}[\gamma,\phi] \exp(-S_E[\gamma,\phi]) \tag{2.3}$$

The above functional integration is supposed to be carried over all compact 3-metrics and all regular "matter" fields in the resulting compact manifolds that have the prescribed values on Σ' and Σ'' (Hartle and Hawking, 1983). However, not only does the indefiniteness of the Euclidean action S_E make the functional integral in (2.3) ill-defined outside the semi-classical approximation, but if past experience with classical stochastic processes is any guide, then the family of all smooth 3-metrics would be expected to be of "measure-zero" *if* a mathematically correct probabilistic interpretation of $|\Psi(\gamma,\phi)|^2$ actually existed. Indeed, the assumption of smoothness of such metrics on the basis that all metrics are smooth in *classical* canonical gravity is as sound as the assumption that *all* paths of a Brownian motion are smooth on the basis that all particle trajectories are smooth in Newtonian mechanics – whereas, as we know from the correct treatment of Brownian motion, the entire family of such paths actually has Wiener measure zero. Hence, it would be *all* "stochastic" metrics, including the "distributional" ones, which would be expected to carry the true quantum content of such canonical models of quantum gravity. However, such an approach to canonical quantum gravity would have to overcome very difficult mathematical obstacles, since "it is not at all clear what is meant by a 'distributional' metric, and how this would affect the ideas of Riemannian geometry" (Isham, 1991, p. 366).

Furthermore, even *if* this technical problem were somehow resolved, the physical problems of interpretation would still have to be faced. Weinberg describes these problems as follows: "Hawking has proposed (1984) that $\exp(-S_E[\gamma,\phi])$ should be regarded as proportional to the probability of a particular metric and matter field history. It is not immediately clear what is meant by this – even supposing that we had a godlike ability to measure the gravitational and matter fields throughout space-time, it would be in a space-time of Lorentzian rather than Euclidean signature. ... [Other] questions still arise

concerning the probabilistic interpretation of Ψ, particularly with regard to normalization ... because in [the] functional integral [that appears in (2.3)] we are summing up possibilities which are not exclusive; if the universe has some γ_1 and ϕ_1 on some 3-surface, then it may also have some other γ_2 and ϕ_2 on some other 3-surface." (Weinberg, 1989, p. 15.)

In contrast, the quantum fluctuations envisaged by the quantum-geometry approach to gravity occur in a truly random fashion, within quantum fibres above base locations in a spacetime. Thus, it is only the mean metrics $\mathbf{g}^{\mathrm{M}}$ in (2.1) that are smooth – as befits macroscopic observations – but not the quantum gravitational fluctuations around those metrics. Moreover, each $\mathbf{g}^{\mathrm{M}}$ is not an *a priori* given background metric, but rather it is a Lorentzian metric perpetually created by the quantum-geometric evolution of gravitational and "matter" fields. Hence, although the quantum-geometry approach shares some basic features with both the canonical as well as with the covariant approaches to the quantization of gravity, it avoids making physically unwarranted assumptions that are difficult to justify from the indeterministic point of view implicit in any form of quantum ontology.

8.3. Superlocal Graviton States and Internal Gauges

Using the quantum-geometric treatment of Yang-Mills fields in the preceding chapter as a guide, we assume that the quantum gravitational state vectors belong to quantum fibres above all coframes $\boldsymbol{u} = \{\boldsymbol{a}^*(x), \boldsymbol{\theta}^i(x)\}$ within the metrized bundles $P^*\mathbf{S}^{\uparrow}(\mathcal{g}^{\mathrm{M}})$ to which the inflow of quantum data gives rise within a base-segment $\mathbf{S}$. We shall refer to state vectors within fibres that lie above $\boldsymbol{u} \in P^*\mathbf{S}^{\uparrow}(\mathcal{g}^{\mathrm{M}})$, rather than directly above base locations $x \in \mathbf{S}$, as *superlocal*.[9] Hence, we shall begin the process of geometro-stochastic quantization of gravity by constructing single-graviton fibres $\mathbf{Z}_{\boldsymbol{u}}$ consisting of the graviton wave functions

$$\boldsymbol{f}(\boldsymbol{u};\zeta) = (2\pi)^{-3/2} \int_{V_0^+} \exp(-i\zeta \cdot k)\, \tilde{\boldsymbol{f}}(\boldsymbol{u};k)\, d\Omega_0(k) \ , \qquad \zeta = q - i\ell v \ , \tag{3.1}$$

that correspond, in accordance with (7.3.13), to all the momentum wave functions

$$\tilde{\boldsymbol{f}}(\boldsymbol{u};k) = \tilde{f}_{ij}(k)\, \boldsymbol{\theta}^i(x) \otimes \boldsymbol{\theta}^j(x) \ , \qquad \tilde{f}_{ij}(k) \equiv \tilde{f}_{ji}(k) \ , \tag{3.2}$$

obtained from 16-tuples $(\tilde{f}_{00}, \tilde{f}_{01}, \ldots, \tilde{f}_{33}) \in \tilde{\mathbf{K}} \otimes \tilde{\mathbf{K}}$ – where $\tilde{\mathbf{K}}$ coincides with the Krein space defined in Sec. 7.2.

The symmetry condition in (3.2) is imposed in order to eliminate the spin-1 components from the Poincaré group representation

$$\tilde{U}(a,\Lambda) \ : \ \tilde{f}_{ij}(k) \ \mapsto \ \tilde{f}'_{ij}(k) = \exp(ia \cdot k) \Lambda_i^{\ i'} \Lambda_j^{\ j'} \tilde{f}_{i'j'}(\Lambda^{-1}k) \ . \tag{3.3}$$

It will also secure the physically required symmetry of superlocal quantum gravitational metric fluctuations. However, it has to be supplemented by the subsidiary condition

$$\mathrm{Tr}\, \tilde{\boldsymbol{f}}(k) := \eta^{ij} \tilde{f}_{ij}(k) \equiv 0 \ , \tag{3.4}$$

[9] In physics literature the term "superlocal" is reserved for locations within supermanifolds. However, as we shall see later in this chapter, the coframes $\boldsymbol{u}$ mark locations within a quantum spacetime supermanifold.

in order to eliminate also the spin-0 components.

Clearly, as a Krein space, $\mathbf{Z}_{\boldsymbol{u}}$ carries the indefinite inner product

$$\langle \boldsymbol{f} | \boldsymbol{f}' \rangle = \int_{V_0^+} \eta^{ii'} \eta^{jj'} \tilde{f}^*_{ij}(k)\, \tilde{f}'_{i'j'}(k)\, d\Omega_0(k) \ , \tag{3.5}$$

and the $J_{\boldsymbol{u}}$-inner product associated with the coframe $\boldsymbol{u} \in P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathbf{M}})$:

$$(\boldsymbol{f}|\boldsymbol{f}')_{J_{\boldsymbol{u}}} = \sum_{i,j=0}^{3} \int_{V_0^+} \tilde{f}^*_{ij}(k)\, \tilde{f}'_{ij}(k)\, d\Omega_0(k) \ . \tag{3.6}$$

The former is left invariant by the representation

$$U(a,\Lambda) \ : \ f_{ij}(q - i\ell v) \ \mapsto \ f'_{ij}(q - i\ell v) = \Lambda_i^{\ i'} \Lambda_j^{\ j'} f_{i'j'}(\Lambda^{-1}(q-a) - i\ell\Lambda^{-1}v) \ , \tag{3.7}$$

$$\boldsymbol{f}(\boldsymbol{u};\zeta) = f_{ij}(\zeta)\, \boldsymbol{\theta}^i(x) \otimes \boldsymbol{\theta}^j(x) \ , \quad f_{ij}(\zeta) = (2\pi)^{-3/2} \int_{V_0^+} \exp(-i\,\zeta \cdot k)\, \tilde{f}_{ij}(k)\, d\Omega_0(k) , \tag{3.8}$$

to which (3.3) gives rise; whereas, the latter is not invariant under Lorentz boosts, but the majorant topology to which it gives rise is left invariant by (3.7). These facts can be used for the latter identification of some of the fibres $\mathbf{Z}_{\boldsymbol{u}}$ under quantum gravitational gauge transformations of the second kind.

The representation (3.7) in $\mathbf{Z}_{\boldsymbol{u}}$ is not irreducible. Indeed, if we introduce generalized Lorenz gauge conditions by analogy with those in (7.9.7), so that for the graviton wave functions in $\mathbf{Z}_{\boldsymbol{u}}$ they assume the form

$$\partial^i f_{ij}(\zeta) = b_j(\zeta) \ , \quad \partial^i = \eta^{ij}\, \partial/\partial q^j \ , \quad i,j = 0,1,2,3 \ , \tag{3.9}$$

and if we require that 4-tuple of functions $b_j(\zeta)$ transforms as follows,

$$U(a,\Lambda) \ : \ b_j(q - i\ell v) \ \mapsto \ b'_j(q - i\ell v) = \Lambda_j^{\ k} b_k(\Lambda^{-1}(q-a) - i\ell\Lambda^{-1}v) \ , \tag{3.10}$$

then the corresponding generalized Lorenz gauge subfibres

$$\mathbf{Z}_{\boldsymbol{u}}^{(b)} = \left\{ \boldsymbol{f} \in \mathbf{Z}_{\boldsymbol{u}} \ \middle| \ \partial^i f_{ij} = b_j \ , \ j = 0,1,2,3 \right\} , \tag{3.11}$$

are obviously mapped into each other by the transformations in (3.7) and (3.10). In particular, the proper Lorenz gauge subfibre, which corresponds to $b_j(\zeta) \equiv 0, j = 0,1,2,3$, is left invariant by (3.7).

On the other hand, the subfibres in (3.11) corresponding to any generalized Lorenz gauge are left invariant by the *internal* Lorenz gauge transformations

$$\boldsymbol{f} = f_{ij}\, \boldsymbol{\theta}^i(x) \otimes \boldsymbol{\theta}^j(x) \ \mapsto \ \boldsymbol{f}' = f'_{ij}\, \boldsymbol{\theta}^i(x) \otimes \boldsymbol{\theta}^j(x) \ , \qquad \boldsymbol{f}, \boldsymbol{f}' \in \mathbf{Z}_{\boldsymbol{u}} \ , \tag{3.12a}$$

$$f'_{ij}(\zeta) = f_{ij}(\zeta) + \partial_i \lambda_j(\zeta) + \partial_j \lambda_i(\zeta) \ , \qquad \partial^i \partial_i \lambda_j(\zeta) \equiv 0 \ . \tag{3.12b}$$

To see that, we observe that zero-trace property (3.4) of the superlocal graviton states $\boldsymbol{f}$ and $\boldsymbol{f}'$ in (3.12) implies that

$$\partial^i \lambda_i(\zeta) = \tfrac{1}{2} \mathrm{Tr}[\boldsymbol{f}'(\boldsymbol{u};\zeta) - \boldsymbol{f}(\boldsymbol{u};\zeta)] \equiv 0 \ , \tag{3.13}$$

so that the functions in (3.12b) resulting from the transformation in (3.12a) satisfy the generalized Lorenz gauge conditions in (3.9) if the original functions satisfy them. We also note that gauge transformations in (3.12) are compatible with the Poincaré transformations in (3.7) only if we also transform the 4-tuple of functions $\lambda_j(\zeta)$ accordingly:

$$U(a,\Lambda) \; : \; \lambda_j(q - i\ell v) \; \mapsto \; \lambda'_j(q - i\ell v) = \Lambda_j{}^k \lambda_k(\Lambda^{-1}(q-a) - i\ell\Lambda^{-1}v) \; . \quad (3.14)$$

The significance of the corresponding functions that appear in electromagnetism and in the theory of Yang-Mills fields is readily understood in terms of external Lorenz gauge fixing procedures in Minkowski space. However, in CGR the only instance where the Minkowski space appears in the presence of a gravitational field is in linearized gravity. There, indeed, the "corrections" $h_{\mu\nu}$ to the Minkowski metric components $\eta_{\mu\nu}$ caused by the presence of a weak gravitational field undergo transformations of the type (3.12b) if an "infinitesimal" coordinate transformation of the globally Minkowski-like coordinates adopted in linearized gravity is performed (cf. Misner *et al.*, 1973, Box 18.2).

To understand, however, the potential physical significance of the functions λ_j and of the internal Lorenz gauge transformations in (3.12) in the presence of a generic Lorentzian metric $\mathbf{g}^{\mathbf{M}}$ that represents the mean metric field

$$\overline{\mathbf{g}}(x) = \eta_{ij}\, \boldsymbol{\theta}^i(x) \otimes \boldsymbol{\theta}^j(x) = \overline{g}_{\mu\nu}(x)\, \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu \; , \qquad x \in \mathbf{S} \; , \quad (3.15)$$

generated by a quantum gravitational field within a base-segment $\mathbf{S}$, let us consider within some neighborhood of a point $x_0 \in \mathbf{S}$ the generic "infinitesimal" coordinate transformation

$$x^\mu \; \mapsto \; x'^\mu = x^\mu + \varepsilon\xi^\mu(x) \; , \qquad x \in \mathcal{N}_{x_0} \subset \mathbf{S} \; , \quad \mu = 0,1,2,3 \quad (3.16)$$

A straightforward computation shows that under such a coordinate transformation the components of the $(0,2)$-tensor field $\overline{\mathbf{g}}$ change as follows (cf. Wheeler, 1979, p. 434),

$$\overline{g}_{\mu\nu} \; \mapsto \; \overline{g}_{\mu\nu} - \varepsilon\xi_{\mu;\nu} - \varepsilon\xi_{\nu;\mu} \; , \qquad \xi_\mu := \overline{g}_{\mu\nu}\xi^\nu \; , \quad (3.17)$$

if the 4-tuple of functions ξ^μ is treated as if they represented the components of a 4-vector field, so that the semicolons denote covariant derivatives,

$$\xi_{\mu;\nu} = \xi_{\mu,\nu} - \overline{\Gamma}^\kappa{}_{\mu\nu}\xi_\kappa \; , \qquad \xi_{\mu,\nu} := \partial\xi_\mu / \partial x^\nu \; , \quad (3.18)$$

with respect to the Levi-Civita connection to which $\overline{\mathbf{g}}$ gives rise in $\mathbf{S}$ (cf. Sec. 2.6):

$$\overline{\Gamma}^\kappa{}_{\mu\nu} = \tfrac{1}{2}\overline{g}^{\kappa\lambda}\left(\overline{g}_{\lambda\mu,\nu} + \overline{g}_{\lambda\nu,\mu} - \overline{g}_{\mu\nu,\lambda}\right) \; , \qquad \overline{g}^{\kappa\lambda}\overline{g}_{\lambda\mu} := \delta^\kappa_\mu \quad (3.19)$$

To endow the above considerations with a mathematically rigorous meaning, let us adopt an active point of view. Thus, let us consider a family of diffeomorphisms $\mathrm{D}_\xi(\varepsilon) : x \mapsto x'$ of the differential manifold $\mathbf{S}$ onto itself, determined by the integral curves of a *bona fide* vector field ξ given in terms of its components ξ^μ in an atlas of coordinate charts over a family of domains M_α that cover $\mathbf{S}$, so that they can be expressed in each of those charts as follows:

$$\mathrm{D}_\xi(\varepsilon) : \; x^\mu \; \mapsto \; x'^\mu = x^\mu + \varepsilon\xi^\mu(x) + O(\varepsilon^2) \; , \quad x \in M_\alpha \; , \quad \mu = 0,1,2,3 \; . \quad (3.20)$$

This gives rise to an active reinterpretation of (3.16) in terms of the maps

$$D_{\varepsilon\xi} \; : \; x \; \mapsto \; x' , \qquad x \in M_\alpha \subset \mathbf{S} \; , \quad x' \in \mathbf{S} , \quad x'^\mu = x^\mu + \varepsilon \xi^\mu(x) \; , \tag{3.21}$$

collectively representing for the chosen atlas of charts an infinitesimal diffeomorphism on $\mathbf{S}$, so that ξ can be regarded as an element of the Lie algebra $L(\mathit{Diff}\,\mathbf{S})$ of the diffeomorphism group $\mathit{Diff}\,\mathbf{S}$ with Lie product given in any of the coordinate charts in (3.21) by

$$(\xi \times \eta)^\mu = \xi^\nu \partial_\nu \eta^\mu - \eta^\nu \partial_\nu \xi^\mu \; , \qquad \xi, \eta \in L(\mathit{Diff}\,\mathbf{S}) \; . \tag{3.22}$$

To establish that these definitions are actually atlas and chart independent, we observe that (3.20) leads in a unique manner to the operators

$$\boldsymbol{X}_\xi \; : \; f \; \mapsto \; f' \; , \qquad f'(x) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left[f(\mathrm{D}_\xi^{-1}(\varepsilon)(x)) - f(x) \right] \; , \tag{3.23}$$

acting on scalar fields f in $\mathbf{S}$, so that according to (2.1.3) they determine a coordinate-independent vector field on $\mathbf{S}$, with the following value at any $x \in \mathbf{S}$:

$$\boldsymbol{X}_\xi(x) = - \xi^\mu(x) \partial_\mu \in T_x \mathbf{S} \; , \qquad \partial_\mu = \partial / \partial x^\mu \; . \tag{3.24}$$

Consequently, the Lie brackets for the family of all these vector fields are also coordinate-independent, and we have[10]

$$\left[\boldsymbol{X}_\xi, \boldsymbol{X}_\eta \right] = - \boldsymbol{X}_{\xi \times \eta} \; , \quad \boldsymbol{X}_\xi = - \xi^\mu \partial_\mu \; , \; \boldsymbol{X}_\eta = - \eta^\mu \partial_\mu \; , \; \boldsymbol{X}_{\xi \times \eta} = - (\xi \times \eta)^\mu \partial_\mu \; , \tag{3.25}$$

so that the atlas-independence of the Lie product in (3.22) is thereby established.

The diffeomorphisms in (3.20) act upon the base manifold $\mathbf{S}$ of any (r,s)-tensor bundle associated with the general linear bundle $\mathit{GL}\mathbf{S}$, and give rise to the transformations $\mathrm{D}^{r,s}(\mathrm{D}_\xi(\varepsilon))$ defined by (1.3) as they act on any (r,s)-tensor field $\boldsymbol{T}$. Consequently, we can say that the infinitesimal transformation[11]

$$\boldsymbol{T} \; \mapsto \; \boldsymbol{T} + \varepsilon \delta_\xi \boldsymbol{T} \; , \qquad \delta_\xi \boldsymbol{T}(x) := \lim_{\varepsilon \to 0} \varepsilon^{-1} \left[\mathrm{D}_\xi(\varepsilon)_* \boldsymbol{T}(\mathrm{D}_\xi^{-1}(\varepsilon)(x)) - \boldsymbol{T}(x) \right] \; , \tag{3.26}$$

represents the change of value of that tensor field under the infinitesimal diffeomorphism in (3.21) determined by some $\xi \in L(\mathit{Diff}\,\mathbf{S})$. In particular, since the mean metric field $\overline{\mathbf{g}}$ can be viewed as a symmetric $(0,2)$-tensor field in $\mathbf{S}$, when this procedure is applied to $\overline{\mathbf{g}}$ it gives rise, after it is expressed in the coordinate charts of the atlas in (3.21), to the following infinitesimal transformation of that metric field:

$$\overline{g}_{\mu\nu} \; \mapsto \; \overline{g}'_{\mu\nu} = \overline{g}_{\mu\nu} - \varepsilon \left(\overline{g}_{\lambda\nu} \partial_\mu \xi^\lambda + \overline{g}_{\mu\lambda} \partial_\nu \xi^\lambda + \xi^\lambda \partial_\lambda \overline{g}_{\mu\nu} \right) . \tag{3.27}$$

This is seen to be in agreement with (3.17), once (3.18) and (3.19) are taken into account.

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[10] An explanation of the origins of the sign difference between Lie product in (3.22) and the commutator bracket in (3.25) can be found in (Isham and Kuchař, 1985, p. 291). This type of commutators was previously considered by Nishijima and Okawa (1978) from the "passive" point of view of the "infinitesimal" coordinate transformations in (3.16). However, in that case one is not dealing with a true Lie algebra, since coordinate transformations do *not* constitute a group, in view of the fact that generically they do not possess the same domain of definition.

[11] These types of transformations appear in all gauge-group treatments of quantum gravity that are based on formal analogies with the quantization of Yang-Mills fields – cf., e.g., Delbourgo and Medrano (1976), Townsend and van Nieuwenhuizen (1977), Nakanishi (1978, 1979), Kugo and Ojima (1978b), Nishijima and Okawa (1978).

However, from the active point of view expounded in Sec. 8.1, the effect of a diffeomorphism $\psi : \mathbf{S} \to \mathbf{S}$ on a cross-section $(\boldsymbol{s}, \mathbf{g}^{\mathrm{M}})$ of $P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$ is expressed by (1.5), with ϕ being in such a case the identity transformation. For the infinitesimal diffeomorphism in (3.21), corresponding to some $\xi \in L(Diff\mathbf{S})$, that effect gives rise to the transition from the vierbein (i.e., physical cross-section of $L^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$ – cf. Sec. 8.1) that appears in (3.15) to a vierbein for the infinitesimally transformed metric in (3.27):

$$\boldsymbol{\theta}^i = \lambda^i{}_\mu \, \boldsymbol{dx}^\mu \quad \mapsto \quad \boldsymbol{\theta}'^i = \lambda^i{}_\mu \, \boldsymbol{dx}^\mu - \varepsilon(\lambda^i{}_\nu \partial_\mu \xi^\nu + \xi^\nu \partial_\nu \lambda^i{}_\mu)\boldsymbol{dx}^\mu \tag{3.28}$$

Hence, (3.27) and (3.28) give rise to an external Cartan gauge transformation from $(\boldsymbol{s}, \mathbf{g}^{\mathrm{M}})$ to $(\boldsymbol{s}', \mathbf{g}'^{\mathrm{M}})$. This external transformation can be transferred in a natural manner to the tangent spaces of the base-segment $\mathbf{S}$ by setting

$$\xi^i_x(q) = \xi^i\left(\exp_x[(q^i + a^i)\boldsymbol{e}_i(x)]\right) , \qquad q = q^i \boldsymbol{e}_i(x) \in T_x\mathbf{S} , \tag{3.29}$$

in any neighborhood of each point $x \in \mathbf{S}$ where the exponential map (cf. (4.2.15)) is one-to-one, where $\{\boldsymbol{e}_i(x)\} \in L\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$ is the Lorentz frame dual to $\{\boldsymbol{\theta}^i(x)\} \in L^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$.

As we shall see in Sec. 8.6, the resulting transformations can be set in one-to-one correspondence with families of functions $\lambda_j(\boldsymbol{u};\zeta), j = 0,1,2,3$, that satisfy the wave equation in (3.12b), and are associated with the coframes $\boldsymbol{u} \in L^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$ above each given $x \in \mathbf{S}$. This transition from diffeomorphisms which are external transformations (in the sense that they act on the base manifold), to Lorenz gauge transformations which are internal (in the sense that they act within fibres) will provide one of the cornerstones in setting the foundations of a gauge supergroup for geometric quantum gravity. This will be ultimately achieved by relating these internal gauge transformations to the formation of "ghost" and "antighost" states corresponding to unphysical degrees of freedom in graviton polarization.

We therefore turn next to the formulation and study of the various types of graviton polarization states within the fibres $\mathbf{Z}_{\boldsymbol{u}}$.

8.4. Linearly and Circularly Polarized Graviton States

The construction of the states of linear graviton polarization can be carried out in very much the same manner as for the case of photons studied in Sec. 7.2, except that, in view of (3.15), it is more convenient to work with the *linear polarization cotetrads* $\{\boldsymbol{\varepsilon}^{(\mu)}(\boldsymbol{u};k)\}$,

$$\boldsymbol{\varepsilon}^{(\mu)}(\boldsymbol{u};k) \cdot \boldsymbol{\varepsilon}^{(\nu)}(\boldsymbol{u};k) = \eta^{\mu\nu} , \qquad \mu, \nu = 0,1,2,3 , \tag{4.1}$$

which are directly related to the Lorentz coframe $\{\boldsymbol{\theta}^i(x)\}$ corresponding to the Poincaré coframe $\boldsymbol{u} = \{\boldsymbol{a}^*(x), \boldsymbol{\theta}^i(x)\} \in P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$ as follows:

$$\boldsymbol{\varepsilon}^{(0)}(\boldsymbol{u};k) = \boldsymbol{\theta}^0(x) , \quad \boldsymbol{\varepsilon}^{(3)}(\boldsymbol{u};k) = |\mathbf{k}|^{-1}\textstyle\sum_{a=1}^{3} k_a \boldsymbol{\theta}^a(x) , \quad k \in \boldsymbol{V}_0^+ , \tag{4.2a}$$

$$\boldsymbol{k} \cdot \boldsymbol{\varepsilon}^{(1)}(\boldsymbol{u};k) = \boldsymbol{k} \cdot \boldsymbol{\varepsilon}^{(2)}(\boldsymbol{u};k) = 0 \ , \ \boldsymbol{k} = k_i \boldsymbol{\theta}^i(x) = k_0 \boldsymbol{\theta}^0(x) + \mathbf{k} \in \boldsymbol{V}^+_{0,\boldsymbol{u}} . \tag{4.2b}$$

The elements of these polarization cotetrads are related by the equations

$$\varepsilon^i_{(\mu)}(\boldsymbol{u};k)\, \varepsilon^{(\nu)}_i(\boldsymbol{u};k) = \delta_\mu{}^\nu , \qquad \varepsilon^i_{(\mu)}(\boldsymbol{u};k)\, \varepsilon^{(\mu)}_j(\boldsymbol{u};k) = \delta^i{}_j , \tag{4.3}$$

to the elements of the corresponding linear polarization tetrads $\{\boldsymbol{\varepsilon}_{(\mu)}(\boldsymbol{u};k)\}$, defined as in (7.2.11) in terms of the Lorentz frame $\{\boldsymbol{e}_i(x)\}$ which has $\{\boldsymbol{\theta}^i(x)\}$ as its dual. As in the case of the latter, their components for transversal polarization are not uniquely determined by (4.2b). Hence, we shall proceed as in Sec. 7.2, and set by analogy with (7.2.14)

$$\boldsymbol{\varepsilon}^{(r)}(\boldsymbol{u};k) = R(\hat{\mathbf{k}})\boldsymbol{\theta}^r(x)\,, \qquad \hat{\mathbf{k}} := \mathbf{k}/|\mathbf{k}| = \boldsymbol{\varepsilon}^{(3)}(\boldsymbol{u};k)\ , \qquad r = 1,2\ , \tag{4.4}$$

where the above rotation is determined from the decomposition

$$\hat{\Lambda}_{\boldsymbol{k}} = R(\hat{\mathbf{k}})\Lambda^{(3)}_{\omega(k)}\ , \qquad \hat{\Lambda}_{\boldsymbol{k}}\,\boldsymbol{k}_{\boldsymbol{u}} = \boldsymbol{k}\ , \qquad \boldsymbol{k}_{\boldsymbol{u}} := \boldsymbol{\theta}^0(x) + \boldsymbol{\theta}^3(x)\ , \tag{4.5}$$

in which the Lorentz boost is uniquely fixed by the following requirements:

$$\Lambda^{(3)}_{\omega(k)}\boldsymbol{k}_{\boldsymbol{u}} = e^{\omega(k)}\boldsymbol{k}_{\boldsymbol{u}}\ , \qquad e^{\omega(k)} = k_0\ , \qquad \Lambda^{(3)}_{\omega(k)}\boldsymbol{\theta}^r(x) = \boldsymbol{\theta}^r(x)\ , \qquad r = 1,2\ . \tag{4.6}$$

As a consequence, those components are uniquely determined except on the set of zero Ω_0-measure, consisting of all the values of $\boldsymbol{k}$ on the forward light-cone in $(\mathbf{R}^4,\boldsymbol{\eta})$ for which $\mathbf{k}$ and $\boldsymbol{\theta}^3(x)$ point in opposite directions, so that the rotation in (4.4) is not well-defined.

Upon replacing in (3.2) the elements of the Lorentz coframe $\{\boldsymbol{\theta}^i(x)\}$ with those of the linear polarization cotetrads $\{\boldsymbol{\varepsilon}^{(\mu)}(\boldsymbol{u};k)\}$, so that we have

$$\tilde{\boldsymbol{f}}(\boldsymbol{u};k) = \tilde{f}_{(\mu\nu)}(k)\,\boldsymbol{\varepsilon}^{(\mu)}(\boldsymbol{u};k)\otimes\boldsymbol{\varepsilon}^{(\nu)}(\boldsymbol{u};k)\ , \tag{4.7}$$

we can express the generic element $\boldsymbol{f}$ of the graviton fibre $\mathbf{Z}_{\boldsymbol{u}}$ in the following form:

$$\boldsymbol{f}(\boldsymbol{u};\zeta) = (2\pi)^{-3/2}\int_{V_0^+}\exp(-i\,\zeta\cdot k)\,\boldsymbol{\varepsilon}^{(\mu)}(\boldsymbol{u};k)\otimes\boldsymbol{\varepsilon}^{(\nu)}(\boldsymbol{u};k)\,\tilde{f}_{(\mu\nu)}(k)\,d\Omega_0(k)\ . \tag{4.8}$$

We can then introduce *graviton linear polarization frames* $\{\boldsymbol{\varepsilon}^{\mu\nu\zeta}\}$ with elements

$$\boldsymbol{\varepsilon}^{\mu\nu\zeta}(\boldsymbol{u};\zeta') = \int_{V_0^+}\exp[i(\bar{\zeta}-\zeta')\cdot k]\,\boldsymbol{\varepsilon}^{(\mu)}(\boldsymbol{u};k)\otimes\boldsymbol{\varepsilon}^{(\nu)}(\boldsymbol{u};k)\,d\Omega_0(k)\ , \tag{4.9}$$

and, by taking mass-zero limits as in Sec. 7.3, we can express the two inner products in (3.5) and (3.6), respectively, as follows:

$$\langle\boldsymbol{f}|\boldsymbol{f}'\rangle = \int\bar{f}^{\mu\nu\zeta}\,f'_{\mu\nu\,\zeta}\,d\tilde{\Sigma}(\zeta)\ , \qquad f^{\mu\nu\zeta} = \left\langle\boldsymbol{\varepsilon}^{\mu\nu\zeta}\middle|\boldsymbol{f}\right\rangle\ , \tag{4.10}$$

$$(\boldsymbol{f}|\boldsymbol{f}')_{J_{\boldsymbol{u}}} = \sum_{\mu,\nu=0}^{3}\int\bar{f}_{\mu\nu\zeta}\,f'_{\mu\nu\,\zeta}\,d\tilde{\Sigma}(\zeta)\ , \qquad f_{\mu\nu\zeta} = \eta_{\mu\kappa}\eta_{\nu\lambda}f^{\kappa\lambda\zeta}\ . \tag{4.11}$$

Hence, in the notation introduced in (4.2.11), the decomposition of any $\boldsymbol{f}\in\mathbf{Z}_{\boldsymbol{u}}$ into linear polarization modes assumes the form

$$\boldsymbol{f} = f_{\mu\nu\zeta}\,\boldsymbol{\varepsilon}^{\mu\nu\zeta} := \int d\tilde{\Sigma}(\zeta)\eta_{\kappa\mu}\eta_{\lambda\nu}f^{\kappa\lambda\zeta}\,\boldsymbol{\varepsilon}^{\mu\nu\zeta}\ , \qquad f^{\kappa\lambda\zeta} = \left\langle\boldsymbol{\varepsilon}^{\kappa\lambda\zeta}\middle|\boldsymbol{f}\right\rangle\ . \tag{4.12}$$

The main features of graviton states that were formulated in Sec. 8.3 in relation to the (external) Lorentz coframes $\{\boldsymbol{\theta}^i(x)\}$ can be reformulated in relation to the (internal) graviton linear polarization frames $\{\boldsymbol{\varepsilon}^{\mu\nu\zeta}\}$. Thus, the symmetry property of the local graviton state vector expressed in (3.2) in terms of ij-indices is seen to be equivalent to

$$f_{\mu\nu\zeta} \equiv f_{\nu\mu\zeta} \ , \qquad \mu,\nu = 0,1,2,3 \ , \qquad \boldsymbol{f} \in \mathbf{Z}_u \tag{4.13}$$

The subfibres in (3.11) corresponding to the generalized Lorenz subsidiary conditions in (3.9) can be characterized as follows:

$$\mathbf{Z}_u^{(b)} = \left\{ \boldsymbol{f} \in \mathbf{Z}_u \;\middle|\; f_{0\nu\zeta} - f_{3\nu\zeta} = b_{\nu\zeta} \ , \quad \nu = 0,1,2,3 \right\} . \tag{4.14}$$

In particular, the Lorenz subfibre is obtained by setting above $b_{\nu\zeta} \equiv 0$, $\nu = 0,1,3,3$. Moreover, we can take advantage of the formal analogies between the internal graviton structure and the linearized theory of classical gravity to arrive at additional TT (transverse traceless) gauge conditions (cf., e.g., Misner *et al.*, 1973, §35.4), which pinpoint the *TT-subfibre*

$$\mathbf{Z}_u^{\mathrm{TT}} = \left\{ \boldsymbol{f} \in \mathbf{Z}_u^{(0)} \;\middle|\; f_{0\nu\zeta} = f_{3\nu\zeta} \equiv 0 \ , \quad \nu = 0,1,2,3 \right\} . \tag{4.15}$$

As in linearized gravity (*ibid.*, §35.6), this singles out the following two independent polarization TT-modes of a local graviton state vector:

$$\boldsymbol{f}^{(1)}(\zeta) = f_{11\zeta}\left(\boldsymbol{\varepsilon}^{11\zeta} - \boldsymbol{\varepsilon}^{22\zeta}\right) , \qquad \boldsymbol{f}^{(2)}(\zeta) = f_{12\zeta}\left(\boldsymbol{\varepsilon}^{12\zeta} + \boldsymbol{\varepsilon}^{21\zeta}\right) . \tag{4.16}$$

Hence, the generic element $\boldsymbol{f}^{\mathrm{TT}}$ of the TT-subfibre in (4.15) can be expressed as a superposition of these two *physical* modes of polarization, so that the TT-subfibre of the graviton fibre $\mathbf{Z}_u$ can be characterized in the following alternative manner:

$$\mathbf{Z}_u^{\mathrm{TT}} = \left\{ \boldsymbol{f}^{(1)} + \boldsymbol{f}^{(2)} \;\middle|\; \boldsymbol{f} \in \mathbf{Z}_u^{(0)} \right\} . \tag{4.17}$$

It follows from (4.7)-(4.11) that a decomposition of any graviton state vector in the Lorenz subfibre of $\mathbf{Z}_u$ can be carried out in a manner that is analogous to the one for the photon state vectors in (7.4.15),

$$\boldsymbol{f} = \boldsymbol{f}^{\mathrm{TT}} + \boldsymbol{f}^{\perp} \in \mathbf{Z}_u^{(0)} \ , \qquad \boldsymbol{f}^{\mathrm{TT}} = \boldsymbol{f}^{(1)} + \boldsymbol{f}^{(2)} \in \mathbf{Z}_u^{\mathrm{TT}} \ , \tag{4.18a}$$

so that as a consequence we have

$$\langle \boldsymbol{f} | \boldsymbol{f} \rangle = \langle \boldsymbol{f}^{\mathrm{TT}} | \boldsymbol{f}^{\mathrm{TT}} \rangle \ , \qquad \langle \boldsymbol{f}^{\perp} | \boldsymbol{f}^{\perp} \rangle = \langle \boldsymbol{f}^{\perp} | \boldsymbol{f}^{\mathrm{TT}} \rangle = 0 \ . \tag{4.18b}$$

This means that the pseudo-orthogonal complement $\boldsymbol{f}^{\perp}$ of $\boldsymbol{f}^{\mathrm{TT}}$ belongs to the null-subfibre

$$\mathbf{N}_u = \left\{ \boldsymbol{f} \in \mathbf{Z}_u^{(0)} \;\middle|\; \langle \boldsymbol{f} | \boldsymbol{f} \rangle = 0 \right\} \tag{4.19}$$

of the graviton fibre $\mathbf{Z}_u$. Indeed, the method used in establishing that the photon subspaces in (7.2.10) and (7.2.13) actually coincide is equally well applicable to gravitons, so that all the null vectors in $\mathbf{Z}_u$ automatically satisfy the Lorenz gauge condition corresponding to $b_j(\zeta) \equiv 0, j = 0,1,2,3$, in (3.9). Taken in conjunction with (4.10)-(4.12), this implies that the generic element of the generalized Lorenz gauge subfibre in (4.14) can be uniquely decomposed into the following three components:

$$\boldsymbol{f} = \boldsymbol{f}^{\mathrm{TT}} + \boldsymbol{f}^{\perp} + \boldsymbol{f}^{b} \in \mathbf{Z}_u^{(b)} \ , \qquad \boldsymbol{f}^{\mathrm{TT}} \in \mathbf{Z}_u^{\mathrm{TT}} \ , \quad \boldsymbol{f}^{\perp} \in \mathbf{N}_u \ . \tag{4.20}$$

However, since under the Poincaré transformations in (3.7) the 4-tuple of gauge fixing functions $b_j(\zeta), j = 0,1,2,3$, transforms as in (3.10), in general such subfibres are not left invariant by all these Poincaré transformations. It is only the Lorenz subfibre of the graviton fibre $\mathbf{Z}_u$ that is left invariant, since $b_j(\zeta) \equiv 0$ in it; but, as was the case with photons in Sec. 7.2, the corresponding decomposition in (4.18a) is not invariant under the Poincaré transformations in (3.7), since the TT-subfibre of $\mathbf{Z}_u$ is not left invariant by Lorentz boosts.

On the other hand, we saw in Sec. 7.2 that when the representation in (7.2.5) was deemed to act on equivalence classes consisting of photon wave functions that were identified modulo null vectors, then the amplitudes resulting from the decompositions of these wave functions with respect to the circular polarization vectors in (7.2.25) transformed in accordance to a mass-0 and spin-1 Wigner-type representation of the Poincaré group. We shall show next that the same type of result holds for gravitons, with spin-1 being replaced by a spin-2 value.

Instead of using circular polarization vectors, as we did in (7.2.25), in the case of gravitons it is more convenient to work with the *circular polarization covectors*

$$\boldsymbol{\varepsilon}^{[\pm 1]}(\boldsymbol{u};k) = \mp\tfrac{1}{\sqrt{2}}\left(\boldsymbol{\varepsilon}^{(1)}(\boldsymbol{u};k) \pm i\boldsymbol{\varepsilon}^{(2)}(\boldsymbol{u};k)\right) = \mp\tfrac{1}{\sqrt{2}}\left(\boldsymbol{\theta}^1(x) \pm i\boldsymbol{\theta}^2(x)\right)\cdot R(\hat{\mathbf{k}}) \ , \quad (4.21a)$$

which together with the pair of covectors

$$\boldsymbol{\varepsilon}^{[+2]}(\boldsymbol{u};k) = k_i\,\boldsymbol{\theta}^i(x) = \boldsymbol{k} \ , \qquad \boldsymbol{\varepsilon}^{[-2]}(\boldsymbol{u};k) = k_0^{-1}\boldsymbol{\theta}^0(x) - \tfrac{1}{2}k_0^{-2}\,\boldsymbol{k} \ , \quad (4.21b)$$

gives rise to a *null polarization cotetrad* within the complexified cotangent space $\overline{T}_x{}^*\mathbf{S}$. This polarization cotetrad constitutes a basis within the linear space $\overline{T}_x{}^*\mathbf{S}$, and consists exclusively of null[12] covectors which obey the following relationships:

$$\boldsymbol{\varepsilon}^{[r]}(\boldsymbol{u};k)\cdot\boldsymbol{\varepsilon}^{[r]}(\boldsymbol{u};k) = 0 \ , \qquad r = \pm 1, \pm 2 \ , \quad (4.22a)$$

$$\boldsymbol{\varepsilon}^{[r]}(\boldsymbol{u};k)\cdot\boldsymbol{\varepsilon}^{[s]}(\boldsymbol{u};k) = 0 \ , \qquad r = \pm 1, \quad s = \pm 2 \ , \quad (4.22b)$$

$$\boldsymbol{\varepsilon}^{[\pm 1]}(\boldsymbol{u};k)\cdot\boldsymbol{\varepsilon}^{[\mp 1]}(\boldsymbol{u};k) = \boldsymbol{\varepsilon}^{[\pm 2]}(\boldsymbol{u};k)\cdot\boldsymbol{\varepsilon}^{[\mp 2]}(\boldsymbol{u};k) = 1 \ . \quad (4.22c)$$

These relationships can be easily verified by observing that, on account of (4.4)-(4.6),

$$\boldsymbol{\varepsilon}^{[r]}(\boldsymbol{u};k) = \hat{\Lambda}_{\boldsymbol{k}}\,\boldsymbol{\varepsilon}^{[r]}(\boldsymbol{u};(1,0,0,1)) \ , \qquad k \in \boldsymbol{V}_0^+ \subset \mathbf{R}^4 \ , \qquad r = \pm 1, \pm 2 \ . \quad (4.23)$$

Let us now expand the graviton state vector in (3.2) in terms of tensor products of the null polarization tetrad in (4.21), so that we have

$$\tilde{\boldsymbol{f}}(\boldsymbol{u};k) = \tilde{f}_{[rs]}(k)\boldsymbol{\varepsilon}^{[r]}(\boldsymbol{u};k)\otimes\boldsymbol{\varepsilon}^{[s]}(\boldsymbol{u};k) \in \mathbf{Z}_u \ , \quad (4.24)$$

if the Einstein summation convention is used for the indices $r,s = \pm 1, \pm 2$. We can then express the Poincaré group representation in (3.3) in the following equivalent form:

$$\tilde{U}(a,\Lambda)\colon\ \tilde{\boldsymbol{f}} \ \mapsto\ \tilde{\boldsymbol{f}}' = \exp(ia\cdot k)\tilde{f}_{[rs]}(\Lambda^{-1}k)\,\Lambda\boldsymbol{\varepsilon}^{[r]}(\boldsymbol{u};\Lambda^{-1}k)\otimes\Lambda\boldsymbol{\varepsilon}^{[s]}(\boldsymbol{u};\Lambda^{-1}k) \ . \quad (4.25)$$

---

[12] The dot-product in (4.22) denotes the bilinear (rather than the sesquilinear) extension of the Lorentzian metric to the complexified cotangent space at x, and the circular polarization covectors in (4.21a) are null only in it.

On the other hand, for any given proper Lorentz transformation Λ, and for any covector $\boldsymbol{k}$ belonging to the forward light cone within $T_x{}^*\mathbf{S}$, the Lorentz transformation

$$D(\Lambda,\boldsymbol{k}) = \hat{\Lambda}^{-1}_{\boldsymbol{k}}\,\Lambda\,\hat{\Lambda}_{\Lambda^{-1}\boldsymbol{k}}\ , \qquad \Lambda \in \mathrm{SO}_0(3,1)\ , \qquad \boldsymbol{k} \in \boldsymbol{V}^+_{0,\boldsymbol{u}}\ , \tag{4.26}$$

leaves the coframe-dependent covector $\boldsymbol{k}_{\boldsymbol{u}}$ in (4.5) invariant, so that it has to belong to its stability group. Hence, this transformation can be characterized by the two parameters

$$\theta = \theta(\Lambda,k) \in \mathbf{R}^1\ , \qquad z = z(\Lambda,\boldsymbol{k}) \in \mathbf{C}^1\ , \tag{4.27}$$

that appear in (7.1.6b). Upon using (4.4)-(4.6) and (4.21) we can compute that

$$D(\Lambda,\boldsymbol{k})\boldsymbol{\varepsilon}^{[+2]}(\boldsymbol{u};\hat{k}) = \boldsymbol{\varepsilon}^{[+2]}(\boldsymbol{u};\hat{k})\ , \qquad \hat{k} = (1,0,0,1), \tag{4.28a}$$

$$D(\Lambda,\boldsymbol{k})\boldsymbol{\varepsilon}^{[+1]}(\boldsymbol{u};\hat{k}) = e^{i\theta}\boldsymbol{\varepsilon}^{[+1]}(\boldsymbol{u};\hat{k}) + z\,\boldsymbol{\varepsilon}^{[+2]}(\boldsymbol{u};\hat{k})\ , \tag{4.28b}$$

$$D(\Lambda,\boldsymbol{k})\boldsymbol{\varepsilon}^{[-1]}(\boldsymbol{u};\hat{k}) = e^{-i\theta}\boldsymbol{\varepsilon}^{[-1]}(\boldsymbol{u};\hat{k}) - \bar{z}\,\boldsymbol{\varepsilon}^{[+2]}(\boldsymbol{u};\hat{k})\ , \tag{4.28c}$$

$$D(\Lambda,\boldsymbol{k})\boldsymbol{\varepsilon}^{[-2]}(\boldsymbol{u};\hat{k}) = \boldsymbol{\varepsilon}^{[-2]}(\boldsymbol{u};\hat{k}) + ze^{-i\theta}\,\boldsymbol{\varepsilon}^{[-1]}(\boldsymbol{u};\hat{k}) - \bar{z}e^{i\theta}\,\boldsymbol{\varepsilon}^{[+1]}(\boldsymbol{u};\hat{k}) + |z|^2\,\boldsymbol{\varepsilon}^{[+2]}(\boldsymbol{u};\hat{k}). \tag{4.28d}$$

It therefore follows from (4.23) that

$$\Lambda\,\boldsymbol{\varepsilon}^{[+2]}(\Lambda^{-1}k) = \boldsymbol{\varepsilon}^{[+2]}(k)\ , \qquad \boldsymbol{\varepsilon}^{[r]}(k) := \boldsymbol{\varepsilon}^{[r]}(\boldsymbol{u};k)\ , \quad r = \pm1,\pm2\ , \tag{4.29a}$$

$$\Lambda\,\boldsymbol{\varepsilon}^{[+1]}(\Lambda^{-1}k) = e^{i\theta}\boldsymbol{\varepsilon}^{[+1]}(k) + z\boldsymbol{\varepsilon}^{[+2]}(k)\ , \tag{4.29b}$$

$$\Lambda\,\boldsymbol{\varepsilon}^{[-1]}(\Lambda^{-1}k) = e^{-i\theta}\boldsymbol{\varepsilon}^{[-1]}(k) - \bar{z}\,\boldsymbol{\varepsilon}^{[+2]}(k)\ , \tag{4.29c}$$

$$\Lambda\,\boldsymbol{\varepsilon}^{[-2]}(\Lambda^{-1}k) = \boldsymbol{\varepsilon}^{[-2]}(k) + ze^{-i\theta}\,\boldsymbol{\varepsilon}^{[-1]}(k) - \bar{z}e^{i\theta}\,\boldsymbol{\varepsilon}^{[+1]}(k) + |z|^2\boldsymbol{\varepsilon}^{[+2]}(k)\ . \tag{4.29d}$$

Upon inserting $\Lambda\Lambda'$ into (4.29b) and (4.29c), and then comparing coefficients after the execution the group multiplications, we obtain the relation

$$\exp\left[\pm i\theta(\Lambda\Lambda',k)\right] = \exp\left[\pm i\theta(\Lambda,k)\right]\exp\left[\pm i\theta(\Lambda',\Lambda^{-1}k)\right], \qquad \Lambda,\Lambda' \in \mathrm{SO}_0(3,1), \tag{4.30}$$

which leads to the representation in (7.2.27). Consequently, when they are taken in conjunction with (4.25), the relations in (4.29) can be used to establish in the same manner as in Sec. 7.2 the equivalence between the spin-2 counterpart of the Wigner-type representation in (7.1.13) and the restriction of the indefinite-metric representation in (3.3) to equivalence classes of graviton momentum wave functions in the Lorenz gauge.

On the other hand, upon inserting (4.24) into (3.1) we obtain

$$\boldsymbol{f}(\boldsymbol{u};\zeta) = (2\pi)^{-3/2}\int_{V_0^+}\exp(-i\,\zeta\cdot k)\,\boldsymbol{\varepsilon}^{[r]}(\boldsymbol{u};k)\otimes\boldsymbol{\varepsilon}^{[s]}(\boldsymbol{u};k)\,\tilde{f}_{[rs]}(k)\,d\Omega_0(k), \tag{4.31}$$

as an alternative to (4.8). This suggests making the transition from the linear polarization frame in (4.9) to the *graviton polarization null-frame* whose elements are given by

$$\boldsymbol{\varepsilon}^{[rs]\zeta}(\boldsymbol{u};\zeta') = \int_{V_0^+}\exp[i(\bar{\zeta}-\zeta')\cdot k]\;\boldsymbol{\varepsilon}^{[r]}(\boldsymbol{u};k)\otimes\boldsymbol{\varepsilon}^{[s]}(\boldsymbol{u};k)\,d\Omega_0(k)\ , \tag{4.32}$$

so that (4.12) becomes replaced with

$$\boldsymbol{f} = f_{[rs]\zeta}\,\boldsymbol{\varepsilon}^{[rs]\zeta} \quad , \qquad f_{[rs]\zeta} = \eta_{rr'}\eta_{ss'}\left\langle \boldsymbol{\varepsilon}^{[r's']\zeta} \middle| \boldsymbol{f} \right\rangle . \tag{4.33}$$

It is then easily verified that the TT-subfibre of $\mathbf{Z}_{\boldsymbol{u}}$ in (4.17) can be expressed as

$$\mathbf{Z}_{\boldsymbol{u}}^{\mathrm{TT}} = \left\{ \sum\nolimits_{r=-1}^{+1} f_{[rr]\zeta}\,\boldsymbol{\varepsilon}^{[rr]\zeta} \;\middle|\; f_{[\pm1\pm1]\zeta} = f_{11\zeta} \mp i f_{12\zeta} \;,\;\; \boldsymbol{f} = f_{\mu\nu\zeta}\,\boldsymbol{\varepsilon}^{\mu\nu\zeta} \in \mathbf{Z}_{\boldsymbol{u}}^{(0)} \right\} ; \tag{4.34}$$

whereas the null-subfibre in (4.19) can be specified in the alternative form

$$\mathbf{N}_{\boldsymbol{u}} = \left\{ \sum\nolimits_{r=-1}^{+2} f_{[r2]\zeta}\,\boldsymbol{\varepsilon}^{[r2]\zeta} + \sum\nolimits_{s=-1}^{+1} f_{[2s]\zeta}\,\boldsymbol{\varepsilon}^{[2s]\zeta} \;\middle|\; \boldsymbol{f} \in \mathbf{Z}_{\boldsymbol{u}}^{(0)} \right\} . \tag{4.35}$$

Hence, in accordance with (4.18), the Lorenz subfibre of $\mathbf{Z}_{\boldsymbol{u}}$ contains only graviton state vectors for which the polarization amplitudes corresponding $r = \pm2$ and/or $s = \pm2$ in (4.32) are all zero, so that we have

$$\mathbf{Z}_{\boldsymbol{u}}^{(0)} = \left\{ \sum\nolimits_{r,s=-1}^{+1} f_{[rs]\zeta}\,\boldsymbol{\varepsilon}^{[rs]\zeta} \;\middle|\; \boldsymbol{f} = f_{[rs]\zeta}\,\boldsymbol{\varepsilon}^{[rs]\zeta} \in \mathbf{Z}_{\boldsymbol{u}} \right\} . \tag{4.36}$$

During the treatment of photons in Secs. 7.2 and 7.3, we had concentrated on state vectors in the Lorenz gauge, primarily because the Gupta-Bleuler formalism favours that gauge. However, when generalized Lorenz gauges are used, photon states that have nonzero components with respect to the $r = -2$ element of a polarization null tetrad also appear. Such components have been referred to in literature as "bad ghosts"; whereas those corresponding to $r = +2$ have been called "good ghosts" (Dürr and Rudolph, 1969). In some more recent studies of BRST theories for quantum Yang-Mills fields the respective terms "parent" and "daughter" are often used instead.[13] Both of them, however, represent only unphysical polarization degrees of freedom in those theories, and the same remains true in quantum gravity. In fact, we shall see in subsequent sections that their elimination in a quantum gravitational fibre is tantamount to obtaining a geometrically local solution of the quantum versions of the Einstein constraint equations described in Sec. 2.7.

In the case of graviton state vectors, the *physical* amplitudes of circular polarization are supplied by the two components

$$\alpha_{\pm}(\zeta) := f_{[\pm1\pm1]\zeta} = f_{11\zeta} \mp i f_{12\zeta} \tag{4.37}$$

in the expansion (4.33) with respect to the null-frame with the elements in (4.32). If we use the representation in (3.7) to construct the *graviton bundle*

$$\mathbf{Grav}(\mathbf{S}, \boldsymbol{g}^{\mathrm{M}}) = P^{*}\mathbf{S}^{\uparrow}(\boldsymbol{g}^{\mathrm{M}}) \times_{\mathrm{ISO}^{\uparrow}(3,1)} \mathbf{Z}_{\boldsymbol{u}_0} \tag{4.38}$$

from the typical fibre $\mathbf{Z}_{\boldsymbol{u}_0}$ corresponding to the standard frame (5.5.12) in $(\mathbf{R}^4, \boldsymbol{\eta})$ and the Poincaré coframe bundle $P^{*}\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$, then these null-frame elements transform according to

[13] Cf. (Nishijima, 1984), or (Horuzhy and Voronin, 1989, p. 682). Covariant quantum gravity shares common features with quantum electrodynamics as well as with Yang-Mills theory. According to Nakanishi (1983, p. 174): "Though it is widely believed that quantum gravity is similar to Yang-Mills theory, I emphasize that *quantum gravity is more similar to quantum electrodynamics*."

$$U(a,\Lambda) : \quad \boldsymbol{\varepsilon}^{[rs]\zeta}(\boldsymbol{u};\zeta') \quad \mapsto \quad \boldsymbol{\varepsilon}'^{[rs]\zeta}(\boldsymbol{u}\cdot(a,\Lambda);\zeta') =$$
$$\int_{V_0^+} \exp[i(\overline{\zeta}-\zeta'+a)\cdot k]\, \boldsymbol{\varepsilon}^{[r]}(\boldsymbol{u}\cdot\Lambda;k)\otimes\boldsymbol{\varepsilon}^{[s]}(\boldsymbol{u}\cdot\Lambda;k)\, d\Omega_0(k)\ , \tag{4.39}$$

under action from the right of $(a,\Lambda)\in \mathrm{ISO}^{\uparrow}(3,1)$ upon $\boldsymbol{u}\in P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$. The null tetrads in (4.39) are also dependent on the Poincaré coframe $\boldsymbol{u}$, and due to their mode of construction in (4.21), we have for any $(0,\Lambda)\cong\Lambda\in \mathrm{SO}_0(3,1)$

$$\boldsymbol{\varepsilon}^{[\pm1]}(\boldsymbol{u}\cdot\Lambda;k) = e^{\mp i\theta(\Lambda,k)}\Lambda\,\boldsymbol{\varepsilon}^{[\pm1]}(\boldsymbol{u};\Lambda^{-1}k)\ , \qquad \boldsymbol{\varepsilon}^{[\pm2]}(\boldsymbol{u}\cdot\Lambda;k) = \Lambda\,\boldsymbol{\varepsilon}^{[\pm2]}(\boldsymbol{u};\Lambda^{-1}k) \tag{4.40}$$

Hence, the two physical polarization amplitudes in (4.37) undergo the following transformations under the action of $(a,\Lambda)\in \mathrm{ISO}_0(3,1)$:

$$\alpha_{\pm}(q-i\ell v) \ \mapsto\ e^{\pm 2i\theta(\Lambda,i\partial)}\alpha_{\pm}(\Lambda^{-1}(q-a)-i\ell\Lambda^{-1}v)\ , \quad \partial:=(\partial_0,\partial_1,\partial_2,\partial_3)\ . \tag{4.41}$$

The *un*physical but "good" modes attached to the polarization null-frames related to the circular polarization of a generic graviton state vector span, in accordance with (4.35), the null fibres of the graviton bundle (4.38). Their amplitudes are therefore given by

$$\gamma_{r2}(\zeta) := f_{[r2]\zeta}\ , \quad r=-1,+1,+2\ , \qquad \gamma_{2s}(\zeta) := f_{[2s]\zeta}\ , \quad s=\pm1\ , \tag{4.42}$$

and they transform in the following manner under the action of $(a,\Lambda)\in \mathrm{ISO}_0(3,1)$:

$$\gamma_{rs}(q-i\ell v) \ \mapsto\ e^{i(\pm\delta_{\pm1r}\pm\delta_{\pm1s})\theta(\Lambda,i\partial)}\gamma_{rs}(\Lambda^{-1}(q-a)-i\ell\Lambda^{-1}v) \tag{4.43}$$

The unphysical "bad" modes attached to the same polarization null-frames for the expansion of graviton state vectors are the ones that manifest themselves only in generalized Lorenz gauges. In view of (4.20) and (4.34)-(4.36), their amplitudes are given by

$$\beta_r(\zeta) := f_{[rs]\zeta} = f_{[sr]\zeta}\ , \qquad r=\pm1,\pm2\ , \quad s=+2\ , \tag{4.44}$$

and they transform as follows under the action of $(a,\Lambda)\in \mathrm{ISO}_0(3,1)$:

$$\beta_r(q-i\ell v) \ \mapsto\ e^{\pm i\delta_{\pm1r}\theta(\Lambda,i\partial)}\beta_r(\Lambda^{-1}(q-a)-i\ell\Lambda^{-1}v) \tag{4.45}$$

The extension of all these transformations to generic $(a,\Lambda)\in \mathrm{ISO}^{\uparrow}(3,1)$ is obtained from the extension of (4.41) to 3-space reflections. It immediately follows from (4.21) that such a space reflection reverses the orientation of circular polarization from clockwise to counter-clockwise, and vice versa.

8.5. Quantum Gravitational and Berezin-Faddeev-Popov Frames

To carry out the geometro-stochastic second-quantization of gravity, we modify somewhat the pattern followed in the preceding four chapters, and construct a quantum gravitational fibre $\mathcal{Z}_{\boldsymbol{u}}$ over each Poincaré coframe $\boldsymbol{u}\in P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$, rather than directly over base locations $\boldsymbol{x}\in\mathbf{S}$. Thus, we shall assign to each $\boldsymbol{u}\in P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$ a superlocal vacuum state vector $\boldsymbol{\Psi}_{0;\boldsymbol{u}}$, which spans the one-dimensional superlocal vacuum sector $\mathcal{Z}_{0;\boldsymbol{u}}$. We can then use the $\boldsymbol{u}$-dependent $J_{\boldsymbol{u}}$-inner product in (3.6) to add to $\mathcal{Z}_{0;\boldsymbol{u}}$ the $J_{\boldsymbol{u}}$-direct sums of symmetrized ten-

sor products of the graviton fibre $\mathbf{Z}_{\boldsymbol{u}}$, thus obtaining the *quantum gravitational fibre*

$$\mathcal{Z}_{\boldsymbol{u}} = \bigoplus_{J}^{\infty}{}_{n=0}\, \mathcal{Z}_{n;\boldsymbol{u}} \quad , \qquad \mathcal{Z}_{n;\boldsymbol{u}} = \mathbf{Z}_{\boldsymbol{u}} \underset{S}{\otimes} \cdots \underset{S}{\otimes} \mathbf{Z}_{\boldsymbol{u}} \quad . \tag{5.1}$$

Naturally, all the superlocal n-graviton fibres $\mathcal{Z}_{n\boldsymbol{u}}$ over the same base location $x \in \mathbf{S}$ could be identified by using in the single-graviton case $\mathcal{Z}_{1;\boldsymbol{u}}$ the soldering map of the graviton bundle in (4.38), and then extending that map to $\mathcal{Z}_{n\boldsymbol{u}}$ by taking Whitney products of that bundle. However, on account of the various technical problems that emerge in case of the massless gauge fields discussed in the preceding chapter, we shall not carry out such an identification at the present stage. Rather, we shall define the graviton annihilation operators that act on n-graviton state vectors belonging to $\mathcal{Z}_{n\boldsymbol{u}}$ as follows:

$$\left(g^{(-)}_{ij}(\boldsymbol{u};\zeta)\,\Psi_{n;\boldsymbol{u}}\right)_{n-1}(\zeta_1,i_1j_1,\ldots,\zeta_{n-1},i_{n-1}j_{n-1}) = \sqrt{n}\;\Psi_{n;\boldsymbol{u}}(\zeta,ij,\zeta_1,i_1j_1,\ldots,\zeta_{n-1},i_{n-1}j_{n-1}). \tag{5.2}$$

The extension of these operators to $\mathcal{Z}_{\boldsymbol{u}}$ by linearity gives rise to $J_{\boldsymbol{u}}$-closable unbounded operators with well-defined adjoints and $J_{\boldsymbol{u}}$-adjoints (cf. Bognár, 1974, p. 122). These extensions can be therefore used to define *bona fide* graviton creation operators for which

$$\left\langle \boldsymbol{\Psi} \middle| g^{(-)}_{ij}(\boldsymbol{u};\zeta)\,\boldsymbol{\Psi}' \right\rangle = \left\langle g^{(+)}_{ij}(\boldsymbol{u};\zeta)\,\boldsymbol{\Psi} \middle| \boldsymbol{\Psi}' \right\rangle \quad , \qquad \boldsymbol{\Psi}, \boldsymbol{\Psi}' \in \mathcal{Z}_{\boldsymbol{u}} \; , \tag{5.3}$$

for all vectors $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}'$ from their respective dense domains. These creation operators act on n-graviton states in the usual manner, namely so that

$$\begin{aligned} &\left(g^{(+)}_{ij}(\boldsymbol{u};\zeta)\,\Psi_{n;\boldsymbol{u}}\right)_{n+1}(\zeta_1,i_1j_1,\ldots,\zeta_{n+1},i_{n+1}j_{n+1}) \\ &= i(n+1)^{-1/2}\sum\nolimits_{r=1}^{n+1}\eta_{ii_r}\,\eta_{jj_r}\,D^{(+)}(\zeta_r;\zeta)\,\Psi_{n;\boldsymbol{u}}(\zeta_1,i_1j_1,\ldots,\hat{\zeta}_r,\hat{i}_r\hat{j}_r,\ldots,\zeta_{n+1},i_{n+1}j_{n+1}) \quad , \end{aligned} \tag{5.4}$$

where the circumflexes indicate those variables on which the wave functions on the right-hand side of the above relation actually do not depend, and the two-point function coincides with the one in (7.3.29).

By using (5.2) and (5.4) it is easy to verify that

$$\left[g^{(\pm)}_{ij}(\boldsymbol{u};\zeta), g^{(\pm)}_{i'j'}(\boldsymbol{u};\zeta')\right] = 0 \quad , \qquad \left[g^{(-)}_{ij}(\boldsymbol{u};\zeta), g^{(+)}_{i'j'}(\boldsymbol{u};\zeta')\right] = i\eta_{ii'}\,\eta_{jj'}\,D^{(+)}(\zeta;\zeta') \quad . \tag{5.5}$$

Consequently, upon setting by definition

$$g^{(+)}(\boldsymbol{u};\boldsymbol{f}) := \int f^{ij}(\zeta)\,g^{(+)}_{ij}(\boldsymbol{u};\zeta)\,d\tilde{\Sigma}(\zeta) = g^{(-)}(\boldsymbol{u};\boldsymbol{f})^{\dagger} \quad , \qquad f^{ij} := \eta^{ii'}\eta^{jj'}f_{i'j'} \quad , \tag{5.6}$$

we can introduce above each $\boldsymbol{u} = \{\boldsymbol{a}^*(x), \boldsymbol{\theta}^i(x)\} \in P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$ the elements

$$\boldsymbol{\Phi}_{\boldsymbol{f}} = \exp\left[-\tfrac{1}{2}\langle \boldsymbol{f}|\boldsymbol{f}\rangle + g^{(+)}(\boldsymbol{u};\boldsymbol{f})\right]\,\Psi_{0;\boldsymbol{u}} \quad , \qquad \boldsymbol{f} \in \mathbf{Z}^{(0)}_{\boldsymbol{u}} \; , \tag{5.7}$$

of a quantum gravitational frame by following the same pattern as in Sec. 7.4, so that

$$g^{(-)}_{ij}(\boldsymbol{u};\zeta)\,\boldsymbol{\Phi}_{\boldsymbol{f}} = f_{ij}(\zeta)\,\boldsymbol{\Phi}_{\boldsymbol{f}} \quad , \qquad \boldsymbol{f} = f_{ij}\,\boldsymbol{\theta}^i(x) \otimes \boldsymbol{\theta}^j(x) \in \mathbf{Z}^{(0)}_{\boldsymbol{u}} \; . \tag{5.8}$$

Moreover, the decomposition in (4.18a) will give rise, as in (7.4.14)-(7.4.18), to the simi-

lar decomposition

$$\boldsymbol{\Phi}_f = \boldsymbol{\Phi}_{f^{\mathrm{TT}}} + \boldsymbol{\Phi}_f^{\perp} \in \mathcal{Z}_{\boldsymbol{u}}^{(0)} , \qquad \boldsymbol{\Phi}_{f^{\mathrm{TT}}} \in \mathcal{Z}_{\boldsymbol{u}}^{\mathrm{TT}} , \tag{5.9a}$$

$$\left\langle \boldsymbol{\Phi}_{f^{\mathrm{TT}}} \middle| \boldsymbol{\Phi}_{f^{\mathrm{TT}}} \right\rangle = \left\langle \boldsymbol{\Phi}_f \middle| \boldsymbol{\Phi}_f \right\rangle = 1 , \qquad \left\langle \boldsymbol{\Phi}_f^{\perp} \middle| \boldsymbol{\Phi}_f^{\perp} \right\rangle = \left\langle \boldsymbol{\Phi}_f^{\perp} \middle| \boldsymbol{\Phi}_{f^{\mathrm{TT}}} \right\rangle = 0 , \tag{5.9b}$$

corresponding to the following $J_{\boldsymbol{u}}$-orthogonal decomposition,

$$\mathcal{Z}_{\boldsymbol{u}}^{(0)} = \mathcal{Z}_{\boldsymbol{u}}^{\mathrm{TT}} \underset{J}{\oplus} \mathcal{N}_{\boldsymbol{u}} , \qquad \mathcal{N}_{\boldsymbol{u}} = \left\{ \boldsymbol{\Psi} \in \mathcal{Z}_{\boldsymbol{u}}^{(0)} \middle| \left\langle \boldsymbol{\Psi} \middle| \boldsymbol{\Psi} \right\rangle = 0 \right\} , \tag{5.10}$$

of the *Lorenz quantum gravitational fibre* $\mathcal{Z}_{\boldsymbol{u}}^{(0)}$ into a *physical* quantum gravitational fibre above $\boldsymbol{u}$ and a *null* quantum gravitational fibre above $\boldsymbol{u}$. By proceeding as in (7.4.19)-(7.4.24) we then obtain the resolution of the identity

$$\int_{\mathcal{Z}_{\boldsymbol{u}}^{\mathrm{TT}}} \left| \boldsymbol{\Phi}_f \right\rangle \, d\boldsymbol{f} d\bar{\boldsymbol{f}} \left\langle \boldsymbol{\Phi}_f \right| = \mathbf{1}_{\boldsymbol{u}}^{\mathrm{TT}} \tag{5.11}$$

within the physical quantum gravitational fibre above $\boldsymbol{u}$.

The remaining degrees of freedom of graviton polarization, belonging to unphysical modes, will be incorporated into Faddeev-Popov (FP) fibres consisting of ghost and antighost state vectors of the type discussed in the preceding chapter. Thus, we shall introduce above each $\boldsymbol{u} = \{\boldsymbol{a}^*(x), \boldsymbol{\theta}^i(x)\} \in P^* \mathbf{S}^{\dagger}(\mathbf{g}^{\mathrm{M}})$ the *graviton ghost fibre*

$$\mathbf{F}_{\boldsymbol{u}} = \left\{ \mathbf{c} = c_i \, \boldsymbol{\theta}^i(x) \;\middle|\; (c_0, c_1, c_2, c_3) \in \mathbf{K} \right\} , \tag{5.12}$$

as well as the *graviton antighost fibre* [14]

$$\overline{\mathbf{F}}_{\boldsymbol{u}} = \left\{ \overline{\mathbf{c}} = \bar{c}_i \, \boldsymbol{\theta}^i(x) \;\middle|\; (\bar{c}_0, \bar{c}_1, \bar{c}_2, \bar{c}_3) \in \mathbf{K}^{\dagger} \right\} , \tag{5.13}$$

where $\mathbf{K}$ is the Krein space defined in Sec. 7.3, and $\mathbf{K}^{\dagger}$ is its dual, so that we have

$$\partial_j \partial^j c_i(\zeta) = 0 , \qquad \partial_j = \partial / \partial q^j , \qquad \partial^j = \eta^{jk} \partial_k , \tag{5.14}$$

$$\langle \mathbf{c} | \mathbf{c}' \rangle = \int_{V_0^+} \eta^{ij} \bar{c}_i^*(k) \bar{c}_j'(k) \, d\Omega_0(k) , \tag{5.15}$$

$$\left(\mathbf{c} | \mathbf{c}' \right)_{J_{\boldsymbol{u}}} = \sum_{j=0}^{3} \int_{V_0^+} \bar{c}_j^*(k) \bar{c}_j'(k) \, d\Omega_0(k) , \tag{5.16}$$

with the corresponding expressions within $\overline{\mathbf{F}}_{\boldsymbol{u}}$ being obtained by conjugation. We can then carry out for the decomposition in (4.20) the identifications

$$\boldsymbol{f}^{\perp} = f_{ij}^{\perp} \, \boldsymbol{\theta}^i(x) \otimes \boldsymbol{\theta}^j(x) \;\leftrightarrow\; \mathbf{c}^{\perp} \in \mathbf{F}_{\boldsymbol{u}} , \qquad f_{ij}^{\perp} = \partial_i c_j^{\perp} + \partial_j c_i^{\perp} , \quad \partial^i c_i^{\perp} = 0 , \tag{5.17}$$

$$\boldsymbol{f}^b = f_{ij}^b \, \boldsymbol{\theta}^i(x) \otimes \boldsymbol{\theta}^j(x) \;\leftrightarrow\; \mathbf{c}^b \in \mathbf{F}_{\boldsymbol{u}} , \qquad f_{ij}^b = \partial_i c_j^b + \partial_j c_i^b , \quad \partial_j \partial^i c_i^b = b_j , \tag{5.18}$$

in which the additive constant that the general solutions of the above differential equations contain is fixed by the boundary conditions in the q-variables at infinity, which the functions in (5.12) and (5.13) have to obey to secure the existence of the integral in (5.16).

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[14] The bars in (5.13) represent an adaptation of the standard notation for antighost states, whose values will be subsequently related to the Grassmannian variables introduced in Sec. 6.2.

Upon constructing the corresponding *FP-ghost* and *FP-antighost fibres*,

$$\mathcal{F}_{\boldsymbol{u}} = \bigoplus_{n=0}^{\infty} (\mathbf{F}_{\boldsymbol{u}} \underset{A}{\otimes} \cdots \underset{A}{\otimes} \mathbf{F}_{\boldsymbol{u}})_n \ , \qquad \overline{\mathcal{F}}_{\boldsymbol{u}} = \bigoplus_{n=0}^{\infty} (\overline{\mathbf{F}}_{\boldsymbol{u}} \underset{A}{\otimes} \cdots \underset{A}{\otimes} \overline{\mathbf{F}}_{\boldsymbol{u}})_n \ , \tag{5.19}$$

we can define the action of the ghost annihilation operators $C^{(-)}(\boldsymbol{u};\zeta)$ and of the ghost creation operators $C^{(+)}(\boldsymbol{u};\zeta)$, as well as of the conjugate operators $\overline{C}^{(-)}(\boldsymbol{u};\zeta)$ and $\overline{C}^{(+)}(\boldsymbol{u};\zeta)$ acting on antighost states, in the same manner as for the case of fermion fields treated in Sec. 6.1, so that, for example,

$$\left(C_i^{(-)}(\boldsymbol{u};\zeta)\Psi^{\mathrm{gh}}_{n;\boldsymbol{u}}\right)_{n-1}(\zeta_1,i_1,\ldots,\zeta_{n-1},i_{n-1}) = \sqrt{n}\ \Psi^{\mathrm{gh}}_{n;\boldsymbol{u}}(\zeta,i,\zeta_1,i_1,\ldots,\zeta_{n-1},i_{n-1}) \tag{5.20}$$

describes the action of $C^{(-)}(\boldsymbol{u};\zeta) = C_i^{(-)}(\boldsymbol{u};\zeta)\,\boldsymbol{\theta}^i(x)$. However, the full significance of these annihilation and creation operators emerges only after the transition to the respective *BFP (Berezin-Faddeev-Popov) ghost-antighost superfibres* $\mathcal{B}_{\boldsymbol{u}} \otimes \overline{\mathcal{B}}_{\boldsymbol{u}}$ is performed.

Basically, this transition can be executed as in the fermion-antifermion case treated in Sec. 6.2. Thus, for any given choice of J_0-orthonormal basis for $\mathbf{K}$ in (5.12) we shall introduce a corresponding $J_{\boldsymbol{u}}$-orthonormal basis $\{\mathbf{c}_{\boldsymbol{u};1}, \mathbf{c}_{\boldsymbol{u};2}, \ldots\}$ in each $\mathbf{F}_{\boldsymbol{u}}$, and its conjugate in $\overline{\mathbf{F}}_{\boldsymbol{u}}$. Then we can set as in (6.2.15) and (6.2.17)

$$\mathbf{B}_{\boldsymbol{u}} = \left\{ \boldsymbol{c} = \theta_{\boldsymbol{u}}^{\alpha}\mathbf{c}_{\boldsymbol{u};\alpha} \,\middle|\, \theta_{\boldsymbol{u}}^{\alpha} = \left(\mathbf{c}_{\boldsymbol{u};\alpha}|\mathbf{c}\right)_{J_{\boldsymbol{u}}} \vartheta_{\alpha} \ , \ \alpha = 1,2,\ldots \ , \ \mathbf{c} \in \mathbf{F}_{\boldsymbol{u}} \right\} , \tag{5.21a}$$

$$\overline{\mathbf{B}}_{\boldsymbol{u}} = \left\{ \overline{\boldsymbol{c}} = \overline{\mathbf{c}}_{\boldsymbol{u};\alpha}\overline{\theta}_{\boldsymbol{u}}^{\alpha} \,\middle|\, \overline{\theta}_{\boldsymbol{u}}^{\alpha} = \left(\overline{\mathbf{c}}_{\boldsymbol{u};\alpha}|\overline{\mathbf{c}}\right)_{J_{\boldsymbol{u}}} \overline{\vartheta}_{\alpha} \ , \ \alpha = 1,2,\ldots \ , \ \overline{\mathbf{c}} \in \overline{\mathbf{F}}_{\boldsymbol{u}} \right\} , \tag{5.21b}$$

where ϑ_{α} and $\overline{\vartheta}_{\alpha}$ are the generators of the Grassmann algebra Λ described in Sec. 6.2.

The BFP ghost-antighost superfibre $\mathcal{B}_{\boldsymbol{u}} \otimes \overline{\mathcal{B}}_{\boldsymbol{u}}$ can be constructed from the superfibres $\mathbf{B}_{\boldsymbol{u}}$ and $\overline{\mathbf{B}}_{\boldsymbol{u}}$ as in (6.2.19)-(6.2.20). If we then set

$$C_i^{(+)}(\boldsymbol{u};\boldsymbol{c}) = \sum_{i=0}^{3}\int c(\zeta,i)C_i^{(+)}(\boldsymbol{u};\zeta)\,d\tilde{\Sigma}(\zeta) \ , \quad c(\zeta,i) = c_{\boldsymbol{u};\alpha,i}(\zeta)\,\theta_{\boldsymbol{u}}^{\alpha} \ , \tag{5.22a}$$

$$\overline{C}_i^{(+)}(\boldsymbol{u};\overline{\boldsymbol{c}}) = \sum_{i=0}^{3}\int \overline{c}(\zeta,i)\overline{C}_i^{(+)}(\boldsymbol{u};\zeta)\,d\tilde{\Sigma}(\zeta) \ , \quad \overline{c}(\zeta,i) = \overline{c}_{\boldsymbol{u};\alpha,i}(\zeta)\,\overline{\theta}_{\boldsymbol{u}}^{\alpha} \ , \tag{5.22b}$$

we can introduce the ghost and antighost *BFP superframe* elements

$$\boldsymbol{\Phi}_{\boldsymbol{c}} = \exp\left(-\tfrac{1}{2}\sum_{\alpha=1}^{\infty}\overline{\theta}_{\boldsymbol{u}}^{\alpha}\,\theta_{\boldsymbol{u}}^{\alpha}\right)\exp\left(C^{(+)}(\boldsymbol{u};\boldsymbol{c})\right)\boldsymbol{\Psi}_{0;\boldsymbol{u}} \ , \quad \boldsymbol{c} \in \mathbf{B}_{\boldsymbol{u}} \ , \tag{5.23a}$$

$$\boldsymbol{\Phi}_{\overline{\boldsymbol{c}}} = \exp\left(-\tfrac{1}{2}\sum_{\alpha=1}^{\infty}\theta_{\boldsymbol{u}}^{\alpha}\,\overline{\theta}_{\boldsymbol{u}}^{\alpha}\right)\exp\left(\overline{C}^{(+)}(\boldsymbol{u};\overline{\boldsymbol{c}})\right)\boldsymbol{\Psi}_{0;\boldsymbol{u}} \ , \quad \overline{\boldsymbol{c}} \in \overline{\mathbf{B}}_{\boldsymbol{u}} \ , \tag{5.23b}$$

which are are eigenvectors of the ghost and antighost annihilation operators, respectively:

$$C_i^{(-)}(\boldsymbol{u};\zeta)\boldsymbol{\Phi}_{\boldsymbol{c}} = c(\zeta,i)\boldsymbol{\Phi}_{\boldsymbol{c}} \ , \qquad \overline{C}_i^{(-)}(\boldsymbol{u};\zeta)\boldsymbol{\Phi}_{\overline{\boldsymbol{c}}} = \overline{c}(\zeta,i)\boldsymbol{\Phi}_{\overline{\boldsymbol{c}}} \ . \tag{5.24}$$

These superframes supply resolutions of the identity within the corresponding FP ghost and antighost fibres in (5.19), so that

$$\int_{\overline{\mathbf{B}}_{\boldsymbol{u}}\times\mathbf{B}_{\boldsymbol{u}}} \left\langle \boldsymbol{\Psi}\middle|\boldsymbol{\Phi}_{\boldsymbol{c}}\right\rangle d\overline{\boldsymbol{c}}\,d\boldsymbol{c}\left\langle \boldsymbol{\Phi}_{\boldsymbol{c}}\middle|\boldsymbol{\Psi}'\right\rangle = \left\langle \boldsymbol{\Psi}\middle|\boldsymbol{\Psi}'\right\rangle , \qquad \boldsymbol{\Psi},\boldsymbol{\Psi}' \in \mathcal{F}_{\boldsymbol{u}} \ , \tag{5.25a}$$

$$\int_{\mathbf{B}_u \times \bar{\mathbf{B}}_u} \langle \Psi | \Phi_{\bar{c}} \rangle \, dc \, d\bar{c} \, \langle \Phi_{\bar{c}} | \Psi' \rangle = \langle \Psi | \Psi' \rangle , \qquad \Psi, \Psi' \in \bar{\mathcal{F}}_u , \tag{5.25b}$$

where the above Berezin functional integrals are interpreted as in (6.3.20).

The fibres and superfibres constructed in this section supply bundles over $P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$ which are mathematically as well as physically trivial, since their structure groups consist of only the identity operators. Clearly, in order to obtain physically nontrivial bundles it is necessary to identify the fibres that lie over the Poincaré coframes $\boldsymbol{u} \in P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$ which are situated above each given point $x \in \mathbf{S}$ and are interrelated by gauge transformations.

If this were a framework with an external gravitational field of the type studied in the previous four chapters, then these identifications would be carried out by adopting the Poincaré group $\mathrm{ISO}^{\uparrow}(3,1)$ as a structure group (i.e., as a gauge group of the first kind – cf. Sec. 7.7). Hence, they would be performed by constructing equivalence classes of pairs $(\boldsymbol{u}', \Psi')$ and $(\boldsymbol{u}'', \Psi'')$, $\Psi', \Psi'' \in \mathcal{Z}_{u_0}$, in the manner prescribed by the application of the general definition of a $\mathbf{G}$-product in Sec. 4.1 to the case where $\mathbf{G} = \mathrm{ISO}^{\uparrow}(3,1)$, and where the typical fibre $\mathcal{Z}_{u_0}$ is obtained as in (5.1) from the typical graviton fibre $\mathbf{Z}_{u_0}$ corresponding to the standard frame $\boldsymbol{u}_0$ in (5.5.12), whose elements belong to $(\mathbf{R}^4, \eta)$. If the fibre elements $\Psi_u \in \mathcal{Z}_u$ are set in natural correspondence with pairs $(\boldsymbol{u}, \Psi)$, then such a construction amounts to considering the quantum gravitational state vectors $\Psi'_{u'} \in \mathcal{Z}_{u'}$ and $\Psi''_{u''} \in \mathcal{Z}_{u''}$ over the respective Poincaré coframes $\boldsymbol{u}'$ and $\boldsymbol{u}''$ belonging to the same fibre of $P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$ as being elements of the same equivalence class.

However, in addition to having to cope with the mathematical difficulties discussed in the preceding chapter, such a construction would completely ignore the role played by the diffeomorphism group in general relativity. Therefore, it would not lead to a satisfactory coupling of gravity to "matter" fields, whereby the latter would influence the former in a physically consistent manner. This means that a more general construction, based on a quantum gravitational group which incorporates $\mathit{Diff}\mathbf{S}$ in some physically acceptable form, is definitely required.

We turn to the implementation of such a construction in the next section.

8.6. Quantum Gravitational Gauge Supergroups and Quantum Spacetime

As discussed in Sec. 8.2, one of the fundamental ideas of geometric quantum gravity is that the quantum-geometric evolution of quantum matter and radiation gives rise to a *metrized* coframe bundle $P^*\mathbf{S}^{\uparrow}(g^{\mathrm{M}})$ over each base-segment $\mathbf{S}$ corresponding to a QGR gauge orbit $(\mathbf{S}, g^{\mathrm{M}})$. Each such QGR gauge orbit consists of a manifold $\mathbf{S}$ whose boundary is the union of two non-intersecting hypersurfaces Σ' and Σ'', and of an equivalence class g^{M} of Lorentzian mean metrics $\mathbf{g}^{\mathrm{M}}$ in relation to which Σ' and Σ'' are both spacelike.

Let us consider the Poincaré coframe bundle $P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$ corresponding to some mean metric $\mathbf{g}^{\mathrm{M}} \in (\mathbf{S}, g^{\mathrm{M}})$. We shall construct from this principal bundle the $\mathbf{G}$-product

$$\mathcal{E}^{\mathrm{TT}}(\mathbf{S}) = P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}}) \times_{\mathbf{G}} \left(\mathcal{Z}^{\mathrm{TT}}_{u_0} \otimes \bar{\mathcal{F}}_{u_0} \otimes \mathcal{F}_{u_0} \right) , \qquad \mathbf{G} = \mathrm{T}^4 \wedge \mathrm{O}(3) , \tag{6.1}$$

whose structure group is the semi-direct product of the spacetime translation group $\mathbf{T}^4$ and the rotation group O(3), and whose typical fibre is obtained as in (5.10) and (5.19) from the quantum gravitational and BFP typical fibres corresponding to the standard frame $\boldsymbol{u}_0$ in $(\mathbf{R}^4,\boldsymbol{\eta})$. As in the case of $\mathbf{G} = \mathrm{ISO}^{\uparrow}(3,1)$, discussed in the closing paragraphs of the preceding section, this construction is tantamount to a classification of vectors

$$\boldsymbol{\Psi}' \in \mathcal{Z}^{\mathrm{TT}}_{\boldsymbol{u}'} \otimes \overline{\mathcal{F}}_{\boldsymbol{u}'} \otimes \mathcal{F}_{\boldsymbol{u}'} \,, \quad \boldsymbol{\Psi}'' \in \mathcal{Z}^{\mathrm{TT}}_{\boldsymbol{u}''} \otimes \overline{\mathcal{F}}_{\boldsymbol{u}''} \otimes \mathcal{F}_{\boldsymbol{u}''} \,, \quad \boldsymbol{u}', \boldsymbol{u}'' \in \boldsymbol{\Pi}^{-1}(x) \subset P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}}), \tag{6.2}$$

into equivalence classes, for which, however, $\mathbf{G} = \mathrm{T}^4 \wedge \mathrm{O}(3)$ is used as a structure group instead of the orthochronous Poincaré group $\mathrm{ISO}^{\uparrow}(3,1)$.

On the other hand, any orthochronous Poincaré transformation can be uniquely decomposed, together with its inverse, as follows (cf., e.g., Miller, 1972):

$$(a,\Lambda) = (a,R)(0,\Lambda_v) \in \mathrm{ISO}^{\uparrow}(3,1) \,, \quad (a,R) \in \mathrm{T}^4 \wedge \mathrm{O}(3) \,, \quad \Lambda_v \in \mathrm{SO}_0(3,1), \tag{6.3a}$$

$$(a,\Lambda)^{-1} = (0,\Lambda_v^{-1})(-R^{-1}a,R^{-1}) \in \mathrm{ISO}^{\uparrow}(3,1) \,, \quad (-R^{-1}a,R^{-1}) = (a,R)^{-1} \,. \tag{6.3b}$$

Upon performing, in accordance with (5.17) and (5.21a), the identifications

$$\boldsymbol{f}^{\perp} = f^{\perp}_{ij}\,\boldsymbol{\theta}^i(x) \otimes \boldsymbol{\theta}^j(x) \quad \leftrightarrow \quad \boldsymbol{c}^{\perp} = \theta^{\alpha}_{\boldsymbol{u}}(\boldsymbol{c}^{\perp})\mathbf{c}_{\boldsymbol{u};\alpha} \in \mathbf{B}_{\boldsymbol{u}} \,, \tag{6.4a}$$

$$\boldsymbol{f}^{b} = f^{b}_{ij}\,\boldsymbol{\theta}^i(x) \otimes \boldsymbol{\theta}^j(x) \quad \leftrightarrow \quad \boldsymbol{c}^{b} = \theta^{\alpha}_{\boldsymbol{u}}(\boldsymbol{c}^{b})\mathbf{c}_{\boldsymbol{u};\alpha} \in \mathbf{B}_{\boldsymbol{u}} \,, \tag{6.4b}$$

we find that the action of the Lorentz boost Λ_v upon the coframe $\boldsymbol{u} = \{\boldsymbol{a}^*(x),\boldsymbol{\theta}^i(x)\} \in P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$ gives rise to the actions

$$U(0,\Lambda_v) : \boldsymbol{c}^{\perp} = \theta^{\alpha}_{\boldsymbol{u}}\mathbf{c}_{\boldsymbol{u};\alpha} \mapsto \theta'^{\alpha}_{\boldsymbol{u}}\mathbf{c}_{\boldsymbol{u};\alpha} \,, \quad \theta'^{\alpha}_{\boldsymbol{u}} = \left(\mathbf{c}_{\boldsymbol{u};\alpha}\middle|\boldsymbol{c}^{\perp}\right)_{J_{\boldsymbol{u}}} U^{\beta}_{\alpha}(0,\Lambda_v)\vartheta_{\beta} \in \Lambda^1 \,, \tag{6.5a}$$

$$U(0,\Lambda_v) : \boldsymbol{c}^{b} = \theta^{\alpha}_{\boldsymbol{u}}\mathbf{c}_{\boldsymbol{u};\alpha} \mapsto \theta'^{\alpha}_{\boldsymbol{u}}\mathbf{c}_{\boldsymbol{u};\alpha} \,, \quad \theta'^{\alpha}_{\boldsymbol{u}} = \left(\mathbf{c}_{\boldsymbol{u};\alpha}\middle|\boldsymbol{c}^{b}\right)_{J_{\boldsymbol{u}}} U^{\beta}_{\alpha}(0,\Lambda_v)\vartheta_{\beta} \in \Lambda^1 \,, \tag{6.5b}$$

$$U^{\alpha}_{\beta}(0,\Lambda_v) = \left(\mathbf{c}_{\boldsymbol{u};\beta}\middle|U(0,\Lambda_v)\mathbf{c}_{\boldsymbol{u};\alpha}\right)_{J_{\boldsymbol{u}}} \,, \qquad \alpha,\beta = 1,2,\ldots \,, \tag{6.5c}$$

of the representative of that boost upon the Grassmann-valued wave functions representing superlocal ghost states in any given generalized Lorenz (internal) gauge. Furthermore, in view of (6.3b), analogous adjoint actions can be assigned within antighost superfibres:

$$U(0,\Lambda_v^{-1}) : \overline{\boldsymbol{c}}^{\perp} = \overline{\theta}^{\alpha}_{\boldsymbol{u}}\overline{\mathbf{c}}_{\boldsymbol{u};\alpha} \mapsto \overline{\theta}'^{\alpha}_{\boldsymbol{u}}\overline{\mathbf{c}}_{\boldsymbol{u};\alpha} \,, \quad \overline{\boldsymbol{c}}^{b} = \overline{\theta}^{\alpha}_{\boldsymbol{u}}\mathbf{c}_{\boldsymbol{u};\alpha} \mapsto \overline{\theta}'^{\alpha}_{\boldsymbol{u}}\overline{\mathbf{c}}_{\boldsymbol{u};\alpha} \,. \tag{6.5d}$$

On the other hand, as we saw in Sec. 8.1, the actions of a diffeomorphisms $\psi : \mathbf{S} \to \mathbf{S}$ can be set in one-to-one correspondence with certain vertical automorphisms $\boldsymbol{\psi}$ acting on cross-sections of the metrized coframe bundle $P^*\mathbf{S}^{\uparrow}(\mathcal{g}^{\mathrm{L}})$, and can be then combined as in (1.5) with Poincaré gauge transformations $\boldsymbol{\phi} \in \mathcal{G}(\mathbf{P}(\mathbf{S},\mathrm{ISO}^{\uparrow}(3,1))$ represented by various $\phi \in \mathrm{Aut}_{\mathbf{S}}P^*\mathbf{S}^{\uparrow}(\mathcal{g}^{\mathrm{L}})$. In this manner a gauge group of the second kind is obtained, which for a given CGR gauge orbit $(\mathbf{S},\mathcal{g}^{\mathrm{L}})$ over $\mathbf{S}$ is provided by the semidirect product

$$\mathcal{G}(\mathbf{S},\mathcal{g}^{\mathrm{L}}) = \mathrm{G}_A(\mathit{Diff}\mathbf{S}) \wedge \mathcal{G}(\mathbf{P}(\mathbf{S},\mathrm{ISO}^{\uparrow}(3,1)) \,. \tag{6.6}$$

This semidirect product can be viewed as the gauge group of CGR in $\mathbf{S}$, since it leaves the

classical geometrodynamic evolution invariant as it acts on the *physical* moving coframes $(\boldsymbol{s},\boldsymbol{g}^{\mathrm{L}})$ within $P^{*}\mathbf{S}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$ for the various diffeomorphically equivalent metrics $\boldsymbol{g}^{\mathrm{L}} \in (\mathbf{S},\mathcal{g}^{\mathrm{L}})$.

In order to apply this idea to geometric quantum gravity, we have to assume that all considered diffeomorphisms $\psi \in \mathit{Diff}\mathbf{S}$ are *quantum-geometrically causal* in the sense that they leave the inflow hypersurface Σ' as well as the outflow hypersurface Σ'' invariant. Naturally, such an assumption is not required in CGR, since CGR is a strictly deterministic theory, and since causal pasts and futures are determined only in relation to a given Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ over a given manifold $\mathbf{M}$. Hence, in CGR the situation is quite different: a diffeomorphism $\psi : \mathbf{M} \rightarrow \mathbf{M}$ that would map Σ' onto Σ'', and vice versa, is physically acceptable once the *entire* classical spacetime manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ has been computed. Indeed, although such a diffeomorphism would convert Σ' into the outflow hypersurface and Σ'' into the inflow hypersurface of $\mathbf{S}$, that would merely entail a reversal of the classical geometrodynamic notational convention. Of course, physically such a judgement can be reached only *after* all of $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ is known. However, within the confines of a deterministic theory, such as CGR, it can be argued on basis of existence theorems on the Cauchy problem (cf. Hawking and Ellis, 1973, Chapter 7) that a classical spacetime manifold $(\mathbf{M},\boldsymbol{g}^{\mathrm{L}})$ *exists* in its entirety once a complete set of initial data has been provided, by virtue of the fact that every single physical event has been preordained from the instant of its creation. Hence, the physical *reality* envisaged within such a framework leaves no room for any actual experimentation that involves state preparation – and therefore active intervention in the natural course of events – but allows only the passive observation of physical events.

This ontological perspective becomes radically different when an *indeterministic* point of view is adopted: one can then talk only of probabilities for the realization of different *possibilities*; moreover, when the global structure of spacetime is under investigation down to its minutest details, then those probabilities are relative, since one does not, and *cannot*, ever know their *entire* range. Since, in principle, local metric fluctuation of *any* magnitude can occur with the passage of proper time, the only form of causality by which quantum-geometric evolution can abide is the one in which the time-ordering of local events is correlated to the time-ordering of any two hypersurfaces Σ' and Σ'', which constitute the boundary $\partial\mathbf{S}$ of some base-segment $\mathbf{S}$. According to the definition we presented in Sec. 8.2, in the presence of a mean metric $\mathbf{g}^{\mathrm{M}}$ to which the inflow of matter and radiation gives rise, that time-ordering has to be such that the futures of all points on Σ' that are classically causal in relation to the mean metric $\mathbf{g}^{\mathrm{M}}$ intersect Σ''; whereas, the pasts of all points on Σ'' that are classically causal in relation to the same mean metric $\mathbf{g}^{\mathrm{M}}$ intersect Σ'.

We shall denote by $\mathit{Diff}_{0}\mathbf{S}$ the group of all quantum-geometrically causal diffeomorphisms of a base-segment $\mathbf{S}$, which leave the inflow and outflow hypersurfaces of $\mathbf{S}$ invariant, and therefore do not change their time-ordering. Hence, the quantum metrization of the general affine coframe bundle $GA^{*}\mathbf{S}$ selects a metrized coframe bundle $P^{*}\mathbf{S}^{\uparrow}(\mathcal{g}^{\mathrm{M}})$ corresponding to a set of mean metrics $\mathbf{g}^{\mathrm{M}}$ that are diffeomorphically equivalent under $\mathit{Diff}_{0}\,\mathbf{S}$, rather than $\mathit{Diff}\mathbf{S}$, thereby constituting a *QGR gauge orbit* $(\mathbf{S},\mathcal{g}^{\mathrm{M}})$ over the base-segment $\mathbf{S}$. Each quantum-geometrically causal diffeomorphism $\psi \in \mathit{Diff}_{0}\mathbf{S}$ gives rise, in the same manner as in Sec. 8.1, to a vertical automorphism $\boldsymbol{\psi} \in \mathrm{G}_{A}(\mathit{Diff}_{0}\mathbf{S})$ that acts on physical moving coframes given by cross-sections $\boldsymbol{s}$ of the Poincaré coframe bundle $P^{*}\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$, by

mapping them, in accordance with (1.4), into physical moving coframes given by cross-sections $\boldsymbol{s}'$ of some other Poincaré coframe bundle $P^*\mathbf{S}^{\uparrow}(\mathbf{g}'^{\mathbf{M}})$ with $\mathbf{g}'^{\mathbf{M}} \in (\mathbf{S}, \mathcal{g}^{\mathbf{M}})$, so that

$$\boldsymbol{\psi} : \boldsymbol{s} = \{(\boldsymbol{a}^*, \boldsymbol{\theta}^i)\} \mapsto \boldsymbol{s}' = \{(\mathrm{D}^{0,1}(\psi)\boldsymbol{a}, \mathrm{D}^{0,1}(\psi)\boldsymbol{\theta}^i)\} . \tag{6.7}$$

The family $\mathrm{G}_A(\mathit{Diff}_0\mathbf{S})$ of all such vertical automorphisms constitutes a subgroup of the vertical automorphism group $\mathbf{Aut}_{\mathbf{S}}\mathbf{P}(\mathbf{M},\mathbf{G})$, $\mathbf{G} = \mathrm{GA}(4,\mathbf{R})$.

Each $\boldsymbol{\psi} \in \mathrm{G}_A(\mathit{Diff}_0\mathbf{S})$ gives rise above each $x \in \mathbf{S}$ to a new cotetrad with elements

$$\boldsymbol{\theta}'^i(x) = A^i_j(x)\boldsymbol{\theta}^j(x) , \qquad \boldsymbol{u}' = (\boldsymbol{a}'^*, \boldsymbol{\theta}'^i) = \boldsymbol{\psi}(\boldsymbol{a}^*, \boldsymbol{\theta}^i) . \tag{6.8}$$

Consequently, it also gives rise to the graviton state vector transformations

$$f_{ij}\,\boldsymbol{\theta}^i(x) \otimes \boldsymbol{\theta}^j(x) \mapsto f'_{ij}\,\boldsymbol{\theta}'^i(x) \otimes \boldsymbol{\theta}'^j(x) \in \mathbf{Z}_{\boldsymbol{u}'} , \quad f'_{ij} = A^{i'}_i(x)A^{j'}_j(x)f_{i'j'} , \tag{6.9}$$

which, in the case of the decomposition in (4.20), lead to the following groupings into equivalence classes,

$$f^{\mathrm{TT}}_{ij}\,\boldsymbol{\theta}^i(x) \otimes \boldsymbol{\theta}^j(x) \leftrightarrow f'^{\mathrm{TT}}_{ij}\,\boldsymbol{\theta}'^i(x) \otimes \boldsymbol{\theta}'^j(x) , \quad f'^{\mathrm{TT}}_{ij} = A^{i'}_i(x)A^{j'}_j(x)f^{\mathrm{TT}}_{i'j'} , \tag{6.10a}$$

$$f^{\perp}_{ij}\,\boldsymbol{\theta}^i(x) \otimes \boldsymbol{\theta}^j(x) \leftrightarrow f'^{\perp}_{ij}\,\boldsymbol{\theta}'^i(x) \otimes \boldsymbol{\theta}'^j(x) , \quad f'^{\perp}_{ij} = A^{i'}_i(x)A^{j'}_j(x)f^{\perp}_{i'j'} , \tag{6.10b}$$

$$f^{b}_{ij}\,\boldsymbol{\theta}^i(x) \otimes \boldsymbol{\theta}^j(x) \leftrightarrow f'^{b}_{ij}\,\boldsymbol{\theta}'^i(x) \otimes \boldsymbol{\theta}'^j(x) , \quad f'^{b}_{ij} = A^{i'}_i(x)A^{j'}_j(x)f^{b}_{i'j'} , \tag{6.10c}$$

Hence, on account of (5.17), (5.18) and (5.21), external information pertaining to diffeomorphic transformations performed on base-segments is encoded into internal graviton degrees of freedom that are correlated to ghost and antighost states. In particular, according to (5.18) and (5.21b), we have

$$f'^{b}_{ij}\,\boldsymbol{\theta}'^i(x) \otimes \boldsymbol{\theta}'^j(x) \leftrightarrow \boldsymbol{c}'^b \in \mathbf{B}_{\boldsymbol{u}'} , \quad \partial_j\partial^i c'^b_i = b'_j = A^k_j(x)b_k , \tag{6.11}$$

so that such diffeomorphism-originated transformations generically entail changes in the internal generalized Lorenz gauge fixing.

Our next task is to construct for each QGR gauge orbit $(\mathbf{S}, \mathcal{g}^{\mathbf{M}})$ over $\mathbf{S}$ a superfibre bundle based on the quantum gravitational gauge supergroup $\mathcal{QG}(\mathbf{S}, \mathcal{g}^{\mathbf{M}})$, and capable of correlating quantum gravitational states consisting of physical and ghost-antighost graviton state vectors belonging to the equivalence classes emerging from the above constructions.

By definition, a *Lie supergroup* is a supermanifold whose group operations of multiplication and inversion are given by G^∞ functions[15] in any coordinate chart. In turn, for any fixed $m = 1,2,\ldots,\infty$ and $n = 1,2,\ldots,\infty$, an (m,n)-dimensional *supermanifold* $\boldsymbol{M}$ is defined as a Hausdorff topological space which is locally homeomorphic to the (m,n)-dimensional superspace

$$\Lambda^{(m,n)} = (\Lambda^0 \times \Lambda^0 \times \cdots) \times (\Lambda^1 \times \Lambda^1 \times \cdots) , \tag{6.12}$$

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[15] G^∞ functions (cf., e.g., Rogers, 1980, p. 1354) play in the theory of supermanifolds a role similar to that of C^∞ functions in the theory of ordinary manifolds. Different definitions of differentiation of Grassmann-valued functions have been proposed in the literature (cf., e.g., Khrennikov, 1988), but since we shall be dealing only with specific structures, such distinctions, normally made at a general level, are of no significance in the sequel.

namely to the Banach space constructed from the even-grade Banach subalgebra Λ^0 and odd-grade Banach subspace Λ^1 of a Grassmann algebra Λ, such as the one described in Sec. 6.2 – and with the topology in the case of $m = \infty$ and/or $n = \infty$ emerging from the norm-topology for the case of finite m and n. In other words, the definition of a supermanifold is modeled[16] after the definition of an ordinary manifold, given in Sec. 2.1 for the case of real 4-dimensional manifold, with the role played in that case by $\mathbf{R}^4$ (or, in the generic case, by any real or complex Banach space) being taken over by the superspace $\Lambda^{(m,n)}$ in (6.12). This means that a supermanifold $\boldsymbol{M}$ can be covered with an atlas of charts (M_α, ϕ_α) over open sets M_α in $\boldsymbol{M}$ in such a manner that there is a homeomorphism ϕ_α between each M_α and a respective open subset O_α in the superspace $\Lambda^{(m,n)}$, so that the coordinate-transformation maps (2.1.2a) between any two charts in that atlas are smooth.

To take into account the gauge fixing in (6.4), we introduce in the subfibre

$$\mathbf{F}_u^\perp = \left\{ \mathbf{c}^\perp = c_i^\perp \, \theta^i(x) \;\middle|\; \partial^i c_i^\perp = 0 \right\} \subset \mathbf{F}_u \,, \tag{6.13}$$

a J_u-orthonormal basis, in terms of which the corresponding linear subspace of the superfibre $\mathbf{B}_u$ in (5.21a) can be expressed in the form

$$\mathbf{B}_u^\perp = \left\{ \boldsymbol{c}^\perp = \theta^\alpha(\mathbf{c}^\perp)\mathbf{c}_{u;\alpha}^\perp \;\middle|\; \theta^\alpha(\mathbf{c}^\perp) = \left(\mathbf{c}_{u;\alpha}^\perp \middle| \mathbf{c}^\perp\right)_{J_u} \vartheta_\alpha \,, \;\; \mathbf{c}^\perp \in \mathbf{F}_u^\perp \right\}. \tag{6.14}$$

On the other hand, the subfibres

$$\mathbf{F}_u^b = \left\{ \mathbf{c} = c_i \, \theta^i(x) \;\middle|\; \partial_j \partial^i c_i = b_j \right\} \subset \mathbf{F}_u \,, \tag{6.15}$$

corresponding to generalized Lorenz gauges are, in general, affine rather than linear subspaces of $\mathbf{F}_u$. However, according to (4.44), the "bad" modes of graviton polarization that are mapped in (6.4b) into ghost states within them can be identified, for all the various Lorenz gauge-fixing functions b, with the elements of a Hilbert fibre

$$\boldsymbol{H} = \left\{ \beta = (\beta_0, \beta_1, \beta_2, \beta_3) \;\middle|\; \langle \beta | \beta \rangle = 4 \sum_{r=-2}^{+2} \int \overline{\beta}_r(\zeta) \beta_r(\zeta) d\tilde{\Sigma}(\zeta) \;<\; \infty \right\}, \tag{6.16}$$

which can be used to parametrize them. Hence, the ghost superfibre that corresponds to a generalized Lorenz gauge is given by

$$\mathbf{F}_u^b = \left\{ \mathbf{b}(\beta) + \mathbf{c}^\perp \;\middle|\; \mathbf{c}^\perp \in \mathbf{F}_u^\perp \right\} \,, \qquad \mathbf{b}(\beta) = \mathrm{b}^\alpha \mathbf{c}_{u;\alpha} \in \mathbf{F}_u \,. \tag{6.17}$$

Naturally, all these observations remain valid in the case of antighost superfibres.

We shall now introduce at each $\boldsymbol{u} \in P^* \mathbf{S}^\uparrow(\mathcal{g}^M)$ the odd Grassmann variables

$$\theta^\alpha = \left(\mathbf{c}_{u;\alpha}^\perp \middle| \mathbf{c}\right)_{J_u} \vartheta_\alpha \,, \quad \mathbf{c} \in \mathbf{F}_u^\perp \,, \quad \overline{\theta}^\alpha = \left(\overline{\mathbf{c}}_{u;\alpha}^\perp \middle| \overline{\mathbf{c}}\right)_{J_u} \overline{\vartheta}_\alpha \,, \quad \overline{\mathbf{c}} \in \overline{\mathbf{F}}_u^\perp \,, \quad \alpha = 1,2,\ldots \,. \tag{6.18}$$

If we set in accordance with (6.5)

[16] Cf. (Rogers, 1980) and (Jadczyk and Pilch, 1981) for comprehensive introductions, and (Berezin, 1987) for all the basic definitions and results on superanalysis, supermanifolds, Lie superalgebras and supergroups. DeWitt (1984a) presents an approach to the same subjects aimed primarily at physicists. However, a detailed knowledge of these subjects is not absolutely necessary for an understanding of the main ideas presented in this chapter.

$$\theta^{\alpha}(v) = U^{\alpha}_{\beta}(0,\Lambda_v)\theta^{\beta}\ , \qquad \overline{\theta}^{\alpha}(v) = \overline{\theta}^{\beta}U^{\alpha}_{\beta}(0,\Lambda_v^{-1})\ , \qquad v \in \boldsymbol{V}^{+}, \tag{6.19}$$

$$U^{\alpha}_{\beta}(0,\Lambda_v) = \left(\mathbf{c}^{\perp}_{\boldsymbol{u};\beta}\middle|U(0,\Lambda_v)\mathbf{c}^{\perp}_{\boldsymbol{u};\alpha}\right)_{J_{\boldsymbol{u}}}\ , \qquad \alpha,\beta = 1,2,\ldots\ , \tag{6.20}$$

then, for any given cross-section $\boldsymbol{s}$ of the Poincaré coframe bundle $P^{*}\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$, the values assumed by θ^{α} and $\overline{\theta}^{\alpha}$ can be used to label all the coframes within $P^{*}\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$ that are obtained, in accordance with (6.3), by imparting Lorentz boosts to the coframes $\boldsymbol{s}(x)$ in $\boldsymbol{s}$, and by reversing those boosts, respectively.

By using the method expounded in Sec. 6.3 we can now construct from the superfibres in (5.21) the quantum gravitational ghost and antighost superfibres

$$\mathcal{B}_{\boldsymbol{u}} = \bigoplus\nolimits_{n=0}^{\infty}\mathcal{B}_{\boldsymbol{u},n}\ , \qquad \overline{\mathcal{B}}_{\boldsymbol{u}} = \bigoplus\nolimits_{n=0}^{\infty}\overline{\mathcal{B}}_{\boldsymbol{u},n}\ , \qquad \mathcal{B}_{\boldsymbol{u},n} = \mathbf{B}_{\boldsymbol{u}}\underset{S}{\otimes}\cdots\underset{S}{\otimes}\mathbf{B}_{\boldsymbol{u}}\ . \tag{6.21}$$

We can then introduce within the duals of the ensuing quantum gravitational superfibres

$$\mathcal{Z}^{\Diamond}_{\boldsymbol{u}} = \mathcal{Z}^{\mathrm{TT}}_{\boldsymbol{u}} \otimes \overline{\mathcal{B}}_{\boldsymbol{u}} \otimes \mathcal{B}_{\boldsymbol{u}} \tag{6.22}$$

the quantum gravitational coframes[17]

$$\boldsymbol{\Phi}^{\dagger}_{f,\bar{\mathbf{c}},\mathbf{b},\mathbf{c}} = \boldsymbol{\Phi}^{\dagger}_{f} \otimes \boldsymbol{\Phi}_{\bar{\mathbf{c}}} \otimes \boldsymbol{\Phi}^{\dagger}_{\mathbf{b},\mathbf{c}}\ , \qquad f \in \mathbf{Z}^{\mathrm{TT}}_{\boldsymbol{u}}\ , \qquad \mathbf{b} \in \mathbf{F}_{\boldsymbol{u}}\ , \qquad \bar{\mathbf{c}} \in \overline{\mathbf{F}}^{\perp}_{\boldsymbol{u}}\ , \qquad \mathbf{c} \in \mathbf{F}^{\perp}_{\boldsymbol{u}}\ , \tag{6.23}$$

in which $\boldsymbol{\Phi}^{\dagger}_{f}$ denotes the dual of the quantum frame element $\boldsymbol{\Phi}_{f}$ that appears in (5.7)-(5.9); whereas, in accordance with (5.22), (5.23) and (6.18),

$$\boldsymbol{\Phi}^{\dagger}_{\mathbf{b},\mathbf{c}} = \boldsymbol{\Psi}^{\dagger}_{0;\boldsymbol{u}}\cdot\exp\left(\overline{\theta}^{\alpha}C^{(-)}_{\alpha,\mathbf{b}}(\boldsymbol{u}) + \tfrac{1}{2}\textstyle\sum_{\alpha=1}^{\infty}\theta^{\alpha}\overline{\theta}^{\alpha}\right), \quad \mathbf{c} \leftrightarrow \boldsymbol{c} = \theta^{\alpha}\mathbf{c}^{\perp}_{\boldsymbol{u};\alpha} \in \mathbf{B}_{\boldsymbol{u}}\ , \tag{6.24a}$$

$$\boldsymbol{\Phi}_{\bar{\mathbf{c}}} = \exp\left(-\tfrac{1}{2}\textstyle\sum_{\alpha=1}^{\infty}\overline{\theta}^{\alpha}\theta^{\alpha} + \theta^{\alpha}\overline{C}^{(+)}_{\alpha}(\boldsymbol{u})\right)\boldsymbol{\Psi}_{0;\boldsymbol{u}}, \quad \bar{\mathbf{c}} \leftrightarrow \bar{\boldsymbol{c}} = \overline{\theta}^{\alpha}\bar{\mathbf{c}}^{\perp}_{\boldsymbol{u};\alpha} \in \overline{\mathbf{B}}_{\boldsymbol{u}}\ , \tag{6.24b}$$

$$C^{(-)}_{\alpha,\mathbf{b}}(\boldsymbol{u}) = C^{(-)}(\boldsymbol{u};\mathbf{c}^{\perp}_{\boldsymbol{u};\alpha}) + \overline{\theta}^{\alpha}B^{(-)}(\boldsymbol{u};\mathrm{b}^{\alpha}\mathbf{c}_{\boldsymbol{u};\alpha})\ , \quad \mathrm{b}^{\alpha} = \left(\mathbf{c}_{\boldsymbol{u};\alpha}\middle|\theta^{j}(x)\partial_{j}\partial^{i}c_{i}\right)_{J_{\boldsymbol{u}}}\ , \tag{6.24c}$$

where B denotes a bosonic gauge-fixing field defined by the general method described in Sec. 5.2, and where no summation over the index α is to be carried out in the last term of the first equation in (6.24c).

Let us now introduce, for each value assumed by α, the following ghost and antighost frame fields

$$C_{\alpha,\mathbf{b}}(\boldsymbol{u}) = C^{(-)}_{\alpha,\mathbf{b}}(\boldsymbol{u}) + C^{(+)}_{\alpha,\mathbf{b}}(\boldsymbol{u})\ , \qquad \overline{C}_{\alpha}(\boldsymbol{u}) = \overline{C}^{(-)}_{\alpha}(\boldsymbol{u}) + \overline{C}^{(+)}_{\alpha}(\boldsymbol{u})\ . \tag{6.25}$$

The addition of ghost creation operators and antighost annihilation operators to the annihilation and creation operators in (6.24a) and (6.24b), respectively, contributes[18] a factor that is given by an exponential in second powers of the above Grassmann variables. Hence that factor equals the identity operator, and does not change the values of the above frame ele-

[17] Duals of frames and of superframes are used in order to obtain action from the right of the resulting quantum gravitational supergroup introduced in (6.29).

[18] This is most easily established by using the well-known equation $\exp(A+B) = \exp(-[A,B]/2)\exp A\exp B$, which holds when $[A,B]$ commutes with both operators A and B. This equation is known in physics literature as the Baker-Campbell-Haussdorf formula, but it was actually first proven by Weyl (1928).

ments, so that we can express those elements as follows:

$$\boldsymbol{\Phi}_{\bar{\mathbf{c}}} \otimes \boldsymbol{\Phi}^{\dagger}_{\mathbf{b},\mathbf{c}} = \exp\left(\theta^{\alpha}\overline{C}_{\alpha}(\boldsymbol{u})\right) \boldsymbol{\Psi}_{0;\boldsymbol{u}} \otimes \boldsymbol{\Psi}^{\dagger}_{0;\boldsymbol{u}} \cdot \exp\left(\sum\nolimits_{\alpha=1}^{\infty} \theta^{\alpha}\overline{\theta}^{\alpha} + \overline{\theta}^{\alpha} C_{\alpha,\mathbf{b}}(\boldsymbol{u})\right) . \quad (6.26)$$

Furthermore, if we write in abbreviated notation[19]

$$\overline{\theta}C(\boldsymbol{u}) = \overline{\theta}^{\alpha}C_{\alpha,\mathbf{0}}(\boldsymbol{u}), \quad \theta\overline{C}(\boldsymbol{u}) = \theta^{\alpha}\overline{C}_{\alpha}(\boldsymbol{u}), \quad \theta\overline{\theta}B(\boldsymbol{u}) = \sum\nolimits_{\alpha=1}^{\infty} \theta^{\alpha}\overline{\theta}^{\alpha}B_{\alpha}(\boldsymbol{u}) , \quad (6.27a)$$

$$\theta\overline{\theta}C(\boldsymbol{u}) \times \overline{C}(\boldsymbol{u}) = \sum\nolimits_{\alpha=1}^{\infty} \theta^{\alpha}\overline{\theta}^{\alpha}\left(4 + \{C_{\alpha,\mathbf{0}}(\boldsymbol{u}), \overline{C}_{\alpha}(\boldsymbol{u})\}\right), \quad B_{\alpha}(\boldsymbol{u}) = B(\boldsymbol{u}; \mathrm{b}^{\alpha}\mathbf{c}_{\boldsymbol{u};\alpha}) , \quad (6.27b)$$

then we can reexpress them in the following succinct form:

$$\boldsymbol{\Phi}^{\dagger}_{\bar{\mathbf{c}},\mathbf{b},\mathbf{c}} := \boldsymbol{\Phi}_{\bar{\mathbf{c}}} \otimes \boldsymbol{\Phi}^{\dagger}_{\mathbf{b},\mathbf{c}} = \boldsymbol{\Psi}_{0;\boldsymbol{u}} \otimes \boldsymbol{\Psi}^{\dagger}_{0;\boldsymbol{u}} \cdot \mathbf{U}(\boldsymbol{u}; \overline{\boldsymbol{c}}, \boldsymbol{b}, \boldsymbol{c}) , \quad (6.28)$$

$$\mathbf{U}(\boldsymbol{u}; \overline{\boldsymbol{c}}, \boldsymbol{b}, \boldsymbol{c}) = \exp\left(\theta\overline{C}(\boldsymbol{u}) + \overline{\theta}C(\boldsymbol{u}) + \theta\overline{\theta}\left[B(\boldsymbol{u}) + \tfrac{1}{2}C(\boldsymbol{u}) \times \overline{C}(\boldsymbol{u})\right]\right) \quad (6.29)$$

Let us now consider any physical moving frame $(\boldsymbol{s}, \mathbf{g}^{\mathrm{M}})$, given by a cross-section $\boldsymbol{s}$ of the Poincaré coframe bundle $P^{*}\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$ corresponding to some Lorentzian mean metric $\mathbf{g}^{\mathrm{M}} \in (\mathbf{S}, \mathcal{g}^{\mathrm{M}})$. The $\boldsymbol{\psi}$-component of an external Cartan gauge transformation, defined in accordance with (1.5) and (1.6) by

$$(\boldsymbol{\psi}, \boldsymbol{\phi}) : \quad (\boldsymbol{s}, \mathbf{g}^{\mathrm{M}}) \mapsto (\boldsymbol{s}', \mathbf{g}'^{\mathrm{M}}) , \quad \boldsymbol{\psi} \in \mathrm{G}_{A}(\mathit{Diff}_{0}\mathbf{S}) , \quad \boldsymbol{\phi} \in \mathcal{G}(\mathbf{P}(\mathbf{S}, \mathrm{ISO}^{\uparrow}(3,1)) , \quad (6.30)$$

can be transferred into a family of internal generalized Lorenz gauge changes determined by $\boldsymbol{\psi}$ as in (6.7)-(6.11); moreover, according to (7.7.9) and (6.3), we have

$$\boldsymbol{\phi}(\boldsymbol{s}(x) \cdot (a(x), \Lambda(x))) = (0, \Lambda_{v}^{-1}(x))\, \boldsymbol{\phi}(\boldsymbol{s}(x) \cdot (a(x), R(x)))(0, \Lambda_{v}(x)) , \quad x \in \mathbf{S}, \quad (6.31)$$

so that its $\boldsymbol{\phi}$-component can be related to a supergauge transformation that is based on (6.29). Thus, if we start from a TT-gauge fixing with respect to $(\boldsymbol{s}, \mathbf{g}^{\mathrm{M}})$ that corresponds to a choice of quantum gravitational coframes whose elements are the duals of the coherent states $\boldsymbol{\Phi}_{\boldsymbol{f}}$ for $\boldsymbol{f} \in \mathbf{F}_{\boldsymbol{s}(x)}$ in the TT-gauge everywhere in $\mathbf{S}$, then this kind of supergauge transformation will be provided by

$$\mathbf{U}_{\boldsymbol{s}}(\boldsymbol{\psi}(x), \boldsymbol{\phi}(x)) = \boldsymbol{U}(a(x), R(x)) \otimes \mathbf{U}(\boldsymbol{s}(x); \overline{\boldsymbol{c}}(x), \boldsymbol{b}(x), \boldsymbol{c}(x)) , \quad (6.32)$$

where the first factor in the above tensor product is supplied by (3.17), (4.37) and (4.41):

$$\boldsymbol{U}(a, R) = \bigoplus\nolimits_{n=0}^{\infty} U(a, R)^{\otimes n} , \qquad (a, R) \in \mathrm{T}^{4} \wedge \mathrm{O}(3) \subset \mathrm{ISO}^{\uparrow}(3,1). \quad (6.33)$$

Therefore, the Cartan gauge transformation in (6.30) will thereby be transferred into a *quantum gravitational gauge transformation* which is provided by the following change of quantum gravitational coframes,

$$\boldsymbol{\Phi}^{\dagger}_{\boldsymbol{f}} \mapsto \boldsymbol{\Phi}^{\dagger}_{\boldsymbol{f}} \cdot \mathbf{U}_{\boldsymbol{s}}(\boldsymbol{\psi}, \boldsymbol{\phi}) = \boldsymbol{\Phi}^{\dagger}_{\boldsymbol{f}} \cdot \boldsymbol{U}(a, R) \otimes \boldsymbol{\Psi}^{\dagger}_{0;\boldsymbol{u}} \cdot \mathbf{U}(\boldsymbol{u}; \overline{\boldsymbol{c}}, \boldsymbol{b}, \boldsymbol{c}) = \boldsymbol{\Phi}^{\dagger}_{U^{-1}(a,R)\boldsymbol{f},\overline{\boldsymbol{c}},\mathbf{b},\mathbf{c}} , \quad (6.34)$$

so that it gives rise to the creation of ghost and antighost states in the original frames.

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[19] This choice of notation is meant to lead in Sec. 8.7 to expressions which formally resemble those in some earlier work on a BRST superformalism – cf. (Bonora *et al.*, 1981b, 1982a), (Hoyos *et al.*, 1982).

The *quantum gravitational supergroup* $\mathcal{QG}(\mathbf{S},g^{\mathbf{M}})$ over $\mathbf{S}$ is a representation of the following restriction,

$$\mathcal{G}(\mathbf{S},g^{\mathbf{M}}) = G_A(Diff_0\mathbf{S}) \wedge \mathcal{G}(\mathbf{P}(\mathbf{S},\mathrm{ISO}^{\uparrow}(3,1)))\,, \tag{6.35}$$

of the CGR gauge group in (6.6) to the QGR gauge orbit $(\mathbf{S},g^{\mathbf{M}})$. This representation is obtained when the transformations in (6.32), as well as all their inverses, are constructed for all physical moving frames $\boldsymbol{s}$ in $P^{*}\mathbf{S}^{\uparrow}(g^{\mathbf{M}})$, all diffeomorphisms $\psi \in Diff_0\mathbf{S}$, and all Poincaré gauge transformations $\phi \in \mathcal{G}(\mathbf{P}(\mathbf{S},\mathrm{ISO}^{\uparrow}(3,1))$, and then combined in accordance with (1.6). Hence, such a quantum gravitational supergroup incorporates not only diffeomorphism but also Poincaré gauge invariance into the quantum-geometric framework for quantum gravity.

This latter type of gauge invariance is ignored by some recent approaches to canonical quantum gravity (Ashtekar, 1991; Smolin, 1991). Since this most fundamental concept is inextricably intertwined with local Poincaré covariance, such formulations of quantum gravity are incapable of incorporating the concept of the graviton as the gravitational gauge quantum which is exchanged in quantum gravitational interactions. On the other hand, some of the earlier work on quantum gravity (Delbourgo *et al.*, 1982; Pasti and Tonin, 1982; Baulieu and Thierry-Mieg, 1984; Falck and Hirshfeld, 1984) displayed a full awareness of the fact that, in order to arrive at a truly consistent theory of quantum gravity, it is mandatory that diffeomorphism invariance be combined with Poincaré gauge invariance.[20] However, when using the type of superfield formalism described in the next section, this earlier work treated Grassmann variables in a purely formal manner, rather than deriving them – as was the case in the present section – from a supersymmetric formulation of diffeomorphism invariance. Consequently, the mathematical as well as the physical aspects of all these treatments have remained very formal, for reasons that are very clearly revealed in the following passage:

"The elegance and efficiency of the unified [BRST superfield] formalism are certainly consequences of a geometrical structure that is still unknown. One may hope that a superfield formalism exists in which the full theory ... would only be expressed in terms of a single generalized gauge field containing classical and ghost components, leading thereby to a true unification of ghost and classical fields in field theory. Unfortunately no one has been able to build such a superformalism, in spite of several attempts. The main problem is of course a deeper understanding of a quantum field theory with a dependence on unphysical variables in θ, $\bar{\theta}$." (Baulieu, 1985, p. 18.)

As we shall demonstrate in the next section, this hereto "unknown geometrical structure" reveals itself very clearly once a quantum gravitational connection is introduced that is based on the quantum gravitational supergroup constructed in this section.

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[20] Some of this earlier work replaced Poincaré gauge invariance with Lorentz gauge invariance. Furthermore, instead of dealing with diffeomorphism gauge invariance, it dealt with "coordinate gauge invariance," asserting that "gravitation theories expressed in terms of vierbein fields are invariant under two local groups: the general co-ordinate transformation group and the internal Lorentz group." (Pasti and Tonin, 1982, p. 99.) Note should be taken, however, of the fact that coordinate transformations corresponding to changes of charts do *not* generically constitute a group, since their domains are chart-dependent.

8.7. Quantum Gravitational Connections and BRST Symmetries

In the semi-classical regime, the fundamental idea of quantum-geometric propagation is that its propagators are provided by path integrals into which the strong equivalence principle is embedded from the outset by virtue of the fact that such propagation proceeds by parallel transport along broken paths that consist of arcs geodesics of Levi-Civita connections, and that such parallel transport is governed by that connection. Clearly, during the process of adaptation of this basic idea to the quantum-gravitational regime, the required Levi-Civita connections will be supplied in each base-segment $\mathbf{S}$ by the mean metrics $\mathbf{g}^{\mathrm{M}} \in (\mathbf{S}, \mathcal{g}^{\mathrm{M}})$. However, in view of the presence in a quantum spacetime of the quantum gravitational supergroup $\mathcal{QG}(\mathbf{S}, \mathcal{g}^{\mathrm{M}})$ introduced in the preceding section, the connection 1-forms of the quantum gravitational connection will obviously not be determined exclusively by the Levi-Civita connection, but also by Maurer-Cartan 1-forms related to the Grassmann degrees of freedom of that supergroup, which emerge from (6.29)-(6.34).

In order to take into account the degrees of freedom contained in the Grassmann variables in (6.18), we shall now associate with each base-segment $\mathbf{S}$ a corresponding *base-supersegment*

$$\boldsymbol{S} = \left\{ (x, \theta, \overline{\theta}) \,\middle|\, x \in \mathbf{S},\ \theta = (\theta^1, \theta^2, \ldots),\ \overline{\theta} = (\overline{\theta}^1, \overline{\theta}^2, \ldots) \right\} . \tag{7.1}$$

In the above definition, θ^α and $\overline{\theta}^\alpha$, $\alpha = 1,2,\ldots$, denote Grassmann variables that assume their values in the Banach subspace Λ^1 of odd supernumbers defined in Sec. 6.2. Since the coordinates of x range only over real numbers, rather than over the Grassmann subalgebra Λ^0 of even supernumbers, $\boldsymbol{S}$ is only a pseudo-supermanifold, rather than a supermanifold in the sense of the conventional definition given in the preceding section. Of course, an extension of $\boldsymbol{S}$ that would accommodate even-valued Grassmann coordinates, rather than just real-valued ones, could be very easily achieved. However, such an extension of the range of those variables is not actually required for the physical purposes of this chapter. In fact, we shall subsequently concentrate on the *quantum spacetime segment*

$$\boldsymbol{S}(\mathcal{g}^{\mathrm{M}}) = \left\{ (x, \theta, \overline{\theta}) \in \boldsymbol{S} \,\middle|\, \theta = (\theta^1_{\boldsymbol{s}(x)}, \ldots),\ \overline{\theta} = (\overline{\theta}^1_{\boldsymbol{s}(x)}, \ldots),\ \boldsymbol{s} \in \Sigma(P^*\mathbf{S}^\uparrow(\mathcal{g}^{\mathrm{M}})) \right\}, \tag{7.2}$$

containing only the Grassmann values that are assumed by the variables θ^α and $\overline{\theta}^\alpha$, $\alpha = 1,2,\ldots$, in (6.18) and (6.19) while $\boldsymbol{s}$ ranges over the family $\Sigma(P^*\mathbf{S}^\uparrow(\mathcal{g}^{\mathrm{M}}))$ of all cross-sections of the metrized coframe bundle $P^*\mathbf{S}^\uparrow(\mathcal{g}^{\mathrm{M}})$.

The first stage in the construction of a quantum gravitational connection over $\boldsymbol{S}(\mathcal{g}^{\mathrm{M}})$ follows the pattern established for gauge fields in the semi-classical regime of the preceding chapter. Thus, we shall first introduce within the quantum gravitational fibres $\mathcal{Z}_{\boldsymbol{u}}$, that were defined in (5.1), the infinitesimal generators

$$\boldsymbol{P}_{j;\boldsymbol{u}} = i \int \eta^{ik} \eta^{i'k'} g^{(+)}_{ii'}(\boldsymbol{u};\zeta)\, \partial_j g^{(-)}_{kk'}(\boldsymbol{u};\zeta)\, d\tilde{\Sigma}(\zeta) , \qquad \partial_j = \partial/\partial q^j , \tag{7.3}$$

of spacetime translations. On the other hand, the considerations that let to the two variants in (7.3.24) for the indefinite inner product in a photon fibre are equally well applicable to

graviton fibres, so that we express the inner product in (3.5) in the alternative form

$$\langle \boldsymbol{f}|\boldsymbol{f}'\rangle = \int_{V_0^+} \eta^{ii'}\eta^{jj'}\tilde{f}^{*}_{ij}(\zeta)\overleftrightarrow{\partial}_k \tilde{f}'_{i'j'}(\zeta)\, d\hat{\sigma}^k(q)\, d\Omega(v) \ , \quad \boldsymbol{f},\boldsymbol{f}' \in \mathbf{Z}_{\boldsymbol{u}} \ . \tag{7.4}$$

By taking advantage of this form of the inner product in $\mathbf{Z}_{\boldsymbol{u}}$ we can adapt the derivation leading in the electromagnetic case from (7.5.3) to (7.5.13) to the restriction of the operators in (7.3) to the Lorenz subfibre of $\mathbf{Z}_{\boldsymbol{u}}$, thus obtaining

$$\boldsymbol{P}_{j;\boldsymbol{u}} = \int\!:T_{jk}[\boldsymbol{g}(\boldsymbol{u};\zeta)]:\, d\hat{\sigma}^k(q)\, d\Omega(v) \ , \quad j = 0,1,2,3 \ , \tag{7.5a}$$

$$\boldsymbol{g} = g_{ij}\,\boldsymbol{\theta}^i(x)\otimes\boldsymbol{\theta}^j(x) \ , \quad g_{ij} = g^{(+)}_{ij} + g^{(-)}_{ij} \ , \quad g^{ij} = \eta^{ii'}\eta^{jj'}g_{i'j'} \ , \tag{7.5b}$$

$$T_{ij}[\boldsymbol{g}] = \tfrac{1}{2}\eta_{ij}\,g_{kl,m}\,g^{kl,m} - g_{km,i}\,g^{km}{}_{,j} \ , \quad g_{ik,j} = \partial g_{ik}/\partial q^j \ , \tag{7.5c}$$

in the Lorenz internal gauge. This adaptation can be further extended to the derivation of the following analogues of (7.5.15) and (7.5.16):

$$\boldsymbol{M}^{ij}_{\boldsymbol{u}} = \int\!:M^{ijk}_{\boldsymbol{u}}[\boldsymbol{g}(\boldsymbol{u};\zeta)]:\, d\hat{\sigma}_k(q)\, d\Omega(v) \ , \tag{7.6a}$$

$$M^{ijk}_{\boldsymbol{u}}[\boldsymbol{g}] = Q^i_{\boldsymbol{u}}T^{jk}[\boldsymbol{g}] - Q^j_{\boldsymbol{u}}T^{ik}[\boldsymbol{g}] + \boldsymbol{g}^{\dagger}\,S^{ijk}_{\boldsymbol{u}}\,\boldsymbol{g} \ . \tag{7.6b}$$

Upon performing the identifications in (6.2), the four infinitesimal generators in (7.5a) and three infinitesimal generators of spatial rotations from (7.6a) can be used to define the following quantum connection

$$\boldsymbol{\Gamma}^{\mathrm{TT}} = \boldsymbol{d} + \boldsymbol{\Gamma}^{\mathrm{TT}} \ , \quad \boldsymbol{\Gamma}^{\mathrm{TT}} = -i\tilde{\boldsymbol{\theta}}^j_s\boldsymbol{P}_{j;s} + \tfrac{i}{2}\tilde{\boldsymbol{\omega}}^s_{ab}\,\boldsymbol{M}^{ab}_s \ , \quad j = 0,1,2,3 \ , \quad a,b = 1,2,3 \ , \tag{7.7}$$

within the fibre bundle in (6.1), which has $\mathbf{G} = \mathrm{T}^4 \wedge \mathrm{O}(3)$ as structure group. Naturally, on one hand, full Poincaré covariance thereby becomes lost but, on the other hand, the mathematical difficulties encountered already in Sec. 7.4 when attempting a manifestly Poincaré covariant treatment of such a massless gauge field as the quantum electromagnetic field are thereby avoided. Even more importantly, the possibility is thereby created to compensate for the loss of manifest covariance under Lorentz boosts by adding to the quantum TT-connection in (7.7) Grassmann-dependent terms related to the unphysical degrees of freedom that emerge under such boosts, so that (7.7) would transcend into the *quantum gravitational connection*

$$\boldsymbol{\nabla} = \mathbf{d} + \boldsymbol{\Gamma} + \boldsymbol{C} + \overline{\boldsymbol{C}} \ , \qquad \mathbf{d} = \boldsymbol{d} + \boldsymbol{\delta} + \overline{\boldsymbol{\delta}} \ . \tag{7.8}$$

In keeping with the general ideas discussed in Sec. 7.8, this connection incorporates the quantum gravitational BRST and anti-BRST operators $\boldsymbol{\delta}$ and $\overline{\boldsymbol{\delta}}$, so that

$$\boldsymbol{d} = \boldsymbol{\theta}^j\,\partial_j = \boldsymbol{dx}^{\mu}\,\partial/\partial x^{\mu} \ , \quad \boldsymbol{\delta} = \boldsymbol{d}\overline{\theta}^{\alpha}\,\partial/\partial\overline{\theta}^{\alpha} \ , \quad \overline{\boldsymbol{\delta}} = \boldsymbol{d}\theta^{\alpha}\,\partial/\partial\theta^{\alpha} \ . \tag{7.9}$$

According to the general results presented in (7.8.2)-(7.8.4), the connection form on the quantum frame bundle with elements (6.23), that would have the quantum gravitational supergroup $\mathcal{QG}(\mathbf{S},\mathcal{G}^{\mathrm{M}})$ of Sec. 8.7 as its gauge group of the second kind, would have to

satisfy for any gauge transformation the relations

$$ {}^{g}\omega \;=\; \mathrm{Ad}_{g^{-1}}\,\omega + g^{-1}dg\;, \qquad g=(\psi,\phi)\in\; \mathcal{G}(\mathbf{S},\mathcal{g}^{\mathrm{M}}), \tag{7.10} $$

where $g^{-1}dg$ represents the Maurer-Cartan form of that gauge group. With this fact in mind, let us consider the quantum gravitational gauge transformation in (6.34) for a physical moving frame $(\boldsymbol{s},\mathbf{g}^{\mathrm{M}})$, so that at each $x\in\mathbf{S}$

$$ \boldsymbol{s}(x)=(\boldsymbol{a}^{*}(x),\boldsymbol{\theta}^{i}(x))\in P^{*}\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})\;, \qquad \boldsymbol{e}_{j}(x)=\lambda_{j}{}^{\mu}(x)\partial_{\mu}\;, \quad \partial_{\mu}=\partial/\partial x^{\mu}\;. \tag{7.11} $$

Furthermore, let $(\psi(x),\phi(x))$ denote the representative at $x\in\mathbf{S}$ of the corresponding gravitational gauge transformation in (7.10), so that, in accordance with (1.5), (6.7) and (6.8), we can consider $(\psi(x),\phi(x))$ to provide the map

$$ (\psi(x),\phi(x)):\quad (\boldsymbol{s}(x),\mathbf{g}^{\mathrm{M}})\;\mapsto\;(\boldsymbol{s}'(x),\mathbf{g}'^{\mathrm{M}}(x))\in P^{*}\mathbf{S}^{\uparrow}(\mathcal{g}^{\mathrm{M}})\;, \tag{7.12} $$

where, in general, at different base-locations $x\in\mathbf{S}$ the coframes $\boldsymbol{s}'(x)$ might correspond to different Lorentzian metrics $\mathbf{g}'^{\mathrm{M}}\in(\mathbf{S},\mathcal{g}^{\mathrm{M}})$. Then, on account of (6.28) and (6.29), the family consisting of the following operators,

$$ \mathbf{U}_{s}(\psi(x),\phi(x))=\boldsymbol{U}(a(x),R(x))\otimes\mathbf{U}_{s}(x,\theta,\bar{\theta})\;, \tag{7.13a} $$

$$ \mathbf{U}_{s}(x,\theta,\bar{\theta})=\exp\Big[\theta\overline{C}(x)+\bar{\theta}C(x)+\theta\bar{\theta}\Big(B(x)+\tfrac{1}{2}C(x)\times\overline{C}(x)\Big)\Big]\;, \tag{7.13b} $$

at all $x\in\mathbf{S}$, provides the quantum gravitational gauge transformations in (7.10), where, in accordance with (6.25) and (6.27),

$$ B_{\alpha}(x)=B(\boldsymbol{s}(x);\mathrm{b}^{\alpha}(x)\mathbf{c}_{\boldsymbol{s}(x);\alpha}),\;\; C_{\alpha}(x)=C_{\alpha}(\boldsymbol{s}(x))\;,\;\; \overline{C}_{\alpha}(x)=\overline{C}_{\alpha,\mathbf{0}}(\boldsymbol{s}(x)), \tag{7.14} $$

with no summation over α being carried out in the first term of (7.14) – as was the case in (6.24c). Thus, if we assume that the $\boldsymbol{e}_{j}$-components of the operators for covariant differentiation were provided in the originally chosen moving frame $\boldsymbol{s}$ by the quantum TT-connection in (7.7), then, according to the general relation in (7.8.7),

$$ \boldsymbol{\nabla}_{j}=\partial_{j}+\boldsymbol{\Gamma}_{j}\;, \qquad \boldsymbol{\Gamma}_{j}=\mathrm{Ad}_{\mathbf{U}^{-1}(g)}\,\boldsymbol{\Gamma}_{j}^{\mathrm{TT};\boldsymbol{s}}+\mathbf{U}^{-1}(g)\lambda_{j}{}^{\mu}\partial_{\mu}\mathbf{U}(g)\;. \tag{7.15} $$

Thus, by explicitly computing[21] the above transformation $\boldsymbol{\Gamma}_{j}$ at a base-location $x\in\mathbf{S}$, we see that in relation to $\boldsymbol{s}\in\Sigma(P^{*}\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}}))$ we have

$$ \boldsymbol{\Gamma}_{j}(x,\theta,\bar{\theta})=\boldsymbol{\Gamma}_{j}^{\mathrm{TT};\boldsymbol{s}(x)}+\theta D_{j}\overline{C}(x)+\bar{\theta}D_{j}C(x)+\theta\bar{\theta}\Big(D_{j}B(x)+D_{j}C(x)\times\overline{C}(x)\Big)\;, \tag{7.16} $$

$$ D_{j}=\partial_{j}+[\boldsymbol{\Gamma}_{j}^{\mathrm{TT};\boldsymbol{s}},\cdot]\;, \qquad \partial_{j}:=\partial_{\boldsymbol{e}_{j}}=\lambda_{j}{}^{\mu}\partial_{\mu}\;, \tag{7.17} $$

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[21] This type of computations, as well as those leading to the remaining relations in this section, were first performed on supermanifolds by Bonora *et al.* (1981b, 1982a) and Hoyos *et al.* (1982). For that reason, we adapt the present notation to match the one used in those treatments of BRST symmetries. These formal analogies can be further extended to cover also other past approaches. For example, if the vierbeins in a physical moving frame s and the 1-forms defined by (6.18) are combined into "vielbein" 1-forms, one arrives at equations which share some formal similarities with the BRST equations of Baulieu and Thierry-Mieg (1984) in quantum gravity.

at each quantum spacetime superlocation identifiable, on account of (6.3), (6.5), (6.7) and (6.19), with the coframe $\boldsymbol{s}'(x)$ in (7.12):

$$\boldsymbol{s}'(x) \quad \leftrightarrow \quad X = (x, \theta, \bar{\theta}) \in \boldsymbol{S}(g^{\mathrm{M}}). \tag{7.18}$$

In addition to the original degrees of freedom inherited from the base-segment $\mathbf{S}$, the base-supersegment $\boldsymbol{S}$ in (7.1) incorporates the Grassmannian degrees of freedom labeled by θ and $\bar{\theta}$, so that the connection form in (7.10) gives rise also to the following *quantum gravitational covariant BRST* and *anti-BRST connection forms*:

$$s = \boldsymbol{\delta} + U^{-1}(g)\, \partial_{\bar{\theta}} U(g)\, \boldsymbol{d}\bar{\theta} \quad , \qquad \partial_{\bar{\theta}} = \partial / \partial \bar{\theta} \ , \tag{7.19a}$$

$$\bar{s} = \bar{\boldsymbol{\delta}} + U^{-1}(g)\, \partial_{\theta} U(g)\, \boldsymbol{d}\theta \ , \qquad \partial_{\theta} = \partial / \partial \theta \ . \tag{7.19b}$$

If we set, by definition,

$$\bar{B}(x) = -B(x) - C(x) \times \bar{C}(x) \ , \tag{7.20}$$

then the corresponding gauge potentials that are implicitly present within the quantum gravitational connection in (7.8) are seen to be equal to

$$C_{\alpha}(x, \theta, \bar{\theta}) = C_{\alpha}(x) + \theta^{\alpha} \bar{B}_{\alpha}(x) - \tfrac{1}{2} \bar{\theta}^{\alpha} \{C_{\alpha}(x), C_{\alpha}(x)\} + \theta^{\alpha} \bar{\theta}^{\alpha} [\bar{B}_{\alpha}(x), C_{\alpha}(x)] \ , \tag{7.21b}$$

$$\bar{C}_{\alpha}(x, \theta, \bar{\theta}) = \bar{C}_{\alpha}(x) + \bar{\theta}^{\alpha} B_{\alpha}(x) - \tfrac{1}{2} \theta^{\alpha} \{\bar{C}_{\alpha}(x), \bar{C}_{\alpha}(x)\} - \theta^{\alpha} \bar{\theta}^{\alpha} [B_{\alpha}(x), \bar{C}_{\alpha}(x)] \ . \tag{7.21b}$$

Hence, if we carry out in the Lorenz gauge in (7.16) and (7.21) the following translations for each Grassmannian degree of freedom separately,

$$\bar{\theta}^{\alpha} \mapsto \bar{\theta}^{\alpha} + \delta\bar{\theta} \ , \qquad \theta^{\alpha} \mapsto \theta^{\alpha} + \delta\theta, \qquad \alpha = 1,2,\ldots \ , \tag{7.22}$$

while keeping in the process that gauge fixed, then we observe that those superlocation X whose Grassmannian coordinates are zero they result in the following changes of the values assumed by the gauge potentials for the quantum gravitational connection $\boldsymbol{\nabla}$:

$$\boldsymbol{\delta} \Gamma_j(x) = \delta\bar{\theta}\, D_j C_{\alpha}(x) \quad , \qquad \boldsymbol{\delta} B_{\alpha}(x) = \mathbf{0} \ , \tag{7.23a}$$

$$\boldsymbol{\delta} C_{\alpha}(x) = -\tfrac{1}{2} \delta\bar{\theta} \{C_{\alpha}(x), C_{\alpha}(x)\} \ , \qquad \boldsymbol{\delta} \bar{C}_{\alpha}(x) = \delta\bar{\theta} B_{\alpha}(x) \ , \tag{7.23b}$$

$$\bar{\boldsymbol{\delta}} \Gamma_j(x) = \delta\theta\, D_j \bar{C}_{\alpha}(x) \quad , \qquad \bar{\boldsymbol{\delta}} \bar{B}_{\alpha}(x) = \mathbf{0} \ , \tag{7.23a}$$

$$\bar{\boldsymbol{\delta}} \bar{C}_{\alpha}(x) = -\tfrac{1}{2} \delta\theta \{\bar{C}_{\alpha}(x), \bar{C}_{\alpha}(x)\} \ , \qquad \boldsymbol{\delta} C_{\alpha}(x) = \delta\theta B_{\alpha}(x) \ . \tag{7.23b}$$

Hence, their outcome is formally analogous to the BRST and anti-BRST symmetries of ghost and antighost fields that were described in Sec. 7.8, but were not based there on a superfield formalism.[22]

The parallel transport to which the quantum gravitational connection $\boldsymbol{\nabla}$ in (7.8) gives rise can be carried out in arbitrary directions in a quantum spacetime segment $\boldsymbol{S}(g^{\mathrm{M}})$. In

[22] On the other hand, as can be seen from Sec. 3 in (Baulieu, 1984), the introduction of "a pair of Grassmann coordinates" has decided advantages even in conventional QFT.

particular, when it is carried out in the directions marked by the Grassmann coordinates of points within the segment, then the covariant derivatives in those directions are provided by the covariant BRST and anti-BRST connection forms in (7.19), which in abbreviated notation can be expressed as follows:

$$s = \delta + C \ , \qquad C = C(x,\theta,\bar{\theta})d\bar{\theta} = C_\alpha(x,\theta,\bar{\theta})d\bar{\theta}^\alpha \ . \tag{7.24a}$$

$$\bar{s} = \bar{\delta} + \bar{C} \ , \qquad \bar{C} = \bar{C}(x,\theta,\bar{\theta})d\theta = \bar{C}_\alpha(x,\theta,\bar{\theta})d\theta^\alpha \tag{7.24b}$$

This notation is meant to underline the fact that the above operators are the geometric counterparts of the nilpotent BRST and anti-BRST charge operators originally employed by Curci and Ferrari (1976c), Kugo and Ojima (1978, 1979), and others, to define physical states in conventional QFT for gauge fields as being those states which belong to the kernel of such operators (Henneaux, 1985, 1986). As we shall see in Sec. 8.9, this same role will be played by the operators in (7.24) during the course of quantum-geometric evolution.

8.8. Superlocal Quantum Gravitational and Matter Fields

In view of the considerations in the preceding two sections, the specification of a quantum gravitational gauge within a QGR gauge orbit $(\mathbf{S},\mathcal{g}^{\mathrm{M}})$ entails a choice of a physical moving frame $\boldsymbol{s} \in \Sigma(P^*\mathbf{S}^\dagger(\mathbf{g}^{\mathrm{M}}))$ as well as of a generalized internal Lorenz gauge above each base-location $x \in \mathbf{S}$. In the case of a choice of the proper Lorenz gauge at all base locations, the quantum gravitational coframes in (6.23) can be expressed in the form

$$\boldsymbol{\Phi}_f^{\boldsymbol{s}\dagger}(X) = \boldsymbol{\Phi}_f^\dagger(x) \otimes \boldsymbol{\Psi}_{0;\boldsymbol{s}(x)}^\dagger \cdot \mathbf{U}_{\boldsymbol{s}}(X) \ , \qquad X = (x,\theta,\bar{\theta}) \in \boldsymbol{S}(\mathcal{g}^{\mathrm{M}}) \ , \qquad \boldsymbol{f} \in \mathbf{Z}_{\boldsymbol{s}(x)}^{\mathrm{TT}} \ , \tag{8.1}$$

where, in accordance with (6.26), (6.29) and (7.13),

$$\mathbf{U}_{\boldsymbol{s}}(X) = \exp\left(\theta\bar{C}(x) + \bar{\theta}C(x) + \tfrac{1}{2}\theta\bar{\theta}C(x)\times\bar{C}(x)\right) \tag{8.2}$$

Hence, in order to take into account the Grassmannian degrees of freedom marked by the ghost and antighost states used to construct these quantum gravitational coframes, we have extrapolate the quantum gravitational coframe field

$$\boldsymbol{g}(x;\zeta) = g_{ij}(\boldsymbol{s}(x);\zeta)\boldsymbol{\theta}^i(x) \otimes \boldsymbol{\theta}^j(x) \ , \qquad g_{ij} = g_{ij}^{(+)} + g_{ij}^{(-)} \ , \qquad x \in \mathbf{S} \ , \tag{8.3}$$

into the quantum gravitational superframe field resulting from (5.6) and (6.27):

$$g(X;\boldsymbol{f}) = g^{(+)}(X;\boldsymbol{f}) + g^{(-)}(X;\boldsymbol{f}) \ , \qquad X \in \mathbf{S} \ , \quad \boldsymbol{f} \in \mathbf{Z}_{\boldsymbol{s}(x)}^{\mathrm{TT}} \ , \tag{8.4a}$$

$$g^{(+)}(X;\boldsymbol{f}) = g^{(+)}(x;\boldsymbol{f}) + \theta\bar{C}^{(+)}(x) + \bar{\theta}C^{(+)}(x) \ , \tag{8.4b}$$

$$g^{(-)}(X;\boldsymbol{f}) = g^{(-)}(x;\boldsymbol{f}) + \theta\bar{C}^{(-)}(x) + \bar{\theta}C^{(-)}(x) \ . \tag{8.4c}$$

According to (5.7), (6.24) and (8.1), in that case the supercoframes in (8.1) become the coherent states generated by the action from the right of the annihilation operators in this field upon the duals of the local vacuum states at each $x \in \mathbf{S}$,

$$\Phi_f^{s\dagger}(X) = \Psi_{0;s(x)}^{\dagger} \cdot \exp\left[-\left(\tfrac{1}{2}\langle f|f\rangle + \sum_{\alpha=1}^{\infty} \bar{\theta}^{\alpha}\theta^{\alpha}\right) + g^{(-)}(X;f)\right]; \tag{8.5a}$$

whereas, the corresponding superframes are obtained by similar action from the left of its creation operators:[23]

$$\Phi_f^{s}(X) = \exp\left[-\left(\tfrac{1}{2}\langle f|f\rangle + \sum_{\alpha=1}^{\infty} \bar{\theta}^{\alpha}\theta^{\alpha}\right) + g^{(+)}(X;f)\right] \Psi_{0;s(x)} . \tag{8.5b}$$

In order to define a superlocal quantum gravitational field, we follow the pattern established in Secs. 5.5, 6.5 and 7.5. Thus, we first introduce the gravitational exciton-mode fibres

$$\hat{\mathbf{Z}}_{\boldsymbol{u}}^{\mathrm{TT}} = \left\{ \hat{f} \,\middle|\, f \in \mathbf{Z}_{\boldsymbol{u}}^{\mathrm{TT}} \right\} , \qquad \hat{f}(v) := f(\boldsymbol{u}; -a(x) - i\ell v) , \qquad v \in \boldsymbol{V}^{+}, \tag{8.6}$$

in which, in accordance with (7.5.22), the inner product is formally given by

$$\left\langle \hat{f} \middle| \hat{f}' \right\rangle = Z_{\ell,0} \int_{V^+} \eta^{ii'} \eta^{jj'} \hat{f}_{ij}^{*}(v)\, \hat{f}'_{i'j'}(v)\, d\Omega(v) = \langle f|f'\rangle . \tag{8.7}$$

Since the above renormalization constant $Z_{\ell,0}$ is infinite, this inner product it is actually determined by a zero-mass limit – as in the case of the corresponding massless gauge exciton states in Chapter 7. The elements of the resulting quantum gravitational field-mode fibres,

$$\hat{\mathcal{Z}}_{\boldsymbol{u}}^{\mathrm{TT}} = \bigoplus_{n=0}^{\infty} \hat{\mathcal{Z}}_{\boldsymbol{u};n}^{\mathrm{TT}} , \qquad \hat{\mathcal{Z}}_{\boldsymbol{u};n}^{\mathrm{TT}} = \hat{\mathbf{Z}}_{\boldsymbol{u}}^{\mathrm{TT}} \underset{S}{\otimes} \cdots \underset{S}{\otimes} \hat{\mathbf{Z}}_{\boldsymbol{u}}^{\mathrm{TT}} , \tag{8.8}$$

can be similarly identified with those of the corresponding TT-subfibres of $\mathcal{Z}_{\boldsymbol{u}}$, so that

$$\hat{\Psi}_n(v_1,\ldots,v_n) = \Psi_n(\boldsymbol{u}; -a(x) - i\ell v_1, \ldots, -a(x) - i\ell v_n) , \qquad \Psi_n \in \hat{\mathcal{Z}}_{\boldsymbol{u};n}^{\mathrm{TT}} . \tag{8.9}$$

Hence, as a counterpart of (5.11), we obtain the following resolution of the identity in the quantum gravitational field-mode fibre for $\boldsymbol{u} = \boldsymbol{s}(x)$ in (8.8):

$$\int_{\mathbf{Z}_{s(x)}^{\mathrm{TT}}} \left|\hat{\Phi}_f\right\rangle \, df\, d\bar{f} \left\langle \hat{\Phi}_f \right| = \hat{\mathbf{1}}_{s(x)}^{\mathrm{TT}} . \tag{8.10}$$

Completely analogous constructions of ghost and antighost field-mode fibres can be carried out. This results in the quantum gravitational field-mode superfibres

$$\hat{\mathcal{Z}}_{s(x)}^{\Diamond} = \hat{\mathcal{Z}}_{s(x)}^{\mathrm{TT}} \otimes \hat{\bar{\mathcal{B}}}_{s(x)} \otimes \hat{\mathcal{B}}_{s(x)} , \tag{8.11}$$

whose state vectors are identifiable with those of the respective proper Lorenz gauge subfibres within the quantum gravitational superfibres in (6.22). Hence, in accordance with (8.10), an extension to these superfibres of state vectors from quantum gravitational field-mode fibres in (8.8) can be achieved by setting

[23] It is at this stage that the present geometric quantum gravity approach differs radically from the conventional approaches to extended BRST symmetry in quantum gravity: in the present approach, it is the *quantum* frames that are modified into quantum superframes; whereas, in the conventional approach, it is the *classical* frames within the vierbein fields that are modified by allowing those fields to become superfields – cf., e.g., (Townsend and van Nieuwenhuizen, 1977, p. 304), (Pasti and Tonin, 1982, p. 101), (Baulieu and Thierry-Mieg, 1984, p. 54).

$$\hat{\Psi}(X) = \hat{\Psi}^f(X)\,\hat{\Phi}^s_f(X) := \int_{Z^{TT}_{s(x)}} df\,d\bar{f}\;\hat{\Psi}^f(X)\,\hat{\Phi}^s_f(X)\;, \tag{8.12a}$$

$$\hat{\Psi}^f(X) = \hat{\Phi}^{s\dagger}_f(X) := \left\langle \hat{\Phi}^s_f(X) \middle| \hat{\Psi} \right\rangle\;, \qquad \hat{\Psi} \in \hat{Z}^{TT}_{s(x)} \tag{8.12b}$$

It should be noted, however, that for any pair of quantum gravitational state vectors that already incorporate ghost modes, so that

$$\hat{\Psi}, \hat{\Psi}' \in \hat{Z}^{TT}_{s(x)} \otimes \hat{\mathcal{F}}_{s(x)}\;, \qquad \hat{\bar{\Psi}} \otimes \hat{\Psi}' \in \hat{Z}^{TT\dagger}_{s(x)} \otimes \hat{\bar{\mathcal{F}}}_{s(x)} \otimes \hat{Z}^{TT}_{s(x)} \otimes \hat{\mathcal{F}}_{s(x)}\;, \tag{8.13}$$

we can derive as in (6.3.19) that

$$\left\langle \hat{\Psi} \middle| \hat{\Psi}' \right\rangle = \int_{Z^{TT}_{s(x)}} df\,d\bar{f} \int d\bar{\theta}\,d\theta \left\langle \hat{\Phi}^s_f(x,\theta,\bar{\theta}) \middle| \hat{\bar{\Psi}} \otimes \hat{\Psi}' \right\rangle\;, \tag{8.14}$$

where the Berezin "integration" is carried out over all the values assumed by θ^α and $\bar{\theta}^\alpha$ in any given Poincaré gauge given by a cross-section $\boldsymbol{s}$ of the Poincaré coframe bundle $P^*\mathbf{S}^\uparrow(\mathbf{g}^M)$. In view of the form of the changes in (6.19) in the values of these Grassmann variables under Lorentz boosts, such changes do not affect the outcome of that "integration" since, according to (6.3.18), the product of the two determinants resulting from them is equal to one.

For each given choice of a physical moving frame $\boldsymbol{s} = \{(\boldsymbol{a}^*(x), \theta^i(x)) |\, x \in \mathbf{S}\}$, let us define the *quantum gravitational superfield* at each base superlocation X as follows:[24]

$$\mathbf{g}^Q(X,v) = g^Q_{ij}(X,v)\,\boldsymbol{\theta}^i(x) \otimes \boldsymbol{\theta}^j(x)\;, \qquad X = (x,\theta,\bar{\theta}) \in \boldsymbol{S}(\mathcal{G}^M)\;, \tag{8.15a}$$

$$g^Q_{ij}(X,v) = \int_{Z^{TT}_{s(x)}} df\,d\bar{f} \int_{Z^{TT}_{s(x)}} df'\,d\bar{f}' \left| \hat{\Phi}^s_f(X) \right\rangle g^Q_{ijff'}(x,v) \left\langle \hat{\Phi}^s_{f'}(X) \right|\;, \tag{8.15b}$$

$$g^Q_{ijff'}(x,v) = \left\langle \hat{\Phi}_f \middle| \left(\eta_{ij} + g_{ij}(\boldsymbol{s}(x); -i\ell v)\right) \hat{\Phi}_{f'} \right\rangle\;, \qquad \hat{\Phi}_f, \hat{\Phi}_{f'} \in \hat{Z}^{TT}_{s(x)}\;. \tag{8.15c}$$

In view of (5.8) and (8.1), we have

$$g^{(-)}_{ij}(\boldsymbol{s}(x); -i\ell v)\,\hat{\Phi}_f = \hat{f}_{ij}(v)\,\hat{\Phi}_f\;, \qquad \hat{f} \in \hat{\mathbf{Z}}^{TT}_{s(x)}\;, \tag{8.16a}$$

$$\hat{\Phi}^\dagger_f \cdot g^{(+)}_{ij}(\boldsymbol{s}(x); -i\ell v) = \hat{f}^*_{ij}(v)\,\hat{\Phi}^\dagger_f\;, \qquad \hat{f} \in \hat{\mathbf{Z}}^{TT}_{s(x)}\;. \tag{8.16b}$$

Consequently, we obtain in general that

$$g^Q_{ijff'}(x,v) = \left(\eta_{ij} + \hat{f}^*_{ij}(v) + \hat{f}'_{ij}(v)\right) \left\langle \hat{\Phi}_f \middle| \hat{\Phi}_{f'} \right\rangle\;, \tag{8.17}$$

where, according to the type of relationship in (5.4.23) (which holds for Glauber coherent

[24] The subsequent relations (8.15) demonstrate very clearly the distinction in the treatment of BRST symmetries in geometric quantum gravity and in covariant quantum gravity (Delbourgo *et al.*, 1976, 1982; Kugo and Ojima, 1978; Falck and Hisshfeld, 1984). Thus, as seen from (8.15a) and (8.15b), in the quantum-geometric gravitational field it is not the elements of the vierbein (namely, the classical frame and coframe elements) that are affected as one moves from the original base-location x to a base-superlocation X; rather, during the transition from a quantum gravitational field to a superfield, it is the elements of the original quantum gravitational frame and coframe in (5.11) that are extended into elements of the quantum gravitational superframe and supercoframe in (8.5).

states in general) we have

$$\left\langle \hat{\boldsymbol{\Phi}}_{\hat{\boldsymbol{f}}} \middle| \hat{\boldsymbol{\Phi}}_{\hat{\boldsymbol{f}}'} \right\rangle = \exp\left[-\tfrac{1}{2}\left(\left\| \hat{\boldsymbol{f}} \right\|^2 + \left\| \hat{\boldsymbol{f}}' \right\|^2 \right) + \left\langle \hat{\boldsymbol{f}} \middle| \hat{\boldsymbol{f}}' \right\rangle \right] , \qquad \hat{\boldsymbol{f}}, \hat{\boldsymbol{f}}' \in \hat{\mathbf{Z}}^{\mathrm{TT}}_{\boldsymbol{g}(x)} . \tag{8.18}$$

Thus, we see that the quantum gravitational fluctuations around the mean quantum gravitational field values

$$\mathbf{g}^{\mathrm{M}}(x) = \eta_{ij}\, \boldsymbol{\theta}^i(x) \otimes \boldsymbol{\theta}^j(x) , \qquad x \in \mathbf{S} , \tag{8.19}$$

are described by a *quantum gravitational radiation superfield* with the components

$$g_{ij}(X,\upsilon) = \mathbf{U}^{\dagger}_{\boldsymbol{g}}(X)\, g_{ij}(x,\upsilon)\, \mathbf{U}_{\boldsymbol{g}}(X) , \qquad X = (x,\theta,\bar{\theta}) \in \boldsymbol{S}(\boldsymbol{g}^{\mathrm{M}}) , \tag{8.20a}$$

$$g_{ij}(x,\upsilon) = \int_{\mathbf{Z}^{\mathrm{TT}}_{\boldsymbol{g}(x)}} \boldsymbol{df}\, \boldsymbol{d\bar{f}} \int_{\mathbf{Z}^{\mathrm{TT}}_{\boldsymbol{g}(x)}} \boldsymbol{df}'\, \boldsymbol{d\bar{f}}' \left| \hat{\boldsymbol{\Phi}}_{\hat{\boldsymbol{f}}} \right\rangle \left(\hat{f}^{*}_{ij}(\upsilon) + \hat{f}'_{ij}(\upsilon) \right) \left\langle \hat{\boldsymbol{\Phi}}_{\hat{\boldsymbol{f}}} \middle| \, \hat{\boldsymbol{\Phi}}_{\hat{\boldsymbol{f}}'} \right\rangle \left\langle \hat{\boldsymbol{\Phi}}_{\hat{\boldsymbol{f}}'} \right| . \tag{8.20b}$$

The extension of these results to arbitrary generalized Lorenz gauges is straightforward, and does not basically affect any of the above conclusions. Due to the fact that the supernumbers multiplying the gauge-fixing field in (6.27) are even, that field contributes a factor which, after the integrations in the Grassmann variables in (8.14) are carried out, yields only unity.

The above derived relations determine the method of construction of the quantum-geometric "matter" fields in the present quantum gravitational regime, where the gauge group is the quantum gravitational supergroup introduced in Sec. 8.6. These types of quantum fields were considered in the preceding three chapters in the semi-classical regime, where the gauge group was the Poincaré group. Hence, in that context they acted on vectors in quantum field-mode bundles that were originally associated with a Lorentz coframe bundle, such as $L^{*}\mathbf{M}_{0}(\boldsymbol{g}^{\mathrm{L}})$ or $L^{*}\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$, and were subsequently identified with those in bundles that were associated with corresponding Poincaré coframe bundles, such as $P^{*}\mathbf{M}_{0}(\boldsymbol{g}^{\mathrm{L}})$ or $P^{*}\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$, respectively. These bundles were determined by $\mathbf{G}$-products $\mathbf{P} \times_{\mathbf{G}} \mathbf{F}$, where, for example, $\mathbf{P} = P^{*}\mathbf{M}^{\uparrow}(\boldsymbol{g}^{\mathrm{L}})$ and $\mathbf{G} = \mathrm{ISO}^{\uparrow}(3,1)$; whereas the gauge group $\mathrm{ISO}^{\uparrow}(3,1)$ acted on the typical fibre $\mathbf{F}$ via a representation $U(a,\Lambda)$. Thus, according to the general definition of a $\mathbf{G}$-product given in Sec. 4.1, the elements of such bundles were pairs $(\boldsymbol{u},\Psi) \in \mathbf{P} \times \mathbf{F}$ on which $(a,\Lambda) \in \mathrm{ISO}^{\uparrow}(3,1)$ acted from the right as follows:

$$(\boldsymbol{u},\Psi) \cdot (a,\Lambda) = (\boldsymbol{u} \cdot (a,\Lambda) , U^{-1}(a,\Lambda)\Psi) \in \mathbf{P} \times \mathbf{F} . \tag{8.21}$$

In view of (7.10)-(7.13), in the present quantum gravitational regime this action of a Poincaré gauge group has to be replaced by that of the quantum gravitational group. Thus, if we consider, for example, the case of the local boson state vectors of Chapter 5, then (8.21) is replaced by

$$(\boldsymbol{s}(x),f) \quad (\boldsymbol{\psi}(x),\boldsymbol{\phi}(x)) = (\boldsymbol{s}(x) \cdot (\boldsymbol{\psi}(x),\boldsymbol{\phi}(x)) , \mathbf{U}^{-1}_{\boldsymbol{s}}(\boldsymbol{\psi}(x),\boldsymbol{\phi}(x))f) , \tag{8.22}$$

after soldering maps are introduced which relate single-exciton state vectors $\mathbf{f}$ to their counterparts f in typical fibre for a given choice of $\boldsymbol{s}(x) \in P^{*}\mathbf{S}^{\uparrow}(\boldsymbol{g}^{\mathrm{M}})$. If we work, however,

directly with coherent cross-sections $\{(\boldsymbol{s}(x),\mathbf{f}(x))|\, x \in \mathbf{S}\}$ of the Klein-Gordon quantum bundle $\mathbf{E}$, and execute a quantum gravitational gauge transition involving only Lorentz boosts, then we have

$$(\boldsymbol{s}(x),\mathbf{f}(x)) \cdot (\boldsymbol{I},\boldsymbol{\phi}(x)) = (\boldsymbol{s}(x) \cdot (\boldsymbol{I},\boldsymbol{\phi}(x)),\, \mathbf{U}_{\boldsymbol{s}}^{-1}(x,\theta,\bar{\theta})\,\mathbf{f}(x)) \,. \tag{8.23}$$

Consequently, in that case the coherent field-mode sections introduced in Sec. 5.6 are "lifted" from the base-locations x to the base-superlocations X as follows:

$$\hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\varphi(x)} \mapsto \hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{\varphi(X)} \in \hat{\mathcal{F}}^{\mathrm{mat}}_{X} \,, \quad \varphi(x) \in \mathbf{F}^{\mathrm{mat}}_{x} \,, \quad \varphi(X) = \mathbf{U}_{\boldsymbol{s}}^{-1}(X)\,\varphi(x) \in \mathbf{F}^{\mathrm{mat}}_{X}; \tag{8.24}$$

whereas, in the case of gravitational gauge transitions that involve generic Lorentz gauge transformations, additional transformations corresponding to $(a(x),R(x))$ in (6.31) and (6.32) are required in (8.23) and (8.24).

Similar "lifts" of local states from base-locations x to base-superlocations X marked by quantum gravitational ghost and antighost modes can be carried out for all other "matter" fields considered in the preceding three chapters. In this manner the quantum superfield $\mathbf{g}^{\mathrm{Q}}$ takes over the role played in those chapters by the classical field $\boldsymbol{g}^{\mathrm{L}}$.

All these "lifts" can be related to the geometric "lifting" procedure described in Sec. 2.5, whereby vectors $\boldsymbol{X}(t) \in T_{x(t)}\mathbf{M}$ tangent to a smooth curve $\gamma = \{x(t)|\, a \le t \le b\}$ in the base manifold $\mathbf{M}$ of a principal bundle $\mathbf{P}(\mathbf{M},\mathbf{G})$ were lifted to vectors $\boldsymbol{X}^*(t)$ tangent to the total space $\mathbf{P}$ of that principal bundle, which belonged to the horizontal subspaces $H_{\boldsymbol{u}(t)}\mathbf{P}$, $\boldsymbol{u}(t) \in \mathbf{P}$, of a given connection defined on $\mathbf{P}(\mathbf{M},\mathbf{G})$. This meant that any horizontal lift $\gamma^* = \{\boldsymbol{u}(t)|\, a \le t \le b\}$ to $\mathbf{P}$ of the curve γ had everywhere horizontal tangents $\boldsymbol{X}^*(t) \in H_{\boldsymbol{u}}\mathbf{P}$, whose vertical components therefore were all zero. Hence, if $\boldsymbol{\omega}$ was the corresponding connection form, then $\boldsymbol{\omega}(\boldsymbol{X}^*(t)) = \mathbf{0}$ for all $t \in [a,b]$.

According to (7.9), (7.16), (7.19) and (7.24), the quantum gravitational connection in (7.8) can be expressed in the following form:

$$\boldsymbol{V} = \boldsymbol{d} + \boldsymbol{\Gamma} + s + \bar{s} \,, \quad s = \boldsymbol{\delta} + \boldsymbol{C} \,, \quad \bar{s} = \bar{\boldsymbol{\delta}} + \bar{\boldsymbol{C}} \tag{8.25}$$

As seen from (7.19), the above BRST and anti-BRST connection forms govern the motion along the Grassmannian degrees of freedom of a quantum spacetime segment $\boldsymbol{S}(\mathcal{g}^{\mathrm{M}})$. On the other hand, according to (7.2), those degrees of freedom correspond to vertical directions within the metrized coframe bundle $P^*\mathbf{S}^{\uparrow}(\mathcal{g}^{\mathrm{M}})$. Consequently, for each given smooth curve $\gamma = \{x(t)|\, a \le t \le b\}$ in the base-segment $\mathbf{S}$, and given ghost and antighost exciton modes above its initial point $x(a)$, the horizontal lift $\gamma^* = \{\boldsymbol{u}(t) \mid a \le t \le b\}$ of γ to a Poincaré coframe subbundle $P^*\mathbf{S}^{\uparrow}(\mathbf{g}^{\mathrm{M}})$ of $P^*\mathbf{S}^{\uparrow}(\mathcal{g}^{\mathrm{M}})$ can be mapped into a corresponding smooth curve $\gamma^{\uparrow} = \{X(t) \mid a \le t \le b\}$ within $\boldsymbol{S}(\mathcal{g}^{\mathrm{M}})$ with tangents

$$\boldsymbol{Y} = \dot{x}^{\mu}\partial_{\mu} + \dot{\bar{\theta}}\partial_{\bar{\theta}} + \dot{\theta}\partial_{\theta} \in T_X\,\boldsymbol{S}(\mathcal{g}^{\mathrm{M}}) \,, \quad X = (x,\theta,\bar{\theta}) \in \gamma^{\uparrow} \,, \tag{8.26}$$

related to the tangents $\boldsymbol{X}^* \in H_{\boldsymbol{u}}\mathbf{P}$ of γ^*. We shall therefore require that quantum gravitational *physical* parallel transport within a base supermanifold should take place only along curves for which the following subsidiary conditions are satisfied:

$$s_{\mathbf{Y}(t)}\,\hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{f(t)} = C_\alpha(X(t))\,\dot{\bar{\theta}}^{\alpha}(t)\,\hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{f(t)} = \mathbf{0}\ , \tag{8.27a}$$

$$\bar{s}_{\mathbf{Y}(t)}\,\hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{f(t)} = \overline{C}_\alpha(X(t))\,\dot{\theta}^{\alpha}(t)\,\hat{\boldsymbol{\Phi}}^{\boldsymbol{s}}_{f(t)} = \mathbf{0} \tag{8.27b}$$

This type of subsidiary conditions is encountered in the conventional quantum field theory of gauge fields in the form of the Kugo-Ojima (1979) conditions, which stipulate that all the *physical* states of such fields must belong to the kernels of BRST and anti-BRST operators. The geometric form of (8.27) renders their physical meaning very transparent: for any fixed choice of gauge for the diffeomorphism group, reflected in a choice of mean metric $\mathbf{g}^{\mathrm{M}}$ from the equivalence class $\mathcal{g}^{\mathrm{M}}$ of diffeomorphically equivalent mean metrics, and for any fixed Poincaré gauge, corresponding to some choice $\boldsymbol{s}$ of cross-section of the Poincaré coframe bundle $P^{*}\mathbf{S}^{\dagger}(\mathbf{g}^{\mathrm{M}})$, all *physical* parallel transport underlying quantum-geometric propagation in quantum spacetime has to take place along paths $\gamma^{\dagger}$ which are the horizontal lifts of paths γ in base-segments $\mathbf{S}$.

8.9. Quantum-Geometric Evolution of Gravitational and Matter Fields

As we have seen in the preceding three chapters, the concept of *intrinsic* time is essential to the quantum-geometric propagation of "matter" fields – by which, as customary, we mean massive fields as well as massless fields describing nongravitational radiation. In the semi-classical regime adopted in those chapters this intrinsic time was determined by the geometrodynamic evolution of an *a priori* given external gravitational field, so that the "matter" fields simply "kept in step" with it, rather than being its primary source. In the present quantum-geometric formulation of gravitational theory, it is those "matter" fields that are the principal *cause* of the physical and intrinsic time in the Universe. Indeed, their presence along inflow supersurfaces $\boldsymbol{\Sigma}$ *creates* the "infinitesimal" gauge orbit $(\mathbf{dS}, \mathcal{g}^{\mathrm{M}})$ over each of the "infinitesimal" base-segments $\mathbf{dS}$ of a quantum spacetime, thus giving rise to the class $\mathcal{g}^{\mathrm{M}}$ of mean metrics in $\mathbf{dS}$. Hence, at the cosmological level, a quantum version of Mach's principle emerges: quantum "matter" throughout the Universe *is* the *cause* of inertia.

Historically, the idea that inertia is determined by the matter distribution in our entire Universe was advanced by Mach in the context of his phenomenalist critique of Newtonian classical mechanics (cf. Bunge, 1966; Bradley, 1971; Blackmore, 1972). This critique had its roots in a positivistic philosophy that tried to anchor all physical concepts in directly perceived "sensations." It was Einstein (1917, 1918) who first elevated this idea to the status of a principle, and who attributed to "Mach's principle" a key role in classical general relativity when he declared the following: "The idea that Mach expressed, that inertia depends upon the mutual action of bodies, is contained, to a first approximation, in the equations of the theory of relativity; it follows from these equations that inertia depends, at least in part, upon the mutual action between masses." (Einstein, 1974, pp. 107-108.)

An "up-dated" version of Mach's principle reads as follows: "Specify everywhere the distribution and flow of mass-energy and thereby determine the inertial properties of every test particle everywhere and at all times." (Misner *et al.*, 1973, p. 546.) Nevertheless, Mach's original idea can still be deemed to be contained only "to a first approximation" in

the CGR equations, since it is only the inertial motion of classical test particles that, according to the geodesic postulate, is determined by the metric of a classical spacetime manifold – which, in turn, is affected by the over-all distribution of "matter" via Einstein's equations in (2.7.1). However, test particles do not constitute an integral part of the CGR field theoretical framework, since they are influenced by the gravitational field, but do not influence it in return.[25]

On the other hand, the four constraint equations out of the total of ten Einstein equations are equivalent to the canonical subsidiary conditions in (2.7.19). Geometrically, they represent the Gauss-Codacci equations relating the extrinsic curvature field $\boldsymbol{K}$ over each Σ_t to the curvature scalar $R(\boldsymbol{\gamma})$ for the Riemannian 3-metric $\boldsymbol{\gamma}$ along that Σ_t (cf., e.g., Hawking and Ellis, 1973, p. 47; Choquet-Bruhat *et al.*, 1987, p. 315). It is these four constraint equations that contain the local proper energy density ρ and the components j_a of the proper 3-momentum $\mathbf{j}$ for all nongravitational sources with respect to the $\boldsymbol{\gamma}$-orthonormal spatial frames $\{\boldsymbol{e}_a(x) | \, a = 1,2,3\}$. In fact, these equations can be expressed in the following form (cf. Choquet-Bruhat and York, 1980, p. 137),

$$R(\boldsymbol{\gamma}) - \boldsymbol{K}\cdot\boldsymbol{K} + (\mathrm{Tr}\,\boldsymbol{K})^2 = 16\pi\rho\,, \qquad \boldsymbol{K}\cdot\boldsymbol{K} := K_{ab}K^{ab}, \quad \mathrm{Tr}\,\boldsymbol{K} := \gamma_{ab}K^{ab} = K^a{}_a\,, \tag{9.1a}$$

$$\mathbf{D}\cdot\boldsymbol{K} - \mathrm{D}\,\mathrm{Tr}\,\boldsymbol{K} = 8\pi\mathbf{j}\,, \qquad (\mathbf{D}\cdot\boldsymbol{K})_a := \mathrm{D}_b K^b{}_a\,, \quad (\mathrm{D}\,\mathrm{Tr}\,\boldsymbol{K})_a := \mathrm{D}_a K^b{}_b\,, \tag{9.1b}$$

where D_a denotes the $\boldsymbol{e}_a$-component of the operator $\mathbf{D}$ for the covariant differentiation determined by the Levi-Civita connection to which the 3-metric $\boldsymbol{\gamma}$ gives rise.

By its very definition, the CGR initial-value problem consists of constructing an initial-data 3-surface Σ_0, a Riemannian 3-metric tensor $\boldsymbol{\gamma}$, and an extrinsic curvature tensor $\boldsymbol{K}$, so that the equations in (9.1) are satisfied. However, once this is achieved, in CGR all these constraint equations are *automatically* satisfied by $\boldsymbol{\gamma}$ and $\boldsymbol{K}$ along all other reference hypersurfaces Σ_t. Thus, the earlier cited "up-dated" version of Mach's principle hinges on a *strict* determinism, which, at the cosmological level, does not allow even the minutest departures from a physically preordained evolution of the 3-space $(\Sigma_t, \boldsymbol{\gamma}_t)$ and of all the "matter" fields. This is, of course, totally at odds with basic quantum principles.

Clearly, the probabilistic nature of those principles can be reconciled with the essence of Mach's principle only if the over-all distribution of "matter" determines "inertia" at each location in spacetime only in a mean probabilistic sense, which allows stochastic fluctuations of various kinds around the *mean* inertial paths that transcend into the most likely paths to be observed at the macro-level. This is in complete accord with Feynman's path-integral conceptualization of quantum evolution.

In geometric quantum gravity, the inertial paths are geodesics *in relation* to the mean metrics $\mathbf{g}^{\mathrm{M}}$ determined by the *total mean* (local) proper energy density

$$\overline{\rho}^{\mathrm{tot}}(x) = \left\langle \Psi_X^{\mathrm{tot}} \middle| \rho^{\mathrm{tot}}(X)\Psi_X^{\mathrm{tot}} \right\rangle\,, \qquad \left\langle \Psi_X^{\mathrm{tot}} \middle| \Psi_X^{\mathrm{tot}} \right\rangle^{\mathrm{body}} = 1\,, \tag{9.2}$$

and by the *total mean* (local) proper 3-momentum with components

[25] This objection would be removed if any of the many attempts to derive that postulate from Einstein's field equations had succeeded at a rigorous level. However, as pointed out in Footnote 17 to Chapter 2, that is not actually the case.

$$\bar{\boldsymbol{j}}_a^{\text{tot}}(x) = \left\langle \boldsymbol{\Psi}_X^{\text{tot}} \middle| \boldsymbol{j}_a^{\text{tot}}(X)\boldsymbol{\Psi}_X^{\text{tot}} \right\rangle , \quad X = (x,\theta,\bar{\theta}) \in \boldsymbol{\Sigma} , \quad a = 1,2,3 , \tag{9.3}$$

in relation to classical Lorentz frames $\{\boldsymbol{e}_i(x) | \, i = 0,1,2,3\}$ at base-locations x corresponding to base-superlocation X along inflow supersurfaces $\boldsymbol{\Sigma}$ in a quantum spacetime. The special geodesic $\gamma_0(x)$ whose tangent is given at x, in proper-time parametrization, by the timelike vector $\boldsymbol{n}(x)$ that points in the *future* direction and indicates relative *mean* stochastic rest for the energy distribution around x, is such that in the Lorentz frames with $\boldsymbol{e}_0(x) = \boldsymbol{n}(x)$ the total mean 3-momentum in (9.3) vanishes. Hence, any object that propagates along $\gamma_0(x)$ is a natural "clock" that measures the *mean intrinsic time* marking the evolution of a quantum spacetime. According to (2.7.20), the components K_{ab} of extrinsic curvature tensor $\boldsymbol{K}$ then determine the rate of change with respect to intrinsic time of the mean 3-metric components γ_{ab} in such frames.

As we have seen in Secs. 5.5, 6.5 and 7.5, the proper energy-momentum operators of the quantum-geometric "matter" fields are well-defined at all base-locations of a classical spacetime. As described in the preceding section, the extension of these fields to the base-superlocations X of a quantum spacetime provides corresponding well-defined proper energy-momentum operators

$$\hat{\boldsymbol{P}}_{j;s}^{\text{mat}}(X) = n^k(x) \int_{V_x^+} :T_{jk}^{\text{mat}}[\boldsymbol{\varphi}^{\text{mat}}(X,v)]: \, d\Omega(v) , \quad j = 0,1,2,3 \tag{9.4}$$

In the above expression $\boldsymbol{\varphi}^{\text{mat}}$ is a collective notation for all the quantum-geometric "matter" fields present. The above total stress-energy tensor for all the fields, containing terms for all the interactions between them, is obtained[26] from the corresponding terms in the stress-energy tensor of the semi-classical regime by replacing in the latter all the Poincaré-covariant derivatives with the quantum gravitationally covariant derivatives

$$\begin{aligned} \boldsymbol{\nabla}_j^{\text{mat}} &= \partial_j - i\tilde{\boldsymbol{\theta}}_s^j \boldsymbol{P}_{j;s}^{\text{mat}} + \tfrac{i}{2}\tilde{\boldsymbol{\omega}}_{ab}^s \boldsymbol{M}_{\text{mat};s}^{ab} \\ &+ \theta D_j \bar{C}(x) + \bar{\theta} D_j C(x) + \theta\bar{\theta}\left(D_j B(x) + D_j C(x) \times \bar{C}(x) \right) . \end{aligned} \tag{9.5}$$

These covariant derivatives result from (7.8) when the infinitesimal generators of the quantum gravitational field are replaced in (7.7) with those for the respective matter fields. Naturally, during this transition from the classical to the quantum general relativistic regime ordering ambiguities might arise – as is the case in classical relativity theory during the transition from the special relativistic to the general relativistic regime. It is clear, however, that once a choice of ordering is made in (9.4), the resulting energy-momentum operators will determine the proper energy and 3-momentum density operators for the totality of all "matter" fields, so that in Lorentz frames with $\boldsymbol{e}_0(x) = \boldsymbol{n}(x)$

$$\boldsymbol{\rho}^{\text{mat}}(X) = \hat{\boldsymbol{P}}_{0;s}^{\text{mat}}(X) , \quad \boldsymbol{j}_a^{\text{mat}}(X) = \hat{\boldsymbol{P}}_{a;s}^{\text{mat}}(X) , \quad a = 1,2,3 . \tag{9.6}$$

To secure agreement with the predictions of CGR models at the macroscopic level, we have to assume that the expectation values of these density operators are the principal contribu-

[26] Note that in those expressions the partial derivatives with respect to the (internal) gauge q_j-variables can be replaced with the (external) Lorentz-covariant derivatives in the $\boldsymbol{e}_j$-directions.

tors to the *total* mean proper energy density and 3-momentum in (9.2) and (9.3), respectively. The question remains, however, whether the *quantum* gravitational radiation also contributes to that total.

In CGR, as a consequence of the strong equivalence principle, the energy of a classical gravitational field cannot be localized (cf., e.g., Misner *et al.*, 1973, §20.4): at any given base-location x the local gravitational energy can be always "gauged away" by making the transition to an inertial frame, so that in the normal coordinates associated with that frame (cf. Sec. 2.6) all the Christoffel symbols vanish at x. However, even in the CGR regime, "[a]t issue is not the existence of gravitational energy, but the localizability of gravitational energy" (*ibid.*, p. 467). Furthermore, the formulation of classical gravitational radiation requires the decomposition $\boldsymbol{g}^{\mathrm{L}} = \boldsymbol{g}^{\mathrm{B}} + \boldsymbol{h}$ of the Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ of a classical spacetime into a "background" metric $\boldsymbol{g}^{\mathrm{B}}$ and a radiative "correction" $\boldsymbol{h}$, so that in relation to that "background" the gravitational radiation energy is (topologically) localizable within a few wavelengths (*ibid.*, §§35.7-35.15).

The most common treatment of classical gravitational radiation occurs in the context of the linearized theory of gravity (*ibid.*, §§35.2-35.4), where that "background" metric is the Minkowski metric. However, the basic ideas of such a treatment can be easily extended to generic background metrics by working with generic Cartan moving frames $\{\boldsymbol{e}_j\}$,

$$\boldsymbol{e}_j(x) = \lambda_j{}^{\mu}(x)\partial_{\mu} \ , \qquad \partial_{\mu} = \partial/\partial x^{\mu} \ , \tag{9.7}$$

and expressing the Lorentzian metric $\boldsymbol{g}^{\mathrm{L}}$ in the corresponding coframes $\{\boldsymbol{\theta}^j\}$ as follows:

$$\boldsymbol{g}^{\mathrm{L}} = g_{ij}\,\boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j, \qquad g_{ij} = \eta_{ij} + h_{ij} \ , \qquad \boldsymbol{\theta}^j = \lambda^j{}_{\mu}\,\boldsymbol{dx}^{\mu} \tag{9.8}$$

Naturally, if a Cartan moving frame $\{\boldsymbol{e}_j\}$ is orthonormal with respect to an already given "background" metric $\boldsymbol{g}^{\mathrm{B}}$, and the field of vectors $\boldsymbol{e}_0$ is timelike in it, then the first term in the above decomposition gives rise to $\boldsymbol{g}^{\mathrm{B}}$. However, in general, on account of (2.6.5) and (2.6.6) we have

$$\Gamma^i{}_{jk} = g^{il}\Gamma_{ljk} \ , \qquad g^{ij}g_{jk} = \delta^i_k \ , \tag{9.9a}$$

$$\Gamma_{ijk} = \tfrac{1}{2}\big(h_{ij,k} + h_{ik,j} - h_{jk,i} + g_{il}\lambda_{jk}{}^{l} + g_{jl}\lambda_{ik}{}^{l} + g_{kl}\lambda_{ij}{}^{l}\big) \ , \tag{9.9b}$$

where, in accordance with (9.7) and (9.8),

$$h_{ij,k} = \lambda_k{}^{\mu}\partial_{\mu}h_{ij} \ , \qquad [\boldsymbol{e}_i, \boldsymbol{e}_j] = \lambda_{ij}{}^{k}\,\boldsymbol{e}_k \tag{9.10}$$

It then immediately follows from (2.7.5), (9.7) and (9.10) that

$$R^i{}_{jkl} = \Gamma^i{}_{lj,k} - \Gamma^i{}_{kj,l} + \Gamma^i{}_{km}\Gamma^m_{lj} - \Gamma^i{}_{lm}\Gamma^m_{kj} - \lambda_{kl}{}^{m}\Gamma^i{}_{mj} \ , \tag{9.11a}$$

$$\Gamma^i{}_{jk,l} = \lambda_l{}^{\mu}\partial_{\mu}\Gamma^i{}_{jk} \ , \qquad \lambda_{ij}{}^{k} = (\lambda_i{}^{\mu}\partial_{\mu}\lambda_j{}^{\nu} - \lambda_j{}^{\mu}\partial_{\mu}\lambda_i{}^{\nu})\lambda^k{}_{\nu} \ . \tag{9.11b}$$

This yields expressions for the components of the Einstein tensor constituting the left-hand side of the Einstein equations in (2.7.1),

$$G_{ij} = R_{ij} - \tfrac{1}{2}g_{ij}R \ , \qquad R_{ij} = R^k{}_{ikj} \ , \qquad R = g^{ik}R_{ki} \ , \tag{9.12}$$

that are valid in an arbitrary Cartan moving frame.

In the case where the background metric $\boldsymbol{g}^{\mathrm{B}}$ is equal to the Minkowski metric $\boldsymbol{\eta}$, and where the moving frame is holonomic and orthonormal in it, with $\boldsymbol{e}_0$ timelike, so that

$$\lambda_j{}^{\mu} = \delta_j{}^{\mu} \ , \quad h_{\lambda\mu,\nu} = \partial_\nu h_{\lambda\mu} \ , \quad [\boldsymbol{e}_i, \boldsymbol{e}_j] = 0 \ , \quad \lambda_{ij}{}^{k} = 0 \ , \tag{9.13}$$

then, upon neglecting all non-linear terms in the resulting expressions for the components of the Einstein tensor in that moving frame, we obtain the approximations

$$G_{\mu\nu}^{\mathrm{lin}} = \tfrac{1}{2}\big(h_{\mu\nu,\lambda}{}^{\lambda} - h_{\mu\lambda,\nu}{}^{\lambda} - h_{\nu\lambda,\mu}{}^{\lambda} + h_{,\mu\nu} + \eta_{\mu\nu}(h_{\kappa\lambda}{}^{,\kappa\lambda} - h_{,\lambda}{}^{\lambda})\big) \ , \tag{9.14}$$

used in linearized gravity. In the TT-gauge these equations assume the very simple form

$$G_{\mu\nu}^{\mathrm{TT,lin}} = \tfrac{1}{2} h_{\mu\nu,\lambda}{}^{\lambda} = \tfrac{1}{2}\eta^{\lambda\kappa}\,\partial^2 h_{\mu\nu}\big/\partial x^{\lambda}\partial x^{\kappa} \ , \tag{9.15}$$

involving only the action of the D'Alembert operator upon the ten independent components of the "correction" of the Minkowski metric used in linearized gravity.

It was observed by Deser (1970) that if one sets, by definition,

$$\sqrt{-g}\, g^{\mu\nu} := \eta^{\mu\nu} + h^{\mu\nu} \ , \qquad g = \det\|g_{\mu\nu}\| \ , \tag{9.16}$$

$$T_{\mu\nu}^{\mathrm{s.int}} := (1/16\pi)\big(\Gamma^{\lambda}{}_{\mu\nu}\Gamma^{\kappa}{}_{\kappa\lambda} - \Gamma^{\lambda}{}_{\kappa\mu}\Gamma^{\kappa}{}_{\nu\lambda}\big) \ , \tag{9.17}$$

then the well-known Einstein-Hilbert action can be formally decomposed as follows,

$$S_{\mathrm{EH}} := (1/16\pi)\int R\sqrt{-g}\, d^4x = S_{\mathrm{EH}}^{\mathrm{lin}} + \int h^{\mu\nu} T_{\mu\nu}^{\mathrm{s.int}}\, d^4x \ , \tag{9.18}$$

where the first term in the decomposition is the quadratic action leading to the field equations (9.14) of the linearized theory; whereas the second term incorporates the nonlinear terms which result when the Einstein tensor is constructed from (9.11a). Thus, the formal effect is the one obtained by transferring the nonlinear terms on the left-hand side of the Einstein field equations (9.12) to their right-hand side, so that they would assume the form

$$G_{\mu\nu}^{\mathrm{lin}} = -8\pi\big(T_{\mu\nu}^{\mathrm{s.int}} + T_{\mu\nu}^{\mathrm{mat}}\big) \tag{9.19}$$

As a consequence of this formulation, the resulting tree graph structure of covariant quantum gravity with a Minkowski background incorporates graviton-graviton interaction terms that describe quantum gravitational self-interactions. The conclusion is then reached that "*a quantum particle description of local (noncosmological) gravitational phenomena necessarily leads to a classical limit which is just a metric theory of gravity. As long as only helicity ±2 gravitons are included, the theory is precisely Einstein's general relativity*" (Boulware and Deser, 1975, p. 195).

The framework for quantum gravity presented in this chapter postulates no background metrics – and, in fact, it does not assume that a spacetime manifold of any kind is *a priori* given. However, it does take for granted that, in accordance with observational data, mean metrics constituting QGR gauge orbits perpetually emerge during the process of

quantum-geometric evolution as a result of the metrization of classical frame and coframe bundles. Hence, although the decomposition in (9.19) is mathematically and physically meaningless in the quantum-geometric gravitational regime, the idea of self-interactions of the quantum gravitational radiation field in (8.20) can be implemented by interpreting that decomposition at a geometrically superlocal level, namely as taking place in the in the superfibres *above* each superlocation X. This means that the total stress-energy tensor for the energy-momentum of quantum gravitational radiation is then given by

$$T_{ij}^{\text{rad}}[\mathbf{g}] = T_{ij}[\mathbf{g}] + T_{ij}^{\text{s.int}}[\mathbf{g}] \ , \tag{9.20}$$

where the first term is derived by supergauge transformations from (7.5),

$$T_{ij}[\mathbf{g}(X,v)] = \tfrac{1}{2}\eta_{ij}g_{kl,m}\,g^{kl,m} - g_{km,i}g^{km}{}_{,j} \ , \quad X = (x,\theta,\overline{\theta}) \in \mathbf{S}(\mathcal{G}^{\text{M}}) \ , \tag{9.21a}$$

$$g_{ik,j}(X,v) = \mathbf{U}_{\mathbf{s}}^{-1}(x,\theta,\overline{\theta})\, g_{ik,j}(\mathbf{s}(x);-i\ell v)\mathbf{U}_{\mathbf{s}}(x,\theta,\overline{\theta}) \ , \tag{9.21b}$$

and where the second term is the superlocal counterpart of (9.17):

$$T_{ij}^{\text{s.int}}[\mathbf{g}] = (1/16\pi)\left(\hat{\Gamma}^{l}{}_{ij}\hat{\Gamma}^{k}{}_{kl} - \hat{\Gamma}^{l}{}_{ki}\hat{\Gamma}^{k}{}_{jl}\right) \ , \tag{9.22a}$$

$$\hat{\Gamma}^{i}{}_{jk} = \tfrac{1}{2}\eta^{il}\left(g_{lj,k} + g_{lk,j} - g_{jk,l}\right) . \tag{9.22b}$$

With all the main ingredients of the quantum-geometric framework for gravity now available, the formulation of the quantum-geometric propagator

$$K(f(X'');f(X')) = \lim_{\varepsilon\to+0}\prod_{n=N}^{1}{}'\int \mathcal{D}f_n \left\langle \hat{\boldsymbol{\Phi}}^{\mathbf{s}}_{f_n}(X_n)\right| e^{-i\varepsilon\int d\sigma(x_n)\rho^{\text{tot}}(X_n)}\ \hat{\boldsymbol{\Phi}}^{\mathbf{s}}(X_n;f_{n-1})\Big\rangle \tag{9.23}$$

for gravitational and "matter" superfields, whose exciton modes are collectively denoted by f, can be obtained by suitable adaptation of the type of iterative procedures considered in Secs. 5.6, 6.6 and 7.5. To arrive at such an adaptation, we note that the functional "measure" element in (9.23) results from the following natural generalization of (5.6.3):

$$\mathcal{D}f_n = \prod_{x_n\in\Sigma_{t_n}} \mathcal{D}[\mathrm{g}_n(x_n)]\,\mathcal{D}[\overline{c}_n(x_n)]\,\mathcal{D}[c_n(x_n)]\mathcal{D}[\varphi_n^{\text{mat}}(X_n)] \ . \tag{9.24}$$

The Grassmannian values assumed by the ghost and antighost exciton modes above each base location $x_n\in\Sigma_{t_n}$ are those corresponding to various superlocations $X_n\in\boldsymbol{\Sigma}_{t_n}$.

In the present quantum-gravitational regime, the base-segments $\mathbf{S}_n$, $n = 1,\ldots,N$, are no longer *a priori* given; rather, they are *generated* by the inflow physical data for each particular segment. Thus, the energy-momentum operators for all "matter" fields, together with the the energy-momentum operators

$$\hat{\boldsymbol{P}}^{\text{rad}}_{j;\mathbf{s}}(X) = n^k(x)\int_{V_x^+} :T^{\text{rad}}_{jk}[\mathbf{g}(X,\boldsymbol{v})]:\ d\Omega(\boldsymbol{v}) \ , \quad j = 0,1,2,3 \ , \tag{9.25}$$

of quantum gravitational radiation resulting from (9.20), give rise to the total proper energy density operator in (9.23), as well as to the corresponding total proper 3-momentum den-

sity operators. The mean values in (9.2) and (9.3) of these operator along each inflow hypersurface $\Sigma_{t_{n-1}}$ then determine via (9.1) the QGR gauge orbit $(\mathbf{S}_n, g^{\mathrm{M}})$, including the outflow hypersurface Σ_{t_n} at the proper time separation $t_n - t_{n-1} = (t'' - t')/N = \varepsilon$. Consequently, the quantum spacetime segment $\mathbf{S}_n(g^{\mathrm{M}})$ is thereby determined via (7.1) and (7.2).

The quantum-geometric propagation proceeds along those paths $\gamma^{\uparrow}(X_{n-1}, X_n)$ within the base-supersegments $\mathbf{S}_n(g^{\mathrm{M}})$ which, as described in the preceding section, are the lifts of the geodesic arcs $\gamma(x_{n-1}, x_n)$ connecting each point x_{n-1} on the inflow hypersurface $\Sigma_{t_{n-1}}$ with the points x_n on the outflow hypersurface Σ_{t_n} of each base-segment $\mathbf{S}_n$. The *physical* parallel transport in (9.23),

$$\boldsymbol{\Phi}^{\mathbf{s}}(X_n; f_{n-1}) = \mathcal{T}_{\gamma^{\uparrow}}(X_n, X_{n-1})\, \boldsymbol{\Phi}^{\mathbf{s}}_{f_{n-1}}(X_{n-1}) \ , \qquad \gamma^{\uparrow} = \gamma^{\uparrow}(X_{n-1}, X_n) \ , \tag{9.26}$$

is then performed under the type of subsidiary conditions in (8.27), so that

$$\nabla_{\mathbf{Y}} \boldsymbol{\Phi}^{\mathbf{s}} = \dot{x}^{\mu}\Big(\partial_{\mu} + \boldsymbol{\Gamma}_{\mu}^{\mathrm{TT};\mathbf{s}} + \theta D_{\mu}\overline{C} + \overline{\theta} D_{\mu} C + \theta\overline{\theta}(D_{\mu}B + D_{\mu}C \times \overline{C})\Big)\boldsymbol{\Phi}^{\mathbf{s}} = \mathbf{0} \tag{9.27}$$

is satisfied along each given smooth superpath segment $\gamma^{\uparrow}(X_{n-1}, X_n)$.

The quantum gravitational propagator for the parallel transport along $\gamma^{\uparrow}(X_{n-1}, X_n)$,

$$K_{\gamma^{\uparrow}}(\boldsymbol{f}_n, X_n; \boldsymbol{f}_{n-1}, X_{n-1}) = \Big\langle \boldsymbol{\Phi}^{\mathbf{s}}_{f_n}(X_n) \Big| \mathcal{T}_{\gamma^{\uparrow}}(X_n, X_{n-1})\, \boldsymbol{\Phi}^{\mathbf{s}}_{f_{n-1}}(X_{n-1}) \Big\rangle \ , \tag{9.28}$$

can be expressed as follows for small separations in the superlocations X_{n-1} and X_n:

$$\begin{aligned} K_{\gamma^{\uparrow}}(\boldsymbol{f}_n, X_n; \boldsymbol{f}_{n-1}, X_{n-1}) = \Big\{ & 1 - i\Big(\delta x_n^j P_j - \tilde{\boldsymbol{\omega}}_{ab}(\boldsymbol{\delta} x_n) M^{ab}\Big)(f(x_n), f(x_{n-1})) \\ & + \delta x_n^{\mu}\Big(\overline{\theta}_n D_{\mu} C + \theta_n D_{\mu}\overline{C} + \theta_n\overline{\theta}_n (D_{\mu}B + D_{\mu}C \times \overline{C})\Big)(X_n, X_{n-1})\Big\} \\ & \times \Big\langle \boldsymbol{\Phi}^{\mathbf{s}}_{f_n}(X_n) \Big| \boldsymbol{\Phi}^{\mathbf{s}}_{f_{n-1}}(X_{n-1}) \Big\rangle + O\Big((\delta\tau_n)^2\Big) \ . \end{aligned} \tag{9.29}$$

The expressions on the first line of the above equation can be computed by an adaptation of the type of method applied to the propagators for the parallel transport in Sec. 5.4. Thus, according to (5.8), (7.5) and (8.5), we have within the typical TT-fibre $\mathcal{Z}^{\mathrm{TT}}$ that

$$P_j(f(x_n), f(x_{n-1})) = \int T_{jk}[\bar{f}(x_n; \zeta) + f(x_{n-1}; \zeta)]\, d\hat{\sigma}^k(q) d\Omega(v) \ , \tag{9.30}$$

and a similar expression is obtained from (7.6) for last term in the first line of the equation in (9.29). The expressions on the second line of the same equation result from the ghost and antighost states created by the parallel transport from X_{n-1} to X_n, and can be computed by using the fact that the superframe and supercoframe elements in (8.5) are eigenstates of the respective ghost and antighost creation and annihilation operators, so that, for example,

$$-i\int\Big(c_{\alpha}^{\dagger}(x_n; \zeta)\tilde{\partial}_k \cdot D_{\mu} c_{\alpha}(x_{n-1}; \zeta) + D_{\mu} c_{\alpha}^{\dagger}(x_n; \zeta)\tilde{\partial}_k \cdot c_{\alpha}(x_{n-1}; \zeta)\Big)\, d\hat{\sigma}^k(q) d\Omega(v) \ , \tag{9.31}$$

determine its first term on account of (5.21)-(5.24) and (6.24)-(6.29). The inner product in

the third line of the equation in (9.29) is taken between the superframe representatives

$$\Phi^{s}_{f_n}(X_n) = \sigma^{s}_{X_n}\Phi^{s}_{f_n}(X_n) \quad , \qquad \Phi^{s}_{f_{n-1}}(X_{n-1}) = \sigma^{s}_{X_n}\Phi^{s}_{f_{n-1}}(X_{n-1}) \, , \tag{9.32}$$

within the typical superfibre of the quantum gravitational superfibre bundle over the considered base-supersegment.

The affine parameter τ in (9.29) can be correlated to the local parameter t for the stochastic relative rest of massive quantum frames in a coherent flow undergoing free fall in the course of quantum-geometric evolution only for the timelike geodesics representing physical paths in the semi-classical regime. Consequently, there will be many quantum-geometrically causal paths for which $\delta\tau_n$ does not tend to zero as $\delta t_n = t_n - t_{n-1} \to +0$. For them the counterpart of (5.4.5), which follows from the reproducing properties of the quantum gravitational propagator, has to be applied in order to compute that propagator by means of (9.29).

In this manner, the iterative procedure for the computation of the quantum-geometric propagator in (9.23) is unambiguously determined at the operator level, once the initial data for all "matter" fields and for the quantum gravitational superfield are provided.

In the semi-classical regime the derivation of an action integral for free-fall quantum-geometric propagators relied on the fact that, due to the strong equivalence principle, all single-exciton states were mapped again into single-exciton states during such evolution. In the presence of the subsidiary conditions in (9.27), however, the ghost-antighost degrees of freedom become intertwined with the physical degrees of freedom. Hence, in deriving an action integral for the quantum-geometric gravitational propagator, the method of Secs. 5.6 and 7.5 has to take this fact into account.

That derivation can be implemented by noting that, in view of (5.8), (8.5) and (8.20),

$$\mathbf{g}^{(-)}_{,j}(X,v)\hat{\mathbf{\Phi}}^{s}_{\mathbf{g}}(X) = \hat{\mathrm{g}}_{,j}(x,v)\hat{\mathbf{\Phi}}^{s}_{\mathbf{g}}(X) \tag{9.33}$$

for gravitational superframes. Thus, if we limit ourselves to the parallel transport governed by the TT-component (7.6) of the quantum gravitational connection in (7.7), so that

$$\mathrm{g}_{n-1}(x_n) := \tau^{\mathrm{TT}}_{\gamma}(x_n, x_{n-1})\mathrm{g}_{n-1}(x_{n-1}) \in \mathbf{Z}^{\mathrm{TT}}_{x_n} \, , \qquad \mathrm{g}_{n-1}(x_{n-1}) \in \mathbf{Z}^{\mathrm{TT}}_{x_{n-1}} \, , \tag{9.34}$$

then, upon computing the inner products in (9.23) by means of (9.25), we get

$$\begin{aligned} &\left\langle \hat{\mathbf{\Phi}}^{s}_{\mathbf{g}_n}(X_n) \right| : T^{\mathrm{rad}}_{ij}[\mathbf{g}(X_n,v)] : \hat{\mathbf{\Phi}}^{s}_{\mathbf{g}_{n-1}}(X_n) \right\rangle \\ &= T^{\mathrm{rad}}_{ij}[\hat{\bar{\mathrm{g}}}_n(x_n,v) + \hat{\mathrm{g}}_{n-1}(x_n,v)] \left\langle \hat{\mathbf{\Phi}}^{s}_{\mathbf{g}_n}(X_n) \right| \hat{\mathbf{\Phi}}^{s}_{\mathbf{g}_{n-1}}(X_n) \right\rangle . \end{aligned} \tag{9.35}$$

Furthermore, since the quantum gravitational superframe elements are tensor products of Glauber-type and Berezin-type coherent states, we have, as was the case in Sec. 5.6, that

$$\left\langle \hat{\mathbf{\Phi}}^{s}_{\mathbf{g}_n}(X_n) \right| \hat{\mathbf{\Phi}}^{s}_{\mathbf{g}_{n-1}}(X_n) \right\rangle = \exp\left[i\varepsilon \tfrac{i}{2} \left(\left\langle \hat{\mathrm{g}}_n(x_n) \right| \dot{\mathrm{g}}_n(x_n) \right\rangle - \left\langle \dot{\mathrm{g}}_n(x_n) \right| \hat{\mathrm{g}}_{n-1}(x_n) \right\rangle \right] . \tag{9.36}$$

However, on account of (9.26) and (9.27), ghost and antighost are in general created in the course of the physical parallel transport which underlies the quantum-geometric evolution

of the gravitational field, so that

$$\boldsymbol{\Phi}^{s}_{g_{n-1}}(X_n) \neq \boldsymbol{\Phi}^{s}(X_n; g_{n-1}) = \mathcal{T}_{\gamma\uparrow}(X_n, X_{n-1})\, \boldsymbol{\Phi}^{s}_{g_{n-1}}(X_{n-1}) \,, \tag{9.37}$$

since for infinitesimal separations of the corresponding base-segment points we have

$$\boldsymbol{\Phi}^{s}(X_n; g_{n-1}) = e^{\delta x^{\mu}\left(\theta_n D_{\mu}\bar{C} + \bar{\theta}_n D_{\mu} C + \theta_n \bar{\theta}_n (D_{\mu} B + D_{\mu} C \times \bar{C})\right)} \boldsymbol{\Phi}^{s}_{g_{n-1}}(X_n) \,. \tag{9.38}$$

Consequently, the quantum gravitational Lagrangian density derived along the same general lines as in (5.6.13) – (5.6.19) will contain, in addition to the physical contribution resulting from (9.35) and (9.36), also purely quantum gravitational supergauge contributions that result from (9.31) and (9.38).

By using BRST and anti-BRST symmetry relations, such as those in (7.23), these supergauge contribution to the quantum gravitational Lagrangian density can be cast in a variety of forms. However, from the path-integration point of view, all these forms are equivalent, since they unambiguously follow from the geometry of quantum gravitational evolution.[27] This is due to the fact that the *physical* component of quantum-geometric gravitational Lagrangian density is unambiguously determined by the *physical* stress-energy tensor in (9.20), so that all the Grassmann variables that appear in geometric quantum gravity are purely mathematical devices for securing the manifest covariance of the description of quantum gravitational evolution in various Poincaré moving frames.

In this respect, the use of Grassmann variables and supermanifolds in QGR is no different from the use of indefinite metrics in the Gupta-Bleuler formalism: it secures in a mathematically elegant manner the covariance of the formalism, but it does not substantially affect its physical content, in the sense that it does not give rise to observable phenomena. Indeed, there is no experimental test that could distinguish in QED between the "manifestly covariant"[28] Gupta-Bleuler formalism and the "noncovariant" Dirac-Schwinger formalism. The same is the case with general covariance in CGR: as was argued already by Kretchmann (cf. Sec. 1.1), there is no experimental test that can establish that a CGR model is generally covariant in the passive sense of invariance of its laws under the coordinate transformations in (2.1.2); and, as we pointed out in Secs. 2.2 and 7.8, the same remains true if general covariance is interpreted in the active sense of invariance under diffeomorphisms. The reason is basically that such transformations do *not* have an *operational* meaning. And the same is true of the generic supergauge transformations in geometric quantum gravity.

8.10. Summary and Conclusion

The primary concern in this monograph was with QGR physical *principles* and the mathematical techniques required for their implementation. Those principles have their roots in CGR and in orthodox quantum theory; whereas, the techniques have their roots primarily

[27] In contrast, a great variety of inequivalent Lagrangian densities appear in conventional approaches to extended BRST symmetry in quantum gravity. For example, a list of no less than 27 alternatives is provided on pp. 198-199 of (Pasti and Tonin, 1982).

[28] As we have seen, however, in Chapter 7, on account of (7.4.3) the Gupta-Bleuler formalism is not manifestly covariant at a mathematically rigorous level.

in the modern theories of coherent states, of fibre and superfibre bundles, and of connections on such bundles. Their consistent combination mediated the formulation of the key concept of quantum-geometric propagation, which incorporated into the resulting QGR framework the adaptation to the quantum regime of key CGR ideas on *geometric* locality, on local energy-momentum conservation, on general covariance, and on the equivalence of reference frames in free-fall to local inertial frames. All these fundamental principles were incorporated into a mathematically sound extrapolation of the conventional formulation of path integration. Consequently, action integrals for the resulting quantum-geometric propagators were not postulated; rather, they were derived from those basic principles.

Although the existence of a bridge to conventional perturbative QFT in Minkowski space was established and discussed in the concluding section of Chapter 5, the present formulation of quantum-geometric propagation is fundamentally nonperturbative in nature. The development of nonperturbative techniques in QFT and quantum gravity is still in its infancy, but various approaches (cf. e.g., Ashtekar, 1991b, Fröhlich, 1991) have made their appearance in recent years, and the subject is bound to undergo rapid growth. Since quantum-geometric propagators in the semi-classical regime as well as in geometric quantum gravity were provided in terms of path-integrals based on well-defined iterative procedures, of special computational significance to the present QGR program are the various recently developed Monte Carlo methods for "summing over paths."[29] These types of techniques can be readily adapted to the numerical evaluation of the quantum-geometric propagators formulated in this monograph. Thus, the present quantum-geometric framework can provide computationally viable ideas and alternative lines of approach to the development of nonperturbative techniques.

The research methodology followed in developing the present QGR programme is the one advocated by the founders of quantum theory and relativity theory, and most explicitly and consistently by Dirac (1939, 1951, 1963, 1968, 1978, 1987). The philosophy of science on which this traditional methodology rests is that, in the face of the bewildering variety and complexity of the physical phenomena with which a researcher is confronted, the pragmatic development of computational rules for arriving at predictions pertaining to those phenomena might lead to initial short-lived successes in the form of agreement with a select groups of observations; but that, in the long run, it will not supply a coherent theory that can survive the test of time.[30] Rather, during the search for an underlying simplicity in natural laws, a powerful and trustworthy guide is provided by the development of the type of mathematics most suitable for expressing those laws; such mathematics is distinguished by its elegant simplicity and its internal consistency.

In fact, in Dirac's view, the "mathematical beauty" of a mathematical framework for a physical theory is its most desirable feature (cf. Kragh, 1990, Chapter 14), and it cannot be

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[29] Cf. Chapter 11 in (Khandekar *et al.*, 1993) for a comprehensive survey and for further references.

[30] The rapid changes of fashions in elementary particle physics provide a good example: right after the initial numerical successes of the renormalization program in QED in the late 1940s, difficulties were encountered with extending the same formal rules to weak and strong interactions. Hence, there followed vigorous promotions of dispersion relations, Regge poles, current algebras, static quark models, QCD, supersymmetry, grand-unification – in a chain that culminated in the 1980s with the claim that superstrings provided the "Theory Of Everything." The 1990s have, however, brought new fashions to the fore, and the Theory Of Everything is gradually fading away – just as predicted by Feynman and Glashow in the interviews carried out by Davies and Brown (1989).

achieved without the use of "sound mathematics."

The essence of Dirac's advice to researchers in theoretical and mathematical physics emerges most clearly from the following quotation (Dirac, 1939, pp. 124-125):

> The research worker, in his efforts to express the fundamental laws of Nature in mathematical form, should strive mainly for mathematical beauty. He should still take simplicity into consideration in a subordinate way to beauty. ... It often happens that the requirements of simplicity and beauty are the same, but where they clash the latter must take precedence. ... [B]egin by choosing that branch of mathematics which one thinks will form the basis of the new theory. ... Having decided on the branch of mathematics, one should proceed to develop it along suitable lines, at the same time looking for that way in which it appears to lend itself naturally to physical interpretation."

It has been clear to many researchers in the past that the most natural path towards the unification of general relativity and quantum theory required the geometrization of the latter. Hence, the fact that the choice of mathematics for this task was to be found in some form of geometry has been clear from the outset. The main problem was to find the kind of mathematics that would "lend itself naturally to physical interpretation," and would also display in clearly discernable form "mathematical beauty."

The physical interpretation that eventually led to the present form of quantum general relativity was found in the special relativistic extrapolation of nonrelativistic quantum mechanics described in Chapter 3. It was based on *bona fide* Poincaré systems of covariance and on related *bona fide* conserved probability currents. Most significantly, this mathematical framework displayed no divergencies in the QFT regime.[31] It, therefore, provided a suitable tool for implementing Dirac's commendation that: "One must seek a new relativistic quantum mechanics and one's prime concern must be to base it on sound mathematics." (Dirac 1978b, p. 6.)

The mathematics required by the subsequent further extrapolation of this new relativistic quantum mechanics and quantum field theory to the general relativistic regime was described in Chapters 4-8. It has its origins in areas of modern differential geometry which distinguish themselves by the kind of natural elegance and conceptual simplicity that is commonly associated with the idea of "mathematical beauty." Moreover, this type of mathematics provides a most natural extrapolation of the modern mathematical framework for classical general relativity, without sacrificing any basic quantum principles in the process. It is fundamentally nonperturbative in nature, but it can nevertheless be subjected to all the formal manipulations characteristic of the perturbative approach as readily as that of conventional QFT. It, therefore, opens new vistas in our understanding of the laws of the microcosm, without sacrificing the empirically verifiable substance of what has already been achieved in conventional relativistic quantum theory during the second half of this century.

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[31] With the exception of the infrared divergencies in the mass-zero case, which, however, can be handled by performing infinite but well-defined wave function renormalizations, obtained by taking $m \to +0$ limits in the manner explained in Secs. 7.3 and 7.5.

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Principles of Quantum General Relativity

by Eduard Prugovečki

This book explains, in a manner accessible to readers with a knowledge of general relativity and quantum theory acquired from introductory graduate-level courses, the principles of a quantum-geometric unification of these two disciplines. These principles have their roots in the writings of Einstein, Bohr, Born, Dirac, Heisenberg, and other founders of these disciplines. Although their formulation and development has sometimes followed a tortuous path, the origins of this path are clearly visible in those writings, as shown in historical notes provided in appropriate contexts. By taking advantage of recent advances in such areas of modern theoretical physics and mathematics as the fibre bundle theory, gauge fields, group-theoretical methods, coherent states, etc., these principles can be incorporated into quantum geometries that provide the foundations of an extrapolation of general relativity to the quantum regime. The main goal of this volume is to present the resulting geometry of quantum general relativity in a manner which sets emphasis on fundamental physical ideas and their precise mathematical implementation, and which conforms to the methodologies championed in classical general relativity and in relativistic quantum theory by Einstein and by Dirac, as they founded these respective disciplines.

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